AN ALTERNATIVE PROOF OF THE LINEARITY OF THE SIZE-RAMSEY NUMBER OF PATHS

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ABSTRACT. The size-Ramsey number $\hat{r}(F)$ of a graph F is the smallest integer m such that there exists a graph G on m edges with the property that every colouring of the edges of G with two colours yields a monochromatic copy of F. In 1983, Beck provided a beautiful argument that shows that $\hat{r}(P_n)$ is linear, solving a problem of Erdős. In this note, we provide another proof of this fact that actually gives a better bound, namely, $\hat{r}(P_n) < 137n$ for n sufficiently large.

1. Introduction

For given two finite graphs F and G, we write $G \to F$ if for *every* colouring of the edges of G with two colours (say blue and red) we obtain a monochromatic copy of F (that is, a copy that is either blue or red). The *size-Ramsey number* of a graph F, introduced by Erdős, Faudree, Rousseau and Schelp [7] in 1978, is defined as follows:

$$\hat{r}(F) = \min\{|E(G)| : G \to F\}.$$

In this note, we consider the size-Ramsey number of the path P_n on n vertices. It is obvious that $\hat{r}(P_n) = \Omega(n)$ and that $\hat{r}(P_n) = O(n^2)$ (for example, $K_{2n} \to P_n$) but the exact behaviour of $\hat{r}(P_n)$ was not known for a long time. In fact, Erdős [6] offered \$100 for a proof or disproof that

$$\hat{r}(P_n)/n \to \infty \quad \text{and} \quad \hat{r}(P_n)/n^2 \to 0.$$

The problem was solved by Beck [2] in 1983 who, quite surprisingly, showed that $\hat{r}(P_n) < 900n$ for sufficiently large n. A variant of his proof was provided by Bollobás [5] and it gives $\hat{r}(P_n) < 720n$ for sufficiently large n. It is worth mentioning that both of these bounds are not explicit constructions. Later Alon and Chung [1] gave an explicit construction of graphs G on $O(n)$ vertices with $G \to P_n$.

Here we provide an alternative and elementary proof of the linearity of the size-Ramsey number of paths that gives a better bound. The proof relies on a simple observation, Lemma 2.1, which may be applicable elsewhere.

Theorem 1.1. *For n sufficiently large, $\hat{r}(P_n) < 137n$.*

In order to show the result, similarly to Beck and Bollobás, we are going to use binomial random graphs. The *binomial random graph* $G(n,p)$ is the random graph G on vertex set $[n]$ for which for every pair $\{i,j\} \in \binom{[n]}{2}$, $\{i,j\}$ appears independently as an edge in G

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with probability p. Note that $p = p(n)$ may, and usually does, tend to zero as n tends to infinity. All asymptotics throughout are as $n \to \infty$. We say that a sequence of events $\mathcal{E}_n$ in a probability space holds *asymptotically almost surely* (or *a.a.s.*) if the probability that $\mathcal{E}_n$ holds tends to 1 as n goes to infinity. For simplicity, we do not round numbers that are supposed to be integers either up or down; this is justified since these rounding errors are negligible to the asymptomatic calculations we will make.

2. Proof of Theorem 1.1

We start with the following elementary observation.[1]

Lemma 2.1. *Let $c > 1$ be a real number and let $G = (V, E)$ be a graph on cn vertices. Suppose that the edges of G are coloured with the colours blue and red and there is no monochromatic P_n. Then the following two properties hold:*

(i) *there exist two disjoint sets $U, W \subseteq V$ of size $n(c-1)/2$ such that there is no blue edge between U and W,*
(ii) *there exist two disjoint sets $U', W' \subseteq V$ of size $n(c-1)/2$ such that there is no red edge between U' and W'.*

Proof. We perform the following algorithm on G and construct a blue path P. Let v_1 be an arbitrary vertex of G, let $P = (v_1)$, $U = V \setminus \{v_1\}$, and $W = \emptyset$. We investigate all edges from v_1 to U searching for a blue edge. If such an edge is found (say from v_1 to v_2), we extend the blue path as $P = (v_1, v_2)$ and remove v_2 from U. We continue extending the blue path P this way for as long as possible. Since there is no monochromatic P_n, we must reach the point of the process in which P cannot be extended, that is, there is a blue path from v_1 to v_k ($k < n$) and there is no blue edge from v_k to U. This time, v_k is moved to W and we try to continue extending the path from v_{k-1}, reaching another critical point in which another vertex will be moved to W, etc. If P is reduced to a single vertex v_1 and no blue edge to U is found, we move v_1 to W and simply re-start the process from another vertex from U, again arbitrarily chosen.

An obvious but important observation is that during this algorithm there is never a blue edge between U and W. Moreover, in each step of the process, the size of U decreases by 1 or the size of W increases by 1. Finally, since there is no monochromatic P_n, the number of vertices of the blue path P is always smaller than n. Hence, at some point of the process both U and W must have size at least $n(c-1)/2$. Part (i) now holds after removing some vertices from U or W, if needed, so that both sets have sizes precisely $n(c-1)/2$.

Part (ii) can be proved by a symmetric argument; this time the algorithm tries to build a red path. The proof is finished. □

Now, we prove the following straightforward properties of random graphs. For every two disjoint sets S and T, $e(S, T)$ denotes the number of edges between S and T.

