PURSUIT-EVASION IN MODELS OF COMPLEX NETWORKS

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Abstract. Pursuit-evasion games, such as the game of Cops and Robbers, are a simplified model for network security. In these games, cops try to capture a robber loose on the vertices of the network. The minimum number of cops required to win on a graph G is the cop number of G. We present asymptotic results for the game of Cops and Robbers played in various stochastic network models, such as in $G(n,p)$ with non-constant p, and in random power law graphs. We find bounds for the cop number of $G(n,p)$ for a large range of p as a function of n. We prove that the cop number of random power law graphs with n vertices is asymptotically almost surely $\Theta(n)$. The cop number of the core of random power law graphs is investigated, and is proved to be of smaller order than the order of the core.

1. Introduction

Suppose that an intruder is loose on the vertices of a network, and travels between adjacent vertices. The intruder could represent a virus or hacker, or some other malicious agent intent on avoiding capture. A set of searchers are attempting to capture the intruder. Although placing a searcher on each vertex guarantees the capture of the intruder, it is a more interesting (and more difficult) problem to find the minimum number of searchers required to capture the intruder. A motivation for minimizing the number of searchers comes from the fact that fewer searchers require fewer resources. Networks that require a smaller number of searchers may be viewed as more secure than those where many searchers are needed.

A pursuit-evasion games, such as Cops and Robbers, may be viewed of as a simplified model for such network security problems. The game of *Cops and Robbers*, introduced independently by Nowakowski and Winkler [16] and Quilliot [18] over twenty years ago, is played on a fixed graph G, and is our focus in this study. We will always assume that G is undirected, simple, and finite. There are two players, a set of k *cops* (or *searchers*), where $k > 0$ is a fixed integer, and the *robber*. The cops begin the game by occupying a set of k vertices. The robber then chooses a vertex, and the cops and robber move in alternate rounds. The players use edges to move from vertex to vertex. More than one cop is allowed to occupy a vertex, and the players may remain on their current vertex. The players know each others current locations. The cops win and the game ends if at least one of the cops can eventually occupy the same vertex as the robber; otherwise, the robber wins. As placing a cop on each vertex guarantees that the cops

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win, we may define the *cop number*, written $c(G)$, which is the minimum number of cops needed to win on G. The cop number was introduced by Aigner and Fromme [1] who proved (among other things) that if G is planar, then $c(G) \leq 3$. For a survey of results on vertex pursuit games such as Cops and Robbers, the reader is directed to the surveys [2, 11, 12].

Over the last few years there was an explosion of mathematical research related to stochastic models of real-world networks, especially for models of the web graph. Many technological, social, biological networks have properties similar to those present in the web, such as power law degree distributions and the small world property. Following [9], we refer to these networks as *complex*. For example, power laws have been observed in protein-protein interaction networks, and social networks such as the one formed by scientific collaborators. While much of the earlier mathematical work on complex networks focused on designing models satisfying certain properties such as power law degree distributions, new approaches are constantly emerging. For additional background on complex networks and their models, see [3, 9].

In this paper, which is the full version of [7], we study vertex pursuit games in random graph models, including models for complex networks. While Cops and Robbers have been extensively studied in highly structured deterministic graphs such as graph products (see [15]), our work is the first to consider such games in models of complex networks.

All asymptotics throughout are as $n \to \infty$. We say that an event in a probability space holds *asymptotically almost surely* (*a.a.s.*) if the probability that it holds tends to 1 as n goes to infinity. For $p \in (0,1)$ or $p = p(n)$ tending to 0 with n, define $\mathbb{L}n = \log_{\frac{1}{1-p}} n$. We denote the incomplete gamma function by $\Gamma(\cdot, \cdot)$.

We consider Erdős, Rényi $G(n,p)$ random graphs and their generalizations used to model complex networks. The *random graph* $G(n,p)$ consists of the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω is the set of all graphs with vertex set $[n] = \{1, 2, \ldots, n\}$, $\mathcal{F}$ is the family of all subsets of Ω, and for every $G \in \Omega$

$$\mathbb{P}(G) = p^{|E(G)|}(1-p)^{\binom{n}{2} - |E(G)|}.$$

This space may be viewed as $\binom{n}{2}$ independent coin flips, one for each pair of vertices, where the probability of success (that is, drawing an edge) is equal to p. Note that $p = p(n)$ can tend to zero as n tends to infinity.

The cop number of $G(n,p)$ was studied in [6], where the following result was proved.

