CLASSICAL THEORY OF MARTINGALES IN DISCRETE TIME

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We collect the basic facts on martingales in discrete time. The main references are Chung, *A Course in Probability Theory* and Meyer, *Martingales and Stochastic Integrals I*.

1. Basic Notions

1.1. Let the collection of time slots $\mathbb{T}$ be either $\mathbb{N}$ or $\mathbb{N}_\infty := \mathbb{N} \cup \{\infty\}$.

Throughout this note, fix a *complete* probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and a filtration $(\mathscr{F}_n)_{n\in\mathbb{N}}$ of *augmented* sub-σ-fields of $\mathscr{F}$. Put $\mathscr{F}_\infty = \vee_{n\in\mathbb{N}}\mathscr{F}_n$, so that we can write $(\mathscr{F}_n)_{n\in\mathbb{N}_\infty}$.

A ***stochastic process*** is a sequence of random variables $(X_n)_{n\in\mathbb{T}}$. We write such a process by X, dropping the subscript, if there is no ambiguity.

A process X is ***adapted*** if X_n is $\mathscr{F}_n$-measurable for each $n \in \mathbb{T}$. A process $(V_n)_{n\in\mathbb{N}}$ is ***predictable*** if V_1 is $\mathscr{F}_1$-measurable and V_n is $\mathscr{F}_{n-1}$-measurable for all $n \geq 2$.

1.2. Let X be an adapted process such that $\mathbb{E}[|X_n|] < \infty$ for all $n \in \mathbb{T}$. We say that

(1) X is a ***martingale*** if $\mathbb{E}[X_m|\mathscr{F}_n] = X_n$ for all $m, n \in \mathbb{T}$ with $m \geq n$,
(2) X is a ***submartingale*** if $\mathbb{E}[X_m|\mathscr{F}_n] \geq X_n$ for all $m, n \in \mathbb{T}$ with $m \geq n$,
(3) X is a ***supermartingale*** if $\mathbb{E}[X_m|\mathscr{F}_n] \leq X_n$ $m, n \in \mathbb{T}$ with $m \geq n$.

Note that we allow $m = \infty$ in case that $\mathbb{T} = \mathbb{N}_\infty$.

1.3.

(1) If X and Y are supermartingale, then $X \wedge Y := (X_n \wedge Y_n)_{n\in\mathbb{T}}$ is also a supermartingale.
(2) If X is a martingale and ϕ is a convex function on $\mathbb{R}$ such that $\mathbb{E}[|\phi(X_n)|] < \infty$ for all $n \in \mathbb{T}$, then $\phi(X) := (\phi(X_n))_{n\in\mathbb{T}}$ is a submartingale.
(3) If X is a submartingale and ϕ is a convex increasing function on $\mathbb{R}$ such that $\mathbb{E}[|\phi(X_n)|] < \infty$ for all $n \in \mathbb{T}$, then $\phi(X)$ is a submartingale.

1.4. A ***stopping time*** S is an rv $S : \Omega \to \mathbb{N}_\infty$ such that $\{S \leq n\} \in \mathscr{F}_n$ for every $n \in \mathbb{N}$.

(1) If S, T are both stopping times, so are $S \wedge T$ and $S \vee T$.
(2) Constants are stopping times. Thus for a stopping time S and a fixed $n \in \mathbb{T}$, $S \wedge n$ is also a stopping time.
(3) Let $(X_n)_{n\in\mathbb{N}}$ be an adapted process and B be a Borel set in $\mathbb{R}$. Put $H^B(\omega) = \inf\{n \in \mathbb{N} : X_n(\omega) \in B\}$; here $\inf \emptyset = \infty$. Then H^B is a stopping time.

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1.5. Let S be a stopping time. Let $\mathscr{F}_S$ be the collection of all $A \in \mathscr{F}_\infty$ such that $A \cap \{S \leq n\} \in \mathscr{F}_n$ for all $n \in \mathbb{N}$. It is a σ-field.

(1) S is $\mathscr{F}_S$-measurable.
(2) If T is a stopping time and $S \leq T$, then $\mathscr{F}_S \subset \mathscr{F}_T$
(3) Let $(X)_{n\in\mathbb{N}_\infty}$ be an adapted process. Put $X_S(\omega) = X_{S(\omega)}(\omega)$ for any $\omega \in \Omega$. Then X_S is $\mathscr{F}_S$-measurable.
(4) Let $(X)_{n\in\mathbb{N}}$ be an adapted process. Then $X_S \mathbb{1}_{\{S<\infty\}}$ is $\mathscr{F}_S$-measurable.

1.6. Let X be a process and S be a stopping time. The ***process*** X ***stopped at*** S, denoted by X^S, is defined by $X^S_n = X_{S\wedge n}$, $n \in \mathbb{T}$.

