FUNDAMENTAL RESULTS IN STOCHASTIC INTEGRALS FOR MATHEMATICAL FINANCE

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This note collects some foundational results from stochastic calculus that are critical for the deductions in the Black-Scholes model. Sections 1-3 cover the definitions and properties of stochastic integrals. Sections 4-6 cover the Itô formula and its applications. Sections 6-8 cover some results for stochastic partial differential equations. Sections 9 and 10 cover Girsanov transformations and martingale representation via stochastic integrals, respectively.

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Throughout the note, we work on a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with a *standard* filtration $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$, i.e., the filtration is right continuous and $\mathscr{F}_0$, and thus each $\mathscr{F}_t$, contains all the $\mathbb{P}$-null sets in $\mathscr{F}$. In this framework, every martingale has a right continuous version, and henceforth we assume all martingales involved are already right continuous. We adopt the terminology and notation from Chung and Williams, *Introduction to Stochastic Integrals*.

1. SI Part I: σ-fields and the Doléans measure

1.1. The ***predictable field*** $\mathscr{P}$ is the σ-field over $\mathbb{R}_+ \times \Omega$ generated by sets of the form: $\{0\} \times F$ for $F \in \mathscr{F}_0$, and of the form: $(s, t] \times F$ for $s, t \in \mathbb{R}_+, s < t, F \in \mathscr{F}_s$.

1.2. Let $\mathscr{V}$ (resp., $\mathscr{M}$, $\sigma(l.c.r.l.)$, etc) be the σ-field over $\mathbb{R}_+ \times \Omega$ generated by all $\mathscr{B} \times \mathscr{F}$-measurable adapted (resp., progressively measurable, left continuous right limit adapted) processes. Let $\mathscr{O}$ be the σ-field over $\mathbb{R}_+ \times \Omega$ generated by all stochastic intervals. Then

$$\begin{aligned}\mathscr{P} =& \sigma(c.) = \sigma(l.c.r.l.) = \sigma(l.c.)\\ \subset& \mathscr{O} = \sigma(r.c.l.l.) = \sigma(r.c.)\\ \subset& \mathscr{M}\\ \subset& \mathscr{V}\\ \subset& \mathscr{B} \times \mathscr{F}\end{aligned}$$

1.3. For a right continuous L^2-martingale M, put

$$\begin{cases}\mu_M\big((s,t] \times F\big) = \mathbb{E}\big[\mathbb{1}_F(M_t^2 - M_s^2)\big], & s, t \in \mathbb{R}_+, s < t, F \in \mathscr{F}_s\\ \mu_M\big(\{0\} \times F_0\big) = 0 & F_0 \in \mathscr{F}_0\end{cases}.$$

Then μ_M extends to a σ-finite measure on $(\mathbb{R}_+ \times \Omega, \mathscr{P})$, called the ***Doléans measure*** of M. The Doléans measure can also be defined as a σ-finite measure on $(\mathbb{R}_+ \times \Omega, \mathscr{P})$ for a right continuous local L^2-martingale M by

$$\mu_M(A) = \lim_{k\to\infty} \mu_{M^k}(A \cap [0, \tau_k]), \quad \forall A \in \mathscr{P}.$$

Here (τ_k) is a localizing sequence of optional times for M.

1.4. Let M be a right continuous local L^2-martingale. If $\mu_M \ll \lambda\times\mathbb{P}$ where λ is the Lebesgue measure on $\mathbb{R}_+$, then μ_M extends to a measure on $(\mathbb{R}_+ \times \Omega, \mathscr{B} \times \mathscr{F})$. Let $\mathscr{P}^\sim$ and $\mathscr{P}^*$ be the augmentations of $\mathscr{P}$ by the collections of null sets of μ_M and $\mathbb{P} \times \lambda$ in $(\mathbb{R}_+ \times \Omega, \mathscr{B} \times \mathscr{F})$, respectively. Then

$$\mathscr{P} \subset \mathscr{O} \subset \mathscr{M} \subset \mathscr{V} \subset \mathscr{P}^* \subset \mathscr{P}^\sim \subset \mathscr{B} \times \mathscr{F}.$$

1.5. If M is a continuous local martingale, then μ_M has an explicit form (4.1), which also allows to extend it to a σ-finite measure on $(\mathbb{R}_+ \times \Omega, \mathscr{B} \times \mathscr{F})$. Let $\mathscr{P}^\sim$ be the augmentations of $\mathscr{P}$ by the collection of null sets of μ_M in $(\mathbb{R}_+ \times \Omega, \mathscr{B} \times \mathscr{F})$. Then

$$\mathscr{P} \subset \mathscr{O} \subset \mathscr{M} \subset \mathscr{P}^\sim \subset \mathscr{B} \times \mathscr{F}.$$