Theorem 1.1. *Let $0 < p < 1$ be fixed. For every real $\varepsilon > 0$ a.a.s. for $G \in G(n,p)$*

$$(1-\varepsilon)\mathbb{L}n \leq c(G) \leq (1+\varepsilon)\mathbb{L}n.$$

The problem of determining the cop number of $G(n,p)$ where $p = p(n)$ is a function of n was left open in [6]. In [5] it was shown that the cop number of $G(n,p)$ is always bounded from above by $n^{1/2+o(1)}$ and this bound is achieved for sparse random graphs. More precisely, they showed that $c(G(n,p)) \leq 160000\sqrt{n}\log n$ for $np \geq 2.1 \log n$ and

$$c(G(n,p)) \geq \frac{1}{(np)^2} n^{\frac{1}{2}\frac{\log\log(np) - 9}{\log\log(np)}}$$

for $np \to \infty$. Since if either $np = n^{o(1)}$ or $np = n^{1/2+o(1)}$, then a.a.s. $c(G(n,p)) = n^{1/2+o(1)}$, it would be natural to assume that the cop number of $G(n,p)$ is close to $\sqrt{n}$ also for $np = n^{\alpha+o(1)}$, where $0 < \alpha < 1/2$. In [14] it was shown that the actual behaviour of $c(G(n,p))$ is more complicated. The function $f : (0,1) \to \mathbb{R}$ defined as

$$f(x) = \frac{\log \mathbb{E}(c(G(n, n^{x-1})))}{\log n},$$

where $\mathbb{E}(c(G(n,p)))$ denotes the expected value of the cop number for $G(n,p)$. The main result of [14] was that f has an unexpected zigzag shape; see Figure 1.

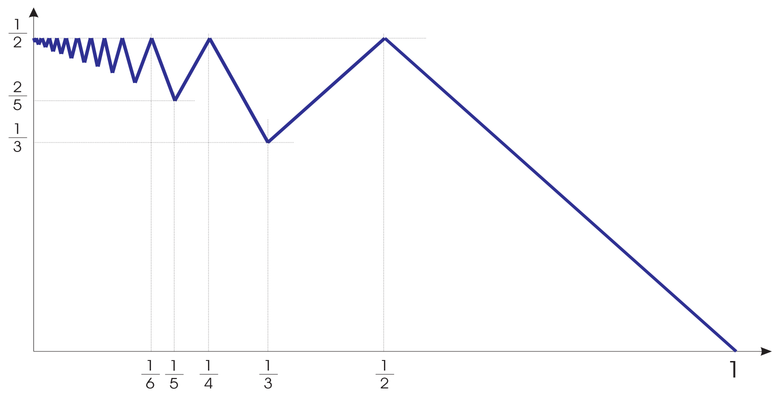

FIGURE 1. The 'zigzag' function f.

In the next subsection, we show that if $np = n^{\alpha+o(1)}$, where $1/2 < \alpha \leq 1$, then a.a.s.

$$c(G(n,p)) = \Theta(\log n/p) = n^{1-\alpha+o(1)}$$

and $c(G(n, n^{-1/2+o(1)})) = n^{1/2+o(1)}$ a.a.s. This result is used in [14] where the main focus is on $0 < \alpha < 1/2$.

Recent work of Chung and Lu [8, 9] supplies an extension of the $G(n,p)$ random graphs to random graphs $G(\mathbf{w})$ with given expected degree sequence $\mathbf{w}$. For example, if $\mathbf{w}$ follows a power law distribution, then $G(\mathbf{w})$ supplies a model for complex networks. We determine bounds on the cop number of random power law graphs as discussed in the next subsection.

1.1. **Results.** We consider the cop number in classical random graphs and for random power law graphs. The proofs of the results in this subsection may be found in Section 2. We consider the cop number of $G(n,p(n))$ when $p(n)$ is a function of n. We will abuse notation and refer to p rather than $p(n)$. For $G(n,p)$ our main results are summarized in the following theorem.

Theorem 1.2.

(1) *Suppose that $p \geq p_0$ where p_0 is the smallest p for which*

$$p^2/40 \geq \frac{\log\left((\log^2 n)/p\right)}{\log n}$$

holds. Then a.a.s. $G \in G(n,p)$ satisfies

$$\mathbb{L}n - \mathbb{L}\big((p^{-1}\mathbb{L}n)(\log n)\big) \le c(G) \le \mathbb{L}n - \mathbb{L}\big((\mathbb{L}n)(\log n)\big) + 2.$$

(2) *If $(2\log n)/\sqrt{n} \le p = o(1)$ and $\omega(n)$ is any function tending to infinity, then a.a.s. $G \in G(n,p)$ satisfies*

$$\mathbb{L}n - \mathbb{L}\big((p^{-1}\mathbb{L}n)(\log n)\big) \le c(G) \le \mathbb{L}n + \mathbb{L}(\omega(n)).$$

The proof may be found in Subsection 2.1 below. By Theorem 1.2, we have the following corollary.