Lemma 2.2. *Let $c = 7.29$ and $d = 5.14$, and consider $G = (V, E) \in G(cn, d/n)$. Then, the following two properties hold a.a.s.:*

(i) $|E(G)| = (1 + o(1))nc^2d/2 < 137n$,
(ii) *for every two disjoint sets of vertices S and T such that $|S| = |T| = n(c-3)/4$ we have $e(S, T) \neq 0$.*

[1] A similar result was independently obtained by Pokrovskiy [10].

Proof. Part (i) is obvious. The expected number of edges in G is $\binom{cn}{2}\frac{d}{n} = (1+o(1))nc^2d/2$, and the concentration around the expectation follows immediately from Chernoff's bound.

For part (ii), let X be the number of pairs of disjoint sets S and T of desired size such that $e(S,T) = 0$. Putting $\alpha = \alpha(c) = (c-3)/4$ for simplicity, we get

$$\begin{aligned} \mathbb{E}[X] &= \binom{cn}{\alpha n}\binom{(c-\alpha)n}{\alpha n}\left(1-\frac{d}{n}\right)^{\alpha n\cdot\alpha n} \\ &\le \frac{(cn)!}{(\alpha n)!(\alpha n)!((c-2\alpha)n)!}\exp\big(-d\alpha^2 n\big). \end{aligned}$$

Using Stirling's formula ($x! = (1+o(1))\sqrt{2\pi x}(x/e)^x$) we get that $\mathbb{E}[X] \le \exp(f(c,d)n)$, where

$$f(c,d) = c\ln c - 2\alpha\ln\alpha - (c-2\alpha)\ln(c-2\alpha) - d\alpha^2.$$

Putting numerical values of c and d into the formula, we get $f(c,d) < -0.008$ and so $\mathbb{E}[X] \to 0$ as $n \to \infty$. (The values of c and d were chosen so as to minimize $c^2d/2$ under the condition $f(c,d) < 0$.) Now part (ii) holds by Markov's inequality. □

Now, we are ready to prove the main result.

Proof of Theorem 1.1. Let $c = 7.29$ and $d = 5.14$, and consider $G = (V,E) \in G(cn, d/n)$. We show that a.a.s. $G \to P_n$ which will finish the proof by Lemma 2.2(i).

For a contradiction, suppose that $G \not\to P_n$. Thus, there is a blue-red colouring of E with no monochromatic P_n. It follows (deterministically) from Lemma 2.1(i) that V can be partitioned into three sets P, U, W such that $|P| = n$, $|U| = |W| = n(c-1)/2$, and there is no blue edge between U and W. Similarly, by Lemma 2.1(ii), V can be partitioned into three sets P', U', W' such that $|P'| = n$, $|U'| = |W'| = n(c-1)/2$, and there is no red edge between U' and W'.

Now, consider $X = U\cap U'$, $Y = U\cap W'$, $X' = W\cap U'$, $Y' = W\cap W'$ and let $x = |X|$, $y = |Y|$, $x' = |X'|$, $y' = |Y'|$ be their sizes, respectively. Observe that

$$x + y = |U \cap (U'\cup W')| = |U\setminus P'| \ge |U| - |P'| = n(c-3)/2. \qquad (1)$$

Similarly, one can show that $x'+y' \ge n(c-3)/2$, $x+x' \ge n(c-3)/2$, and that $y+y' \ge n(c-3)/2$. We say that a set is *large* if its size is at least $n(c-3)/4$; otherwise, we say that it is *small*. We need the following straightforward observation.

Claim. Either both X and Y' are large or both Y and X' are large.

(In fact one can easily show that the constant $(c-3)/4$ in the definition of being large is optimal.)

Proof of the claim. For a contradiction, suppose that at least one of X, Y' is small *and* at least one of Y, X' is small, say, X and Y are small. But this implies that $x + y < n(c-3)/4 + n(c-3)/4 = n(c-3)/2$, which contradicts (1). The remaining three cases are symmetric, and so the claim holds.

Now, let us come back to the proof. Without loss of generality, we may assume that $X = U\cap U'$ and $Y' = W\cap W'$ are large. Since $X \subseteq U$ and $Y' \subseteq W$, there is no blue edge between X and Y'. Similarly, one can argue that there is no red edge between X and Y', and so $e(X,Y') = 0$. On the other hand, Lemma 2.2(ii) implies that a.a.s.

$e(X, Y') \neq 0$, reaching the desired contradiction. It follows that a.a.s. $G \to P_n$ which finishes the proof. □

3. Remarks

In this note we showed that $\hat{r}(P_n) < 137n$. On the other hand, the best known lower bound, $\hat{r}(P_n) \geq (1+\sqrt{2})n - 2$, was given by Bollobás [4] who improved the previous result of Beck [3] that shows that $\hat{r}(P_n) \geq \frac{9}{4}n$. Decreasing the gap between the lower and upper bounds might be of some interest. One approach to improving the upper bound could be to deal with non-symmetric cases in our claim or to use random d-regular graphs instead of binomial graphs.

Another related problem deals with longest monochromatic paths in $G(n,p)$. Observe that it follows from the proof of Theorem 1.1 that for every $\omega = \omega(n)$ tending to infinity arbitrarily slowly together with n we have that a.a.s. any 2-colouring of the edges of $G(n, \omega/n)$ yields a monochromatic path of length $\frac{(1-\varepsilon)}{3}n$ for an arbitrarily small $\varepsilon > 0$. On the other hand, a simple construction of Gerencsér and Gyárfás [8] shows that such path cannot be longer than $\frac{2}{3}n$. We conjecture that actually $(1+o(1))\frac{2}{3}n$ is the right answer for random graphs with average degree tending to infinity.[2]

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[2]The conjecture was recently proved by Letzter [9].