2. SI Part II: Definitions

2.1. Let M be a right continuous L^2-martingale. Let $\mathscr{E}$ be the linear span of rvs of the form: $\mathbb{1}_{\{0\}\times F}$ for $F \in \mathscr{F}_0$, and of the form: $\mathbb{1}_{(s,t]\times F}$ for $s,t \in \mathbb{R}_+, s < t, F \in \mathscr{F}_s$. Define

$$\begin{cases} \int \mathbb{1}_{\{0\}\times F} \mathrm{d}M = & 0 \\ \int \mathbb{1}_{(s,t]\times F} \mathrm{d}M = & \mathbb{1}_F(M_t - M_s) \end{cases}.$$

Extend the definition to $\mathscr{E}$ by linearity. Then the stochastic integral extends to a mapping from $L^2(\mathscr{P}) := L^2(\mathbb{R}_+ \times \Omega, \mathscr{P}, \mu_M)$ to $L^2 := L^2(\Omega, \mathscr{F}, \mathbb{P})$ by linear isometry:

$$L^2(\mathscr{P}) \to L^2; \quad X \mapsto \int X \mathrm{d}M.$$

2.2. Let M be a right continuous L^2-martingale. Let $\Lambda^2(\mathscr{P}, M)$ be the space of all $X \in \mathscr{P}$ such that $\mathbb{1}_{[0,t]}X \in L^2(\mathscr{P})$ for each $t \in \mathbb{R}_+$. For $X \in \Lambda^2(\mathscr{P}, M)$, $\big\{ \int \mathbb{1}_{[0,t]} X \mathrm{d}M : t \in \mathbb{R}_+ \big\}$ is a zero-mean L^2-martingale, which admits a right continuous version, denoted by

$$\int_{[0,t]} X \mathrm{d}M$$

for each $t \in \mathbb{R}_+$. We put $\int_{(s,t]} X \mathrm{d}M = \int_{[0,t]} X \mathrm{d}M - \int_{[0,s]} X \mathrm{d}M$ for $s < t$ in $\mathbb{R}_+$.

If, in addition, M is continuous, then Y has a continuous version. We denote it by

$$\int_0^t X \mathrm{d}M$$

for each $t \in \mathbb{R}_+$ and put $\int_s^t X \mathrm{d}M = \int_0^t X \mathrm{d}M - \int_0^s X \mathrm{d}M$ for $s < t$ in $\mathbb{R}_+$.

2.3. Let M be a right continuous local L^2-martingale. Let $\Lambda(\mathscr{P}, M)$ be the collection of all $X \in \mathscr{P}$ for which there exists a localizing sequence (τ_k) such that, for each $k \in \mathbb{N}$, $M^k = \{M_{\cdot\wedge\tau_k} - M_0\}$ is an L^2-martingale and $\mathbb{1}_{[0,\tau_k]}X \in \Lambda^2(\mathscr{P}, M^k)$. Put $Y^k = \big\{ \int_{[0,t]} \mathbb{1}_{[0,\tau_k]} X \mathrm{d}M^k : t \in \mathbb{R}_+ \big\}$. Then a.s., $Y = \lim_k Y^k$ exists for all $t \in \mathbb{R}_+$ and is right continuous in t. Moreover, a.s., for all $k \in \mathbb{N}$ and all $t \in \mathbb{R}_+$,

$$Y_{t\wedge\tau_k} = Y_t^k.$$

Thus Y is a right continuous local L^2-martingale starting at 0. If M is continuous, then so is Y. Similar notations as before to replace Y_t, e.g., $\int_{[0,t]} X \mathrm{d}M$ and $\int_0^t X \mathrm{d}M$.

2.4. Let M be a right continuous local L^2-martingale satisfying the conditions in 1.4 or 1.5. The definition of stochastic integrals wrt M can be extended to integrands $Z \in \mathscr{P}^\sim$.

Assume first that M is an L^2-martingale. Then $L^2(\mathscr{P}^\sim)$, $\Lambda^2(\mathscr{P}^\sim, M)$, and $\big(\int_{[0,t]} Z \mathrm{d}M \big)_{t\in\mathbb{R}_+}$ can be defined as in 2.1 and 2.2. For the general case, one can properly define $\Lambda(\mathscr{P}, M)$ and then $\big(\int_{[0,t]} Z \mathrm{d}M \big)_{t\in\mathbb{R}_+}$ as in 2.3; (note that the intermediate $\mathscr{P}_k^\sim$ needs to be defined as well, which requires a bit case under the condition in 1.4). For any $Z \in \mathscr{P}^\sim$, there exists $X \in \mathscr{P}$ such that $\mu_M(X \neq Z) = 0$. For so selected X, $Z \in \Lambda(\mathscr{P}^\sim, M)$ iff $X \in \Lambda(\mathscr{P}, M)$, and in this case, $\big\{ \int_{[0,t]} X \mathrm{d}M : t \in \mathbb{R}_+ \big\}$ is indistinguishable to $\big\{ \int_{[0,t]} Z \mathrm{d}M : t \in \mathbb{R}_+ \big\}$.