Corollary 1.3. *If $p = n^{-o(1)}$ and $p < 1$, then a.a.s. $G \in G(n,p)$ satisfies*

$$c(G) = (1+o(1))\mathbb{L}n.$$

Indeed, from part (1) it follows that if p is a constant, then

$$c(G) = \mathbb{L}n - 2\mathbb{L}\log n + \Theta(1) = (1+o(1))\mathbb{L}n.$$

From part (2), for $p = n^{-o(1)}$ tending to zero with n, the lower bound is

$$\begin{aligned}\mathbb{L}n - \mathbb{L}\big((p^{-1}\mathbb{L}n)(\log n)\big) &= \mathbb{L}n - 2\mathbb{L}\big((1+o(1))p^{-1}\log n\big)\\ &= \mathbb{L}n - 2\mathbb{L}\big(n^{o(1)}\big)\\ &= (1+o(1))\mathbb{L}n.\end{aligned}$$

Note also that for $p = n^{-a(1+o(1))}$ $(0 < a < 1/2)$ we do not have a concentration for $c(G)$ but the following bounds hold

$$(1+o(1))(1-2a)\mathbb{L}n \le c(G) \le (1+o(1))\mathbb{L}n.$$

We now describe results for the cop number of random power law graphs. Let

$$\mathbf{w} = (w_1, \dots, w_n)$$

be a sequence of n nonnegative real numbers. We define a random graph model, written $G(\mathbf{w})$, as follows. Typically, vertices are integers in $[n]$. Each potential edge between i and j is chosen independently with probability $p_{ij} = w_i w_j \rho$, where

$$\rho = \frac{1}{\sum_{i=1}^{n} w_i}.$$

We will always assume that

$$\max_i w_i^2 < \sum_{i=1}^{n} w_i,$$

which implies that $p_{ij} \in [0,1)$. The model $G(\mathbf{w})$ is referred to as *random graphs with given expected degree sequence* $\mathbf{w}$. Observe that $G(n,p)$ may be viewed as a special case of $G(\mathbf{w})$ by taking $\mathbf{w}$ to be equal the constant n-sequence $(pn, pn, \dots, pn)$.

Given $\beta > 2$, $d > 0$, and a function $M = M(n) = o(\sqrt{n})$ (with M tending to infinity with n), we consider the random graph with given expected degrees $w_i > 0$, where

$$w_i = c i^{-\frac{1}{\beta-1}} \tag{1}$$

for i satisfying $i_0 \le i < n + i_0$. The term c depends on β and d, and i_0 depends also on M; namely,

$$c = \left(\frac{\beta-2}{\beta-1}\right) dn^{\frac{1}{\beta-1}}, \qquad i_0 = n\left(\frac{d}{M}\left(\frac{\beta-2}{\beta-1}\right)\right)^{\beta-1}. \tag{2}$$

It is not hard to show (see [8, 9]) that a.a.s. the random graphs with the expected degrees satisfying (1) and (2) follow a power law degree distribution with exponent β, average degree $(1+o(1))d$, and maximum degree $(1+o(1))M$.

Our next theorem demonstrates that the cop number of random power law graphs is a.a.s. $\Theta(n)$, and so is of much larger order than the logarithmic cop number of $G(n,p)$ random graphs. Hence, these results are suggestive that in power law graphs, on average a large number of cops are needed to secure the network.

Theorem 1.4. *For a random power law graph $G \in G(\mathbf{w})$ with exponent $\beta > 2$ and average degree d, a.a.s. the following hold.*

(1) *If X is the random variable denoting the number of isolated vertices in $G(\mathbf{w})$, then*

$$\begin{aligned} c(G) &\ge X \\ &= (1+o(1))n\int_0^1 \exp\left(-d\frac{\beta-2}{\beta-1}x^{-1/(\beta-1)}\right)dx \\ &= (1+o(1))(d(\beta-2))^{\beta-1}(\beta-1)^{2-\beta}n\Gamma\left(1-\beta, d\frac{\beta-2}{\beta-1}\right). \end{aligned}$$

(2) *For $a \in (0,1)$, define*

$$f(a) = a + \int_a^1 \exp\left(-d\frac{\beta-2}{\beta-1}a^{(\beta-2)/(\beta-1)}x^{-1/(\beta-1)}\right)dx.$$

Then

$$c(G) \le (1+o(1))n \min_{0<a<1} f(a).$$

The proof of Theorem 1.4 may be found in Subsection 2.3. We note that integrals in the statement of Theorem 1.4 do not possess closed-form solutions in general. We supply numerical values for lower/upper bounds of the cop number of $G(\mathbf{w})$ when $d = 10, 20$ and $\beta = 2.1, 2.7$ (note that the values of $d = 10$ and $\beta = 2.1$ coincide with earlier experimental results found in the web graph; see for example the survey [3]).