1.7. Let V and Y be two processes over $\mathbb{N}$, the first one predictable and the second adapted. Define the new process:

$$\begin{aligned} Z_1 =& V_1 Y_1, \\ Z_n =& V_1 Y_1 + V_2(Y_2 - Y_1) + \cdots + V_n(Y_n - Y_{n-1}),\ n \geq 2. \end{aligned}$$

It is called the ***transform of*** Y ***by*** V, and denoted by $Z = V \cdot Y$.

Let S be a stopping time and Y an adapted process. Then the stopped process Y^S is just the transform of Y by the predictable process $V_n = \mathbb{1}_{\{S\geq n\}}$.

1.8. Let X be an adapted process such that $\mathbb{E}[|X_n|] < \infty$ for all $T \in \mathbb{N}$. Put

$$\begin{aligned} M_1 =& X_1 \\ M_2 =& M_1 + (X_2 - \mathbb{E}[X_2|\mathscr{F}_1]) \\ & \cdots \\ M_n =& M_{n-1} + (X_n - \mathbb{E}[X_n|\mathscr{F}_{n-1}]), \end{aligned}$$

and

$$\begin{aligned} A_1 =& 0 \\ A_2 =& A_1 + (X_1 - \mathbb{E}[X_2|\mathscr{F}_1]) \\ & \cdots \\ A_n =& A_{n-1} + (X_{n-1} - \mathbb{E}[X_n|\mathscr{F}_{n-1}]). \end{aligned}$$

Then $X = M - A$, M is a martingale, A is predictable with $A_1 = 0$. This is called ***Doob's Decomposition*** of X. If, further, X is a supermartingale, then A is increasing.

2. Doob's Theory

2.1.

(1) Let Y be a martingale over $\mathbb{N}$, and V be a predictable process. If $\mathbb{E}[|V \cdot Y|] < \infty$ for all $n \in \mathbb{N}$, then $V \cdot Y$ is a martingale.
(2) Let Y be a submartingale over $\mathbb{N}$, and V be a positive predictable process. If $\mathbb{E}[|V \cdot Y|] < \infty$ for all $n \in \mathbb{N}$, then $V \cdot Y$ is a submartingale.

It follows that

(3) Let X be a submartingale, and T is a stopping time. The stopped process X^T is a submartingale.

2.2. ***Optional Sampling Theorem.***

(1) Let $(X_n)_{n\in\mathbb{N}_\infty}$ be a submartingale, and let S, T be two stopping times such that $S \le T$. Then X_S and X_T are integrable, and $\mathbb{E}[X_T|\mathscr{F}_S] \ge X_S$.

As a consequence,

(2) Let $(X_n)_{n\in\mathbb{N}}$ be a submartingale, and let S, T be two bounded stopping times such that $S \le T$. Then X_S and X_T are integrable, and $\mathbb{E}[X_T|\mathscr{F}_S] \ge X_S$.
(3) Let Y be integrable and put $X_n = \mathbb{E}[Y|\mathscr{F}_n]$ for $n \in \mathbb{N}_\infty$. Then for any stopping time S, $X_S = \mathbb{E}[Y|\mathscr{F}_S]$.

2.3.

(1) Let $(X_n)_{n\in\mathbb{N}}$ be a submartingale. Then for any $n \in \mathbb{N}$ and any $\lambda \in \mathbb{R}$,

$$\begin{aligned}\lambda\mathbb{P}(\max_{1\le i\le n} X_i \ge \lambda) &\le \int_{\{\max_{1\le i\le n} X_i \ge \lambda\}} X_n \mathrm{d}\mathbb{P} \le \mathbb{E}[X_n^+],\\ \lambda\mathbb{P}(\min_{1\le i\le n} X_i \le -\lambda) &\le \mathbb{E}[X_n - X_1] - \int_{\{\max_{1\le i\le n} X_i \le -\lambda\}} X_n \mathrm{d}\mathbb{P} \le \mathbb{E}[X_n^+] - \mathbb{E}[X_1].\end{aligned}$$

Consequently,

(2) Every submartingale over $\mathbb{N}$ is of weak $(1,1)$ type. If $X \ge 0$ or $X \le 0$ or X is a martingale, then the constant is 1; otherwise, 3 in general.
(3) Every positive submartingale (and thus every martingale) over $\mathbb{N}$ is of strong type (p,p), $1 < p \le \infty$, with constant q.

2.4.

(1) If a submartingale $(X_n)_{n\in\mathbb{N}}$ satisfies $\sup_n \mathbb{E}[X_n^+] < \infty$, then it is bounded in L^1.
(2) A martingale $(X_n)_{n\in\mathbb{N}}$ is bounded in L^1 iff it is a difference of two positive martingales.
(3) Let $(X_n)_{n\in\mathbb{N}}$ be a bounded submartingale in L^1. Then X converges a.s.