In particular, under the condition in 1.4, stochastic integrals can be defined for processes in $\Lambda(\mathscr{V}, M)$ ($\mathscr{B} \times \mathscr{F}$-measurable adapted processes), and under the condition in 1.5, stochastic integrals can be defined for processes in $\Lambda(\mathscr{M}, M)$ (progressively measurable processes).

3. SI Part III: Properties

3.1. Let M be a continuous local martingale. If $Z \in \mathscr{P}^{\sim}$ (in particular, if Z is progressively measurable) satisfies that a.s., $\int_0^t Z_s^2 \mathrm{d}[M]_s < \infty$ for all $t \in \mathbb{R}_+$ (see Section 4 for $[M]$), then $Z \in \Lambda(\mathscr{P}^{\sim}, M)$. In particular, if X is a continuous adapted process. Then $X \in \Lambda(\mathscr{P}, M)$.

3.2. By a ***Brownian motion***, we mean a collection of rvs indexed by $t \in \mathbb{R}_+$, $W = \{W_t : t \in \mathbb{R}_+\}$, satisfying (1) For $0 \leq s < t < \infty$, $W_t - W_s \sim \mathrm{N}(0, t-s)$; (2) For $0 \leq t_0 < t_1 < \cdots < t_n < \infty$, $W_{t_0}, W_{t_1} - W_{t_0}, \dots, W_{t_n} - W_{t_{n-1}}$ are independent. A Brownian motion is automatically continuous. By a ***Brownian motion local martingale***, we mean a Brownian motion that is also a local martingale wrt the standard filtration $(\mathscr{F}_t)$. In this case, $W_t - W_s$ is automatically independent of $\mathscr{F}_s$ for $0 \leq s < t < \infty$ and $(W_t - W_0)$ is automatically an L^2-martingale. A Brownian motion W is a Brownian motion local martingale wrt the ***associated standard filtration*** $(\mathscr{F}_t)$, where $\mathscr{F}_t$ is the augmentation of the σ-field generated by $\{W_s : 0 \leq s \leq t\}$, for each $t \in \mathbb{R}_+$.

For a Brownian motion local martingale W, $\mu_W = \lambda \times \mathbb{P}$.

3.3. Let M be a right continuous local L^2-martingale, $X \in \Lambda(\mathscr{P}, M)$. For $s < t \leq t^*$ in $\mathbb{R}_+$, $Z \in b\mathscr{F}_s$, we have $\mathbb{1}_{(s,t]} ZX \in \Lambda(\mathscr{P}, M)$ and

$$\int_{[0,t^*]} \mathbb{1}_{(s,t]} ZX \mathrm{d}M = Z \int_{(s,t]} X \mathrm{d}M.$$

If in addition M is an L^2-martingale and $X \in \Lambda^2(\mathscr{P}, M)$, then LHS $= \int \mathbb{1}_{(s,t]} ZX \mathrm{d}M$.

3.4. Let M be a right continuous local L^2-martingale and $X \in \Lambda(\mathscr{P}, M)$. Let τ be an optional time bounded above by $t^* \in \mathbb{R}_+$. Then

$$\int_{[0,\tau]} X \mathrm{d}M = \int_{[0,t^*]} \mathbb{1}_{[0,\tau]} X \mathrm{d}M.$$

3.5. Let M be a right continuous local L^2-martingale, $X \in \Lambda(\mathscr{P}, M)$, and $Z \in \Lambda(\mathscr{P}, Y)$, where $Y_t = \int_{[0,t]} X \mathrm{d}M$ for $t \in \mathbb{R}_+$. Then $XZ \in \Lambda(\mathscr{P}, M)$ and a.s., for all $t \in \mathbb{R}_+$,

$$\int_{[0,t]} Z \mathrm{d}Y = \int_{[0,t]} XZ \mathrm{d}M.$$

Symbolically, $\mathrm{d}Y = X \mathrm{d}M$. If, in addition, M is an L^2-martingale, $X \in \Lambda^2(\mathscr{P}, M)$ and $Z \in \Lambda^2(\mathscr{P}, Y)$, then $XZ \in \Lambda^2(\mathscr{P}, M)$, and $\mathrm{d}\mu_Y = X^2 \mathrm{d}\mu_M$.

4. IF Part I: Quadratic Variation Processes

4.1. Let M be a continuous local martingale. For $t \in \mathbb{R}_+$, put

$$[M]_t = (M_t)^2 - (M_0)^2 - 2 \int_0^t M \mathrm{d}M.$$

Let (π^n) be a partition of $[0, t]$ such that $\lim_n \|\pi^n\| = 0$. Put $S_t^n = \sum_j (M_{t_{j+1}^n} - M_{t_j^n})^2$ for each $n \in \mathbb{N}$. Then for any $t \in \mathbb{R}_+$, (S_t^n) converges to $[M]_t$ in probability. Moreover, the convergence is in L^2 if M is bounded, and in L^1 if M is an L^2-martingale.