TABLE 1. Upper and lower bounds for the cop number of $G(\mathbf{w})$ for various values of d (top row) and β (left column).

	10	20
2.1	$0.1806/0.2940$	$0.5112 \cdot 10^{-1}/0.1265$
2.7	$0.4270 \cdot 10^{-2}/0.1895$	$0.4205 \cdot 10^{-4}/0.8261 \cdot 10^{-1}$

While Theorem 1.4 suggests a large number of cops are needed to secure complex networks against intruders, by item (1) it is the abundance of isolated vertices that makes the cop number equal to $\Theta(n)$. To overcome the issue with isolated vertices, we consider restricting the movements of the cops and robber to the subgraph induced by sufficiently high degree vertices.

Fix $\beta \in (2,3)$. Define the *core* of a graph G, written $\widehat{G}$, as the subgraph induced by the set of vertices of degree at least $n^{1/\log\log n}$. Random power law graphs with $\beta \in (2,3)$ are referred to as *octopus* graphs in [9], since the core is dense with small diameter $O(\log\log n)$ and the overall diameter is $O(\log n)$. For $G \in G(\mathbf{w})$, since the expected degree of vertex i in G is

$$w_i = \frac{\beta-2}{\beta-1} d n^{1/(\beta-1)} i^{-1/(\beta-1)},$$

vertices with expected degree at least $n^{1/\log\log n}$ have labels at most

$$i_N = \left(\frac{\beta-2}{\beta-1} d\right)^{\beta-1} n^{1-(\beta-1)/\log\log n}.$$

The order of the core is written N. By Chernoff's bound,

$$N = (1+o(1)) i_N - i_0 = (1+o(1)) i_N = \Theta(n^{1-(\beta-1)/\log\log n}),$$

provided that $\log M \gg (\log n)/\log\log n$. Thus,

$$n = N^{1+(\beta-1)/\log\log N + \Theta(1)/\log^2\log N}. \tag{3}$$

We consider the cop number of the core of random power law graphs (so the cop and robber are restricted to movements within the core). As vertices in the core informally represent the *hubs* of the network, one would suspect that the cop number of the core is of smaller order than the core itself. This intuition is made precise by the following theorem, which provides a sublinear upper bound for the cop number of the core. The proof is deferred to Subsection 2.4.

Theorem 1.5. *For a random power law graph $G \in G(\mathbf{w})$ with power law exponent $\beta \in (2,3)$ a.a.s. the cop number of the core $\widehat{G}$ of G satisfies*

$$N^{(1+o(1))(3-\beta)/\log\log N} \le c(\widehat{G}) \le N^{1-(1+o(1))(\beta-1)(3-\beta)/(\beta-2)\log\log n}.$$

As the asymptotic bounds in Theorem 1.5 are not tight, it is an interesting open problem to determine the asymptotic value of the cop number of the core of random power law graphs.

2. Proofs of Main Results

The remainder of the paper is devoted to the proof of our three main theorems. Each subsequent subsection contains a proof of the required bounds in each theorem. In all cases, the upper bound for the cop number is provided by a corresponding bound of the domination number. In all cases except the proof of Theorem 1.4 (which estimates the number of isolated vertices), a lower bound for the cop number is derived by considering an appropriate adjacency property. A minor detour is made in Subsection 2.2, which gives a concentration result for the cop number in sparse $G(n,p)$ random graphs.

2.1. **Proof of Theorem 1.2.** We first prove the upper bound in item (1), and require some background on the domination number of a graph. A set of vertices S is a *dominating set* in G if each vertex not in S is joined to some vertex of S. The *domination number* of G, written $\gamma(G)$, is the minimum cardinality of a dominating set in G. A straightforward observation is that

$$c(G) \leq \gamma(G), \tag{4}$$

(place a cop on each vertex of dominating set with minimum cardinality). However, if $n \geq 1$, then $c(P_n) = 1$ (where P_n is a path with n vertices) and $\gamma(P_n) = \left\lceil \frac{n}{3} \right\rceil$. The bound of (4) while useful, is far from tight in general. Domination in models of complex networks were considered by Cooper et al. in [10].

The upper bound in Theorem 1.2 (1) follows by the following result proved in [19].