2.5. Let U_a^b be the number of upcrossings of $[a, b]$.

(1) Let X be a supermartingale stopped at time N. For any $a < b$,

$$\mathbb{E}[U_a^b] \le \frac{1}{b-a}\mathbb{E}[(X_N - a)^-].$$

(2) Let X be a submartingale stopped at time N. For any $a < b$,

$$\mathbb{E}[U_a^b] \le \frac{\mathbb{E}[(X_N - a)^+] - \mathbb{E}[(X_1 - a)^+]}{b-a} \le \frac{\mathbb{E}[X_N^+] + |a|}{b-a}.$$

2.6. Let $(X_n)_{n\in\mathbb{N}}$ be a submartingale. TFAE:

(1) (X_n) is uniformly integrable.
(2) it converges in L^1.
(3) it converges a.e. to an integrable rv X_∞ such that $(X_n)_{n\in\mathbb{N}_\infty}$ is a submartingale and $\mathbb{E}[X_n] \longrightarrow \mathbb{E}[X_\infty]$.

In case of a martingale, the above conditions are equivalent to:

(4) There exists an integrable rv such that $X_n = \mathbb{E}[X|\mathscr{F}_n]$ for any $n \in \mathbb{N}$.

2.7. We can similarly define a submartingale $\{X_n, \mathscr{F}_n, n \in -\mathbb{N}\}$. Then $\lim_{n\to-\infty} X_n = X_{-\infty} \in [-\infty, \infty)$ a.s. Moreover, TFAE:

(1) (X_n) is uniformly integrable.
(2) it converges to X_∞ in L^1.
(3) $(X_n)_{n\in-\mathbb{N}_\infty}$ is a submartingale.
(4) $(\mathbb{E}[X_n])$ is bounded below.

Clearly, the last conditions are automatically satisfied by a martingale.

3. Burkholder's Theory

All processes in this section are indexed over $\mathbb{N}$.

3.1. Let X be a positive supermartingale. Set $X_\infty = \lim_n X_n$. Then

$$X_n = (X_n - \mathbb{E}[X_\infty|\mathscr{F}_n]) + \mathbb{E}[X_\infty|\mathscr{F}_n]$$

is the only decomposition $X = Y + Z$, where Z is a uniformly integrable martingale, and Y is a positive supermartingale such that $\lim_n Y_n = 0$ a.s.

A positive supermartingale does not dominate any positive martingales except 0 iff it converges to 0 in L^1. In this case, it is called a ***potential***, and the Doob decomposition $X = M - A$ can be written as $X_n = \mathbb{E}[A_\infty|\mathscr{F}_n] - A_n$. Clearly, $M_\infty := \lim_n M_n = A_\infty$.

Let X be a supermartingale such that $\lim_n \mathbb{E}[X_n] > -\infty$. Then X has a unique decomposition $X = Y + Z$, where Z is a martingale, and Y is a potential. Moreover, Z is positive if X is positive. This is the ***Riesz decomposition*** of X.

3.2. Let X be a positive supermartingale.

(1) If X is dominated by a constant c. Then for any $k \in \mathbb{N}$,

$$\mathbb{E}[M_\infty^k] \leq k!c^{k-1}\mathbb{E}[X_1] \leq k!c^k.$$

(2) If $1 < p < \infty$, there exists a constant c_p such that

$$\|M_\infty\|_p = \sup_n \|M_n\|_p \leq c_p \sup_n \|X_n\|_p.$$

3.3. Let X be a martingale and V a predictable process bounded by 1. Then

(1) for every $\lambda > 0$,

$$\lambda\mathbb{P}(\sup_n |(V \cdot X)_n| > \lambda) \leq 18 \sup_n \|X_n\|_1,$$

where the constant 18 can be replaced with 9 if X is positive.

(2) for every $1 < p < \infty$, there exists a constant c_p such that

$$\sup_n \|(V \cdot X)_n\|_p \leq c_p \sup_n \|X_n\|_p.$$

3.4. Let X, Y be any two processes. Define the ***variation process*** $[X, Y]$ by

$$\begin{aligned}[X,Y]_1 &= X_1 Y_1 \\ [X,Y]_n &= X_1 Y_1 + (X_2 - X_1)(Y_2 - Y_1) + \cdots + (X_n - X_{n-1})(Y_n - Y_{n-1}), \quad n \geq 2.\end{aligned}$$

Then

$$[X,Y] = \frac{1}{2}([X+Y, X+Y] - [X,X] - [Y,Y]).$$

Also write

$$Q^X = \sqrt{[X,X]}.$$

3.5. Let X be a martingale.

(1) If $1 < p < \infty$, then
$$\|Q^X_\infty\|_p \sim_p \sup_n \|X_n\|_p.$$

(2) There exists a constant $c > 0$ such that for any $\lambda > 0$,
$$\lambda \mathbb{P}(Q^X_\infty > \lambda) \leq c \sup_n \|X_n\|_1.$$