It follows that $[M]$ is a continuous increasing process, called the ***quadratic variation process of*** M. If, in addition, M is an L^2-martingale, $[M]$ is also integrable.

4.2. Let M, N be two continuous local martingale. Put $[M,N] = \frac{1}{4}\big([M+N]-[M-N]\big)$, called the ***mutual variation process*** of M and N. Then for any $s < t$ in $\mathbb{R}_+$, we have

$$\big|[M,N]\big|_s^t \leq \sqrt{[M]_s^t [N]_s^t}.$$

For $X \in \Lambda(\mathscr{P}, M)$ and $Y \in \Lambda(\mathscr{P}, N)$, write $(X \cdot M)_t = \int_0^t X_s \mathrm{d}M_s$ and similarly define $Y \cdot N$. Then for all $t \in \mathbb{R}_+$,

$$[X \cdot M, Y \cdot N]_t = \int_0^t X_s Y_s \mathrm{d}[M,N]_s.$$

Let further τ be an optional time. For each $t \in \mathbb{R}_+$, let $M_t^\tau = M_{t\wedge\tau}$ and $N_t^\tau = N_{t\wedge\tau}$. Then for each $t \in \mathbb{R}_+$,

$$[M^\tau, N^\tau]_t = [M,N]_{t\wedge\tau}.$$

4.3. A continuous local martingale V that is locally of bounded variation satisfies a.s., $V_t = V_0$ for all $t \in \mathbb{R}_+$. A continuous ***semimartingale*** is the sum of a continuous local martingale M and a continuous adapted process V that is locally of bounded variation. By the preceding result, any continuous semimartingale has a unique such representation if we require $V_0 = 0$.

The product MN of any two continuous local martingales M and N is a continuous semimartingale. The decomposition is given by

$$(MN)_t = \left[(MN)_0 + \frac{1}{2}\int_0^t (M+N)_s \mathrm{d}(M+N)_s - \frac{1}{2}\int_0^t (M-N)_s \mathrm{d}(M-N)_s\right] + [M,N]_t.$$

4.4. Let M be a continuous local martingale. For any $0 \leq X \in \mathscr{P}$,

$$\int_{\mathbb{R}_+\times\Omega} X \mathrm{d}\mu_M = \mathbb{E}\left[\int_0^\infty X \mathrm{d}[M]_t\right]. \tag{4.1}$$

4.5. An m-***dimensional Brownian motion*** is a process $(W^1, \ldots, W^m)$ where each component process W^i is a Brownian motion and the m component processes are independent. An m-***dimensional Brownian motion local martingale*** is an m-dimensional Brownian motion with each component process a local martingale. For an m-dimensional Brownian motion local martingale W, $[W^i, W^j]_t = \delta_{ij} t$ for $t \in \mathbb{R}_+$, $1 \leq i, j \leq m$. In this case, $W_t - W_s$ is automatically independent of $\mathscr{F}_s$ for $0 \leq s < t < \infty$. Similarly define the associated standard filtration of a multi-dimensional Brownian motion, wrt which the Brownian motion becomes a Brownian motion local martingale.

5. IF Part II: Formula and Applications

5.1. Let $m, n \in \mathbb{N}$. Let M^i be a continuous local martingale for $1 \leq i \leq m$ and V^k be a continuous adapted process that is locally of bounded variation for $1 \leq k \leq n$. Suppose that D is an open set in $\mathbb{R}^{m+n}$ such that a.s.

$$Z_t = (M_t^1, \ldots, M_t^m, V_t^1, \ldots, V_t^n) \in D \text{ for all } t \in \mathbb{R}_+.$$

Let $f(x,y)$ be a continuous real-valued function of $(x,y) \in D$ such that $\frac{\partial f}{\partial x_i}$, $\frac{\partial^2 f}{\partial x_i \partial x_j}$, $1 \le i,j \le m$, and $\frac{\partial f}{\partial y_k}$, $1 \le k \le n$, exist and are continuous in D. Then

$$f(Z_t) - f(Z_0) = \sum_{i=1}^m \int_0^t \frac{\partial f}{\partial x_i}(Z_s)\mathrm{d}M_s^i + \sum_{k=1}^n \int_0^t \frac{\partial f}{\partial y_k}(Z_s)\mathrm{d}V_s^k + \frac{1}{2}\sum_{i=1}^m\sum_{j=1}^m \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(Z_s)\mathrm{d}[M^i, M^j]_s.$$

5.2. An m-dimensional process $M = \{M_t : t \in \mathbb{R}_+\}$ is a Brownian motion if and only if there is a standard filtration $\{\mathscr{F}_t\}$ wrt which each component process of M is a continuous local martingale and $[M_i, M_j]_t = \delta_{ij} t$ for all $t \in \mathbb{R}_+$ and $1 \le i,j \le m$.