Theorem 2.1. *Suppose that $p \geq p_0(n)$ where p_0 is the smallest p for which*

$$p^2/40 \geq \frac{\log\big((\log^2 n)/p\big)}{\log n}$$

holds. Then a.a.s. $G \in G(n,p)$ satisfies

$$\lfloor \mathbb{L}n - \mathbb{L}\big((\mathbb{L}n)(\log n)\big) \rfloor + 1 \leq \gamma(G) \leq \lfloor \mathbb{L}n - \mathbb{L}\big((\mathbb{L}n)(\log n)\big) \rfloor + 2.$$

For item (2), the proof follows by the following theorem.

Theorem 2.2. *If $p = o(1)$ and $\omega(n)$ is any function tending to infinity with n, then a.a.s. $G \in G(n,p)$ satisfies*

$$\gamma(G) \leq \lceil \mathbb{L}n + \mathbb{L}(\omega(n)) \rceil .$$

Proof. Since $p = o(1)$ we have that

$$\mathbb{L}n = \frac{\log n}{-\log(1-p)} = (1+o(1))\frac{\log n}{p} . \tag{5}$$

The probability that the domination number of a random graph is at most

$$k = \lceil \mathbb{L}n + \mathbb{L}(\omega(n)) \rceil$$

is bounded from below by the probability that any fixed set of k vertices is a dominating set. But the latter probability is equal to

$$\begin{aligned} \big(1-(1-p)^k\big)^{n-k} &\geq 1-(n-k)(1-p)^k \\ &\geq 1-n(1-p)^k \\ &\geq 1-n(1-p)^{\mathbb{L}n+\mathbb{L}(\omega(n))} \\ &= 1-\frac{1}{\omega(n)} \\ &= 1-o(1) . \end{aligned}$$

□

For the proofs of the lower bounds in Theorem 1.2, we employ the following adjacency property. For a fixed $k > 0$ an integer, we say that G is $(1,k)$-*existentially closed* (or $(1,k)$-*e.c.*) if for each k-set S of vertices of G and vertex $u \notin S$, there is a vertex $z \notin S$ not joined to a vertex in S and joined to u. If G is $(1,k)$-*e.c.*, then $c(G) \geq k$ (the robber may use the property to escape to a vertex not joined to any vertex occupied by a cop).

The lower bound in Theorem 1.2 will follow once we prove the following theorem.

Theorem 2.3. *If* $p > (2\log n)/\sqrt{n}$ *and*

$$k = \left\lfloor \mathbb{L}n - \mathbb{L}\big((p^{-1}\mathbb{L}n)(\log n)\big) \right\rfloor , \tag{6}$$

then a.a.s. $G \in G(n,p)$ *is* $(1,k)$*-e.c.*

Note that we do not use the condition for p in the proof of the theorem. The condition is introduced in order to get a non-trivial result; the value of k is negative otherwise.

Proof. Assume that $p = o(1)$. Then

$$\begin{aligned} k &= \mathbb{L}n - \mathbb{L}\left((1+o(1))\frac{\log^2 n}{p^2}\right) \\ &= \mathbb{L}n - 2\mathbb{L}\left((1+o(1))\frac{\log n}{p}\right). \end{aligned}$$

Fix S a k-subset of vertices and a vertex u not in S. For a vertex $x \in V(G)\backslash(S \cup \{u\})$, the probability that a vertex x is joined to u and to no vertex of S is $p(1-p)^k$. Since edges are chosen independently, the probability that no suitable vertex can be found for this particular S and u is $(1-p(1-p)^k)^{n-k-1}$.

Let X be the random variable counting the number of S and u for which no suitable x can be found. We then have that

$$\begin{aligned} \mathbb{E}(X) &= \binom{n}{k}(n-k)\left(1-p(1-p)^k\right)^{n-k-1} \\ &\leq n^{k+1}\left(1 - \frac{(\mathbb{L}n)(\log n)}{n}\right)^{n(1-(\mathbb{L}n)/n)} \\ &= n^{k+1}\exp\left(-(\mathbb{L}n)(\log n)(1-(\mathbb{L}n)/n)\right)(1+o(1)) \\ &= n^{k+1}\exp\left(-(\mathbb{L}n - (\mathbb{L}n)^2/n)(\log n)(1+o(1))\right) \\ &\leq n^{k+1}\exp\left(-\left(k + \frac{2\log\log n}{p} - \frac{2\log^2 n}{p^2 n}\right)(\log n)(1+o(1))\right) \\ &= n^{k+1}\exp\left(-\left(k + \frac{2\log\log n}{p}\right)(\log n)(1+o(1))\right) \\ &= o(1), \end{aligned}$$

where the second inequality follows by (5). It is also easy to show that the same argument holds for p a constant. The proof now follows by Markov's inequality. $\square$

2.2. **Extreme cases.** It is known that if p is tending to one pretty fast, then the maximum degree of a random graph is very large. Formally, for a fixed non-negative integer k, if $n(1-p) - \log n - k \log\log n \to -\infty$, then the maximum degree is at least $n-1-k$ a.a.s. and clearly $k+1$ is an upper bound for a cop number (one cop can occupy a vertex with maximum degree). In fact

$$p = p(n) = 1 - \left(\frac{k \log n}{n}\right)^{\frac{1}{k}}$$

is the threshold for the cop number $k+1$ (see [17] for more).

We next provide a concentration result for the cop number of the random graphs $G(n,p)$ for p approaching zero very fast. For example, if $p = o(1/n^2)$, a.a.s. $G \in G(n,p)$ is empty. In this range of p a.a.s. the cop number of G is n. We now consider the case when $p = d/n$ for constant $d \in (0,1)$. Bollobás [4] proved the following result.

Theorem 2.4. *Let $0 < d < 1$, $p = d/n$, and let X be the number of tree connected components of $G(n,p)$. Then the expectation of X is*

$$\mathbb{E}(X) = u(d)n + O(1),$$

where

$$u(d) = \frac{1}{d}\sum_{k=1}^{\infty} \frac{k^{k-2}}{k!}(de^{-d})^k.$$

A.a.s. $G(n,p)$ satisfies

$$|X| = (1+o(1))u(d)n.$$

We note that $u(d) \in (0,1)$. A graph is *unicyclic* if it contains exactly one cycle.

Theorem 2.5. *Let $0 < d < 1$ and $p = d/n$. Then a.a.s. $G \in G(n,p)$ is such that every connected component is a tree or a unicyclic graph, and there are at most $\log\log n$ vertices in the unicyclic components.*

Trees are cop-win graphs, while unicyclic graphs have cop number at most 2. Each tree component requires exactly one cop, while there are at most $2\log\log n$ many cops needed for all the unicyclic components. Hence, the number of cops on the unicyclic components becomes negligible in contrast to the number of cops on the tree components. Therefore, from Theorems 2.4 and 2.5 we have the following concentration result.

Corollary 2.6. *Let $0 < d < 1$, $p = d/n$. Then for the graph $G \in G(n,p)$,*

$$\mathbb{E}(c(G)) = u(d)n + O(\log\log n).$$

A.a.s. $G \in G(n,p)$ satisfies

$$c(G) = (1+o(1))u(d)n.$$

2.3. **The proof of Theorem 1.4.**

Proof. For the lower bound, we exploit the fact that the cop number is bounded from below by the number of isolated vertices: one cop is needed per isolated vertex. In general power law graphs, there may exist an abundance of isolated vertices, even as much as $\Theta(n)$ many. We show that this is indeed the case for random power law graphs.

The probability that the vertex i for $i_0 \leq i < n + i_0$ (that is, the vertex i corresponds to the weight w_i) is isolated is equal to

$$\begin{aligned} p_i &= \prod_{j, j \neq i} (1 - w_i w_j \rho) \\ &= \prod_{j, j \neq i} \exp\left(-w_i w_j \rho (1 + o(1))\right) \\ &= \exp\left(-w_i \rho \sum_{j, j \neq i} w_j (1 + o(1))\right) \\ &= \exp\left(-w_i (1 + o(1))\right) . \end{aligned} \tag{7}$$

Let X_i be an indicator random variable for the event that the vertex i is isolated. Then

$$\mathbb{P}(X_i = 1) = 1 - \mathbb{P}(X_i = 0) = p_i$$

for $i_0 \leq i < n + i_0$.

Let X be the number of isolated vertices in $G(\mathbf{w})$. As $X = \sum_{i_0 \leq i < n + i_0} X_i$, it follows from (7) that the expected value of X is

$$\begin{aligned} \sum_{i_0 \leq i < n + i_0} p_i &= (1 + o(1)) n \int_0^1 \exp\left(-c(xn)^{-1/(\beta-1)}\right) dx \\ &= (1 + o(1)) n \int_0^1 \exp\left(-d \frac{\beta - 2}{\beta - 1} x^{-1/(\beta-1)}\right) dx \, . \end{aligned}$$