5.3. Let M and A be continuous adapted processes such that A is increasing starting at 0. For each $\alpha \in \mathbb{R}$, let Z^α be the process defined by

$$Z^\alpha = \exp(\alpha M - \frac{1}{2}\alpha^2 A).$$

Then M is a local martingale and $A = [M]$ iff Z^α is a local martingale for each $\alpha \in \mathbb{R}$.

If M is an L^2-martingale with $[M] = A$ and α is such that $Z_0^\alpha \in L^2$ and $\mathbb{E}\left(\int_0^t (Z_s^\alpha)^2 \mathrm{d}A_s\right) < \infty$ for each $t \in \mathbb{R}_+$, then Z^α is an L^2-martingale.

If A_t is bounded for each $t \in \mathbb{R}_+$ and there exists $\alpha_0 > 0$ such that $\mathbb{E}[\exp(\alpha_0 |M_t|)]$ for all $t \in \mathbb{R}_+$ and Z^α is a martingale for $|\alpha| \le \alpha_0$, then M is an L^2-martingale with $[M] = A$.

5.4. Let M and A be continuous adapted processes such that A is increasing starting at 0. Suppose that A_t is bounded for each $t \in \mathbb{R}_+$ and there exists $\alpha_0 > 0$ such that $\mathbb{E}[\exp(\alpha_0|M_t|)]$ for all $t \in \mathbb{R}_+$ and Z^α is a martingale for $|\alpha| \le \alpha_0$. Write

$$H_n(x,y) = \frac{\mathrm{d}^n}{\mathrm{d}\alpha^n}\Big|_{\alpha=0} \exp\left(\alpha x - \frac{1}{2}\alpha^2 y\right).$$

Then $H_n(M,A)$ is an L^2-martingale for each $n \in \mathbb{Z}_+$.

For $n \in \mathbb{N}$,

$$H_n(M_t, A_t) = n\int_0^t H_{n-1}(M_s, A_s)\mathrm{d}M_s.$$

6. IF Part III/SDE Part I: Linear Case

Let W be an m-dimensional Brownian motion local martingale. Let A, a be 1-dimensional $\mathscr{B}\times\mathscr{F}$-measurable adapted processes and S, σ be m-dimensional $\mathscr{B}\times\mathscr{F}$-measurable adapted processes. Suppose that a.s., for all $t \in \mathbb{R}_+$,

$$\int_0^t (|A(s)| + |a(s)|)\,\mathrm{d}s < \infty,$$

$$\int_0^t (\|S(s)\|^2 + \|\sigma(s)^2\|)\,\mathrm{d}s < \infty.$$

6.1. For any initial rv $Z(0) \in \mathscr{F}_0$, the following SDE has a unique strong solution:

$$\mathrm{d}Z(t) = Z(t)A(t)\mathrm{d}t + Z(t)S(t)'\mathrm{d}W(t). \tag{6.1}$$

The solution is given by

$$Z(t) = Z(0)\exp\left(\int_0^t \big(A(s) - \frac{1}{2}\|S(s)\|^2\big)\mathrm{d}s + \int_0^t S(s)'\mathrm{d}W(s)\right).$$

6.2. For any initial rv $X(0) \in \mathscr{F}_0$, the following SDE has a unique strong solution:

$$\mathrm{d}X(t) = \big(X(t)A(t) + a(t)\big)\mathrm{d}t + \big(X(t)S(t)' + \sigma(t)'\big)\mathrm{d}W(t). \tag{6.2}$$

The solution is given by

$$X(t) = Z(t)\left(X(0) + \int_0^t \frac{1}{Z(s)}\Big(a(s) - S(s)'\sigma(s)\Big)\mathrm{d}s + \int_0^t \frac{1}{Z(s)}\sigma(s)'\mathrm{d}W(s)\right),$$

where Z is the unique solution to (6.1) with initial value 1.

6.3. Putting $A(t) \equiv -\alpha \in \mathbb{R}$, $a(t) \equiv 0$, $S(t) \equiv 0$ and $\sigma(t) \equiv 1$, we obtain that $X(t) = e^{-\alpha t}X(0) + e^{-\alpha t}\int_0^t e^{\alpha s}\mathrm{d}W(s)$ is the unique solution to the SDE $\mathrm{d}X_t = -\alpha X_t \mathrm{d}t + \mathrm{d}W(t)$. The solution is called the ***Ornstein-Uhlenbeck process***.