A sum of independent random variables with large enough expected value is concentrated on its expected value (see, for example, Theorem 2.8 in [13]). Thus, the number of isolated vertices in $G(\mathbf{w})$ is a.a.s. equal to

$$\begin{aligned} X &= (1 + o(1)) n \int_0^1 \exp\left(-d \frac{\beta - 2}{\beta - 1} x^{-1/(\beta-1)}\right) dx \\ &= (1 + o(1)) (d(\beta - 2))^{\beta - 1} (\beta - 1)^{2 - \beta} n \int_{d \frac{\beta - 2}{\beta - 1}}^{\infty} t^{-\beta} e^{-t} dt \\ &= (1 + o(1)) (d(\beta - 2))^{\beta - 1} (\beta - 1)^{2 - \beta} n \Gamma\left(1 - \beta, d \frac{\beta - 2}{\beta - 1}\right) . \end{aligned}$$

Hence, item (1) of Theorem 1.4 follows. We note that another asymptotic expression of the number of isolated vertices in random power law graphs is given in [9] (see Theorem 5.20).

For the proof of the upper bound in item (2) of Theorem 1.4, we give a bound on the domination number. Fix a constant $a \in (0, 1)$ and consider the set $A \subseteq V$ of first

$\lfloor an \rfloor - i_0 + 1 = (1+o(1))an$ vertices; that is, $A = \{i_0, i_0+1, \ldots, \lfloor an \rfloor\}$. Let $B \subseteq V \setminus A$ denote the set of vertices that do not have a neighbour in A. Then $D = A \cup B$ is a dominating set of G, and we estimate the cardinality of D.

Consider the vertex i, where $an < i < n + i_0$. Since $i_0 = o(n)$, there is $b \in (0,1]$ such that $i = bn(1+o(1))$. The probability that i does not have a neighbour in A is equal to

$$\begin{aligned} q_i &= \prod_{j < an + i_0} (1 - w_i w_j \rho) \\ &= \exp\left(-(1+o(1)) w_i \rho \sum_{j < an+i_0} w_j \right) \\ &= \exp\left(-(1+o(1)) c(bn)^{-1/(\beta-1)} (dn)^{-1} n \int_0^a c(xn)^{-1/(\beta-1)} dx \right) \\ &= \exp\left(-(1+o(1)) d \Big(\frac{\beta-2}{\beta-1}\Big)^2 b^{-1/(\beta-1)} \int_0^a x^{-1/(\beta-1)} dx \right) \\ &= \exp\left(-(1+o(1)) d \frac{\beta-2}{\beta-1} b^{-1/(\beta-1)} a^{(\beta-2)/(\beta-1)} \right) . \end{aligned}$$

Thus, using Chernoff's bound, we obtain that a.a.s.

$$|B| = (1+o(1))n \int_a^1 \exp\left(-d \frac{\beta-2}{\beta-1} a^{(\beta-2)/(\beta-1)} x^{-1/(\beta-1)} \right) dx \, ,$$

and that a.a.s.

$$\begin{aligned} |D| &= |A \cup B| \\ &= (1+o(1))n \left(a + \int_a^1 \exp\left(-d \frac{\beta-2}{\beta-1} a^{(\beta-2)/(\beta-1)} x^{-1/(\beta-1)} \right) dx \right) . \end{aligned}$$

As this holds for every $a \in (0,1)$, the proof of item (2) follows. □

2.4. **The proof of Theorem 1.5.**

Proof. We first consider an upper bound for $c(\widehat{G})$ by using a dominating set. The probability that there is an edge between two given vertices in the core is at least

$$\begin{aligned} p_{\min} &= w^2_{i_N(1+o(1))} \rho \\ &= (n^{2/\log\log n}/dn)(1+o(1)) \\ &= (N^{(3-\beta)/\log\log N + \Theta(1)/\log^2 \log n})/N \qquad (8) \\ &= (N^{(1+o(1))(3-\beta)/\log\log N})/N \, . \end{aligned}$$

Hence, $\widehat{G}$ contains a random graph $G(N, p_{\min})$. Thus, using Theorem 2.2, any set of cardinality

$$\begin{aligned}
k_N &= \left\lceil \log_{1/(1-p_{\min})} N + \log_{1/(1-p_{\min})} \omega(N) \right\rceil \\
&= (1+o(1)) \frac{\log N}{p_{\min}} \\
&= \frac{N \log N}{N^{(1+o(1))(3-\beta)/\log\log N}} \\
&= N \exp\left(-(1+o(1))(3-\beta)(\log N)/\log\log N + \log\log N\right) \\
&= N \exp\left(-(1+o(1))(3-\beta)(\log N)/\log\log N\right) \\
&= N/N^{(1+o(1))(3-\beta)/\log\log N} \\
&= o(N)
\end{aligned}$$

is a dominating set a.a.s. (where $\omega(N)$ is any function tending to infinity with N). As this holds for any set of cardinality k_N, we obtain a smaller dominating set by considering only vertices with the largest expected degree. Consider the subset of vertices $U = \{i_0, i_0+1, \ldots, k\}$ of first $k - i_0 + 1$ vertices, $k \gg i_0$. Then

$$\sum_{i=i_0}^{k} \omega_i = c \int_{i_0}^{k} i^{-1/(\beta-1)} di + O(1) = (1+o(1)) c \frac{\beta-1}{\beta-2} k^{(\beta-2)/(\beta-1)}.$$

Hence, the probability that vertex i does not have a neighbour in U is equal to

$$\begin{aligned}
q(i) &= \prod_{j=i_0}^{k} (1 - \omega_i \omega_j \rho) \\
&= \exp\left(-(1+o(1)) \omega_i \rho \sum_{j=i_0}^{k} \omega_j \right) \\
&= \exp\left(-(1+o(1)) \frac{\beta-2}{\beta-1} d n^{(3-\beta)/(\beta-1)} i^{-1/(\beta-1)} k^{(\beta-2)/(\beta-1)} \right).
\end{aligned}$$

It is straightforward to see that for all vertices i in the core,

$$\begin{aligned}
q(i) &\le q\big(i_N(1+o(1))\big) \\
&= \exp\left(-(1+o(1)) n^{(2-\beta)/(\beta-1)+1/\log\log n} k^{(\beta-2)/(\beta-1)}\right). \qquad (9)
\end{aligned}$$

Therefore, in order to make the right hand side of (9) equal to $o(n^{-1})$, it is enough to take

$$\begin{aligned}
k &= n^{1-(\beta-1)/(\beta-2)\log\log n} \log^{2(\beta-1)/(\beta-2)} n \\
&= n^{1-(1+o(1))(\beta-1)/(\beta-2)\log\log n} \\
&= N^{1-(1+o(1))(\beta-1)(3-\beta)/(\beta-2)\log\log n}.
\end{aligned}$$

Now, the expected number of vertices that are not dominated by U is $o(1)$, and the assertion follows from Markov's inequality. The upper bound now follows.

For the lower bound, we can show that the core is a.a.s. $(1,k)$-*e.c.* for

$$k = \log N / \log\log N.$$

We do not present this argument here, but instead prove a better bound by considering a weaker adjacency property. We will show that a.a.s. for all subsets S of vertices of cardinality

$$K = N^{(3-\beta)/\log\log N - 1/\log^{3/2}\log N} = N^{(1+o(1))(3-\beta)/\log\log N}$$

and all vertices $u \notin S$ with $N/3 < u < 2N/3$, there is a vertex $x \notin (S \cup \{u\})$ with $N/3 < x < 2N/3$, not joined to a vertex in S and joined to u. This will yield a lower bound of K for the cop number since the robber can move only on vertices with labels between $N/3$ and $2N/3$ to avoid being captured.

Note that the probability that there is an edge between vertex in the core and vertex with label between $N/3$ and $2N/3$ is at most

$$\begin{aligned} p_{\max} &= M\Theta(n^{1/\log\log n})\rho \\ &\leq n^{-1/2+1/\log\log n} \\ &= N^{-1/2+(3-\beta)/2\log\log N+\Theta(1)/\log^2\log N} \end{aligned}$$

(using (3) and the fact that $M = o(\sqrt{n})$). A lower bound for the edge probability is given by (8).

Let X be the random variable counting the number of ordered pairs of K-sets S and vertices u with $N/3 < u < 2N/3$ for which no suitable vertex x with $N/3 < x < 2N/3$ can be found. Then

$$\begin{aligned} \mathbb{E}(X) &\leq N^{K+1}\left(1 - p_{\min}(1-p_{\max})^K\right)^{\Theta(N)-K-1} \\ &= N^{K+1}\left(1 - N^{-1+(3-\beta)/\log\log N+\Theta(1)/\log^2\log N}(1 - N^{-1/2+\Theta(1)/\log\log N})^K\right)^{\Theta(N)} \\ &= N^{K+1}\left(1 - N^{-1+(3-\beta)/\log\log N+\Theta(1)/\log^2\log N}\right)^{\Theta(N)} \\ &= \exp\left((K+1)\log N - N^{(3-\beta)/\log\log N+\Theta(1)/\log^2\log N}\right) \\ &= \exp\left(N^{(3-\beta)/\log\log N-\Theta(1)/\log^{3/2}\log N} - N^{(3-\beta)/\log\log N+\Theta(1)/\log^2\log N}\right) \\ &= \exp\left(-N^{(1+o(1))(3-\beta)/\log\log N}\right) \\ &= o(1). \end{aligned}$$

The proof now follows by Markov's inequality. □

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