7. SDE Part II: Nonlinear Case

Let $b : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d\times m}$ be continuous functions. Let W be an m-dimensional Brownian motion local martingale. A ***strong solution*** to the following SDE

$$\mathrm{d}X(t) = b(t, X(t))\mathrm{d}t + \sigma(t, X(t))\mathrm{d}W(t) \tag{7.1}$$

is a continuous adapted process such that for all $t \geq 0$,

$$X(t) = X(0) + \int_0^t b(s, X(s))\mathrm{d}s + \int_0^t \sigma(s, X(s))\mathrm{d}W(s).$$

7.1. Suppose that b and σ are bounded and there exists a constant $K > 0$ such that

$$\|b(t,x) - b(t,y)\| + \|\sigma(t,x) - \sigma(t,y)\| \leq K\|x - y\|$$

for any $t \in \mathbb{R}_+$ and $x, y \in \mathbb{R}^d$. Then for any initial random vector $X(0) \in \mathscr{F}_0$, the SDE (7.1) has a unique strong solution.

Moreover, if Y is a strong solution with the initial random vector $Y(0) \in \mathscr{F}_0$ such that $X(0) - Y(0) \in L^2$, then there exists a constant depending on K and m such that for all $t \geq 0$,

$$\mathbb{E}\left[\sup_{0\leq s\leq t}|X(s) - Y(s)|^2\right] \leq \exp\big(C(t + t^2)\big)E\left[|X(0) - Y(0)|^2\right].$$

7.2. Suppose that there exists a constant $K > 0$ such that

$$\|b(t,x) - b(t,y)\| + \|\sigma(t,x) - \sigma(t,y)\| \le K\|x - y\|, \tag{7.2}$$

$$\|b(t,x)\|^2 + \|\sigma(t,x)\|^2 \le K(1 + \|x\|^2), \tag{7.3}$$

for any $t \in \mathbb{R}_+$ and $x, y \in \mathbb{R}^d$. Then for any $\mathscr{F}_0$-measurable initial random vector $X(0) \in L^2$, the SDE (7.1) has a unique strong solution. There exists a constant C depending on K and m such that

$$\mathbb{E}\left[\sup_{0\le s\le t} \|X(s)\|^2\right] \le \exp\big(C(1 + t + t^2)\big)\big(1 + \|X(0)\|_2^2\big).$$

Moreover, if Y is a strong solution with the $\mathscr{F}_0$-measurable initial random vector $Y(0) \in L^2$, then there exists a constant depending on K and m such that for all $t \ge 0$,

$$\mathbb{E}\left[\sup_{0\le s\le t} |X(s) - Y(s)|^2\right] \le \exp\big(C(t + t^2)\big)E\left[|X(0) - Y(0)|^2\right].$$

7.3. Suppose that for each $n \in \mathbb{N}$, there exists a constant $K_n > 0$ such that

$$\|b(t,x) - b(t,y)\| + \|\sigma(t,x) - \sigma(t,y)\| \le K_n\|x - y\|$$

for any $0 \le t \le n$, $\|x\| \le n$, and $\|y\| \le n$, and such that

$$x \cdot b(t,x) + \|\sigma(t,x)\|^2 \le K_n(1 + \|x\|^2)$$

for any $0 \le t \le n$ and any $x, y \in \mathbb{R}^d$. Then for any $\mathscr{F}_0$-measurable random vector $X(0)$, the SDE (7.1) has a unique strong solution. If $X(0) \in L^2$, then $X(t) \in L^2$ for all $t \in \mathbb{R}_+$.

8. SDE Part III: Feynman-Kac Representation

8.1. Let $b : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d\times m}$ be continuous functions satisfying (7.2) and (7.3). Then for any $x \in \mathbb{R}^d$ and $t \ge 0$, there exists a unique continuous process $\big(X_s^{(t,x)}\big)_{s\in[t,\infty)}$ adapted to $(\mathscr{F}_s)_{s\in[t,\infty)}$ such that

$$X_s^{(t,x)} = x + \int_t^s b\big(u, X_u^{(t,x)}\big)\mathrm{d}u + \int_t^s \sigma\big(u, X_u^{(t,x)}\big)\mathrm{d}W_u, \qquad s \in [t,\infty).$$

We may suppress the superscripts (t,x) in $X^{(t,x)}$, and write $\mathbb{E}^{t,x}$ to indicate the expectation computed under these initial conditions.

8.2. Let $T > 0$ be given. Let $f : \mathbb{R}^d \to \mathbb{R}$, $g : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ and $k : [0,T] \times \mathbb{R}^d \to [0,\infty)$ be continuous functions such that

(1) either $|f(x)| \le C(1 + \|x\|^{2\eta})$ for some constants $C > 0$ and $\eta \ge 1$ and all $x \in \mathbb{R}^d$, or $f \ge 0$; and
(2) either $|g(t,x)| \le C(1 + \|x\|^{2\eta})$ for some constants $C > 0$ and $\eta \ge 1$ and all $(t,x) \in [0,T] \times \mathbb{R}^d$, or $g \ge 0$.

8.3. Define the second-order differential operator

$$(A_t h)(x) = \sum_{i=1}^{d} b_i(t,x)\frac{\partial h}{\partial x_i}(x) + \frac{1}{2}\sum_{i=1}^{d}\sum_{k=1}^{d}\Big(\sum_{j=1}^{m}\sigma_{ij}(t,x)\sigma_{kj}(t,x)\Big)\frac{\partial^2 h}{\partial x_i \partial x_k}(x).$$

Let $v : [0,T]\times\mathbb{R}^d \to \mathbb{R}$ be a function in $C([0,T]\times\mathbb{R}^d)\cap C^{1,2}([0,T)\times\mathbb{R}^d)$ such that

$$\begin{aligned} -v_t + kv =& A_t v + g \qquad \text{on } [0,T)\times\mathbb{R}^d \\ v(T,x) =& f(x) \qquad \text{for all } x\in\mathbb{R}^d \end{aligned}$$

8.4. Under the conditions in 8.1-8.2, assuming that the solution v to the PDE in 8.3 satisfies that, for some constants $L>0$ and $\delta\geq 1$,

$$\max_{0\leq t\leq T}|v(t,x)| \leq L(1+\|x\|^{2\delta}) \quad \text{for any } x\in\mathbb{R}^d, \tag{8.1}$$

it holds that for any $t\in[0,T]$ and $x\in\mathbb{R}^d$,

$$\begin{aligned} v(t,x) =& \mathbb{E}^{t,x}\Big[f(X_T)\exp\Big(-\int_t^T k(s,X_s)\mathrm{d}s\Big) \\ &+ \int_t^T g(s,X_s)\exp\Big(-\int_t^s k(u,X_u)\mathrm{d}u\Big)\mathrm{d}s\Big]. \end{aligned}$$

In this case, v is unique.

If b and σ are bounded satisfying (7.2), the conditions in 8.2 hold, and the solution v to the PDE in 8.3 satisfies that, for some constants $L>0$ and $0<\delta<\frac{1}{18K'Td}$,

$$\max_{0\leq t\leq T}|v(t,x)| \leq Le^{\delta\|x\|^2} \quad \text{for any } x\in\mathbb{R}^d, \tag{8.2}$$

where $K' := \sup_{1\leq i\leq d}\sup_{t\in\mathbb{R}_+,x\in\mathbb{R}^d}\Big(|b_i(t,x)|+\sum_{j=1}^m\sigma_{ij}^2(t,x)\Big)$, then the preceding stochastic representation for v still holds. In this case, v is unique.

8.5. Suppose b,σ,k are bounded on $[0,T]\times\mathbb{R}^d$. Suppose f,g have polynomial growth as in 8.2. Suppose b,σ,k,g are uniformly Hölder continuous $[0,T]\times\mathbb{R}^d$. Suppose $\sigma\sigma'\geq\xi\mathrm{Id}_d$ on $[0,\infty)\times\mathbb{R}^d$ for some constant $\xi>0$. Then the PDE in 8.3 has a solution satisfying (8.2). If, in addition, (7.2) is satisfied, then the above stochastic representation holds.

9. Girsanov Transformation

9.1. Let $\mathbb{Q}\sim\mathbb{P}$ on $(\Omega,\mathscr{F})$ with $\rho := \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$. Let (ρ_t) be a right continuous version of $(\mathbb{E}^{\mathbb{P}}[\rho|\mathscr{F}_t])$.

Then for any optional time τ, ρ_τ equals the Radon-Nikodym derivative of $\mathbb{Q}$ relative to $\mathbb{P}$ when restricted to $\mathscr{F}_\tau$. For a right continuous adapted process M, M is a local martingale under $\mathbb{Q}$ if and only if ρM is a local martingale under P.

Let M be a continuous local martingale under $\mathbb{P}$. Suppose that (ρ_t) is continuous. Put $A_t = \int_0^t \rho_s^{-1}\mathrm{d}[\rho,M]_s$ for each $t\in\mathbb{R}_+$. Then a.s., A_t is well defined, finite, and is locally of bounded variation for all $t\in\mathbb{R}_+$. Moreover, $M-A$ is a continuous local martingale under $\mathbb{Q}$ with quadratic variation process the same as that for M under $\mathbb{P}$.

9.2. Let M be a continuous local martingale. Let $b:\mathbb{R}\to\mathbb{R}$ be Borel-measurable such that $b(M)\in\Lambda(\mathscr{P},M)$. For $t\in\mathbb{R}_+$, put

$$\rho_t=\exp\left(\int_0^t b(M_s)\mathrm{d}M_s-\frac{1}{2}\int_0^t b(M_s)^2\mathrm{d}[M]_s\right).$$

For a given $t>0$, if $(\rho_s)_{s\in[0,t]}$ is a martingale, then

$$\left(M_s-\int_0^s b(M_u)\mathrm{d}[M]_u\right)_{s\in[0,t]}$$

is a continuous local martingale under $\mathbb{Q}_t$, where $\mathrm{d}\mathbb{Q}_t=\rho_t\mathrm{d}\mathbb{P}_t$, with quadratic variation process $\big([M]_s\big)_{s\in[0,t]}$ (computed for M under $\mathbb{P}$).

If M is a Brownian motion martingale with $M(0)\in L^2$, b is bounded on each finite interval, and there exists a constant $C>0$ such that $x\cdot b(x)\leq C(1+x^2)$ for all $x\in\mathbb{R}$, then $(\rho_t)_{t\in\mathbb{R}_+}$ is a martingale, and thus, for any $t>0$, $\Big(M_s-\int_0^s b(M_u)\mathrm{d}u\Big)_{s\in[0,t]}$ is a Brownian motion martingale under $\mathbb{Q}_t$ where $\mathrm{d}\mathbb{Q}_t=\rho_t\mathrm{d}\mathbb{P}_t$.

9.3. Let W be an m-dimensional Brownian motion martingale, and X be an m-dimensional $\mathscr{B}\times\mathscr{F}$-measurable adapted process with a.s.

$$\int_0^t\|X(s)\|^2\mathrm{d}s<\infty\quad\text{for all } t\in\mathbb{R}_+.$$

For $t\in\mathbb{R}_+$, put

$$\rho_t=\exp\left(\int_0^t X(s)'\mathrm{d}W_s-\frac{1}{2}\int_0^t\|X(s)\|^2\mathrm{d}s\right).$$

Then ρ is a continuous local martingale. Let $t>0$ be given. If $(\rho_s)_{s\in[0,t]}$ is a martingale, then

$$\left(W_s-\int_0^s X(u)\mathrm{d}u\right)_{s\in[0,t]}$$

is an m-dimensional Brownian motion martingale under $\mathbb{Q}_t$ where $\mathrm{d}\mathbb{Q}_t=\rho_t\mathrm{d}\mathbb{P}_t$.

If $\mathbb{E}\Big[\exp\Big(\frac{1}{2}\int_0^t\|X(s)\|^2\mathrm{d}s\Big)\Big]<\infty$ for a given $t>0$, then $(\rho_s)_{s\in[0,t]}$ is a martingale.

10. Representation of Martingales

10.1. Let M be an L^2-martingale starting at 0, wrt the associated standard filtration of an m-dimensional Brownian W. Then there exists an m-dimensional process $\Psi\in\mathscr{P}$ such that $\mathbb{E}[\int_0^t\|\Psi(s)\|^2\mathrm{d}s]<\infty$ for all $t\in\mathbb{R}_+$ and $\Psi\cdot W$ is a version of M. If Ψ and Ψ' both satisfy this condition, then $\lambda\times\mathbb{P}(\Psi\neq\Psi')=0$. In particular, M has a continuous version. It also follows that every local martingale has a continuous version.

If M is a continuous local martingale starting at 0, $\Psi\in\mathscr{P}$ can be chosen to only satisfy a.s., $\int_0^t\|\Psi(s)\|^2\mathrm{d}s<\infty$ for all $t\in\mathbb{R}_+$. It is still unique up to a $\lambda\times\mathbb{P}$-null set.

10.2. Let M be a continuous local martingale starting at 0 such that $[M]_\infty := \sup_{t\in\mathbb{R}_+}[M]_t = \infty$ a.s. For $t \in \mathbb{R}_+$, put $\tau_t = \inf\{s \in \mathbb{R}_+ : [M]_s > t\}$. Put

$$W_t := M_{\tau_t} \quad \text{for all } t \in \mathbb{R}_+.$$

Then $\left(\mathscr{F}_{\tau_t}\right)$ is a standard filtration wrt which W is a Brownian motion martingale. Moreover, a.s.,

$$M_t = W_{[M]_t} \quad \text{for all } t \in \mathbb{R}_+.$$

Let X be a progressively measurable process such that a.s., $\int_0^\infty X_s^2 \mathrm{d}[M]_s < \infty$. Put

$$Y_t := X_{\tau_t}$$

for all $t \in \mathbb{R}_+$. Then Y is adapted wrt $\left(\mathscr{F}_{\tau_t}\right)$ and a.s., $\int_0^\infty Y_s^2 \mathrm{d}s < \infty$. Moreover, for any $t, s \in \mathbb{R}_+$,

$$\int_0^t X_u \mathrm{d}M_u = \int_0^{[M]_t} Y_v \mathrm{d}W_v,$$

$$\int_0^{\tau_s} X_v \mathrm{d}M_v = \int_0^s Y_v \mathrm{d}W_v.$$

10.3. Let M be an m-dimensional continuous local martingale such that $\sup_{t\in\mathbb{R}_+}[M^i]_t = \infty$ for each $1 \le i \le m$ and $[M^i, M^j]_t = 0$ for any $t \in \mathbb{R}_+$ and any $i \ne j$. For $t \in \mathbb{R}_+$ and $1 \le i \le m$, put $\tau_t^i = \inf\{s \in \mathbb{R}_+ : [M^i]_s > t\}$ and $W_t^i = M^i_{\tau_t^i}$. Then W is a Brownian motion.