FACTS ABOUT THE RIEMANN-STIELTJES AND LEBESGUE-STIELTJES INTEGRALS

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In this short note, we collect a few facts about the Riemann-Stieltjes (RS) and Lebesgue-Stieltjes (LS) integrals and their connections. The reader can find direct proofs of these facts in *Measures and Integrals* by Wheeden and Zygmund or may imitate some proofs there. The reader is assumed to be familiar with Riemann and Lebesgue integrals.

We use $\int_a^b$ to indicate a Riemann-type integral and $\int_A$ for a Lebesgue-type integral.

1. Definition of an RS integral

1.1. Let f and ϕ be two functions on $[a,b]$ and $I \in \mathbb{R}$. For a partition Γ of $[a,b]$: $a = x_0 < x_1 < \cdots < x_n = b$, put $|\Gamma| = \max_{1\le i\le n}(x_i - x_{i-1})$. If for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\Big|\sum_{i=1}^n f(\xi_i)\big(\phi(x_i) - \phi(x_{i-1})\big) - I\Big| < \varepsilon$$

for any partition $\Gamma: a = x_0 < x_1 < \cdots < x_n = b$ with $|\Gamma| < \delta$ and any points $(\xi_i)_{i=1}^n$ such that $\xi_i \in [x_{i-1}, x_i]$ for $i = 1, \dots, n$, we say that I is the Riemann-Stieltjes integral of f with respect to ϕ on $[a,b]$, and write it as $\int_a^b f\mathrm{d}\phi$.

1.2. If f is bounded and ϕ is increasing on $[a,b]$, put

$$L_\Gamma = \sum_{i=1}^n \left(\inf_{\xi\in[x_{i-1},x_i]} f(\xi)\right)\Big(\phi(x_i) - \phi(x_{i-1})\Big),$$

$$U_\Gamma = \sum_{i=1}^n \left(\sup_{\xi\in[x_{i-1},x_i]} f(\xi)\right)\Big(\phi(x_i) - \phi(x_{i-1})\Big).$$

1.3. Let f be bounded and ϕ be increasing on $[a,b]$. The following are equivalent:

(1) $\int_a^b f\mathrm{d}\phi$ exists;
(2) $\lim_{|\Gamma|\to 0}(U_\Gamma - L_\Gamma) = 0$;
(3) $\lim_{|\Gamma|\to 0} U_\Gamma$ and $\lim_{|\Gamma|\to 0} L_\Gamma$ both exist and are equal.

In this case,

$$\lim_{|\Gamma|\to 0} L_\Gamma = \lim_{|\Gamma|\to 0} U_\Gamma = \sup_\Gamma L_\Gamma = \inf_\Gamma U_\Gamma = \int_a^b f\mathrm{d}\phi.$$

See Appendix at the end for more facts on lower and upper sums.

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2. Good sufficient conditions

2.1. If f is continuous on $[a,b]$ and ϕ is of bounded variation on $[a,b]$, then $\int_a^b f\mathrm{d}\phi$ exists. Moreover,

$$\Big|\int_a^b f\mathrm{d}\phi\Big| \leq (\sup_{[a,b]}|f|)\mathrm{V}[\phi;a,b].$$

2.2. If $\int_a^b f\mathrm{d}\phi$ exists, then so does $\int_a^b \phi\mathrm{d}f$, and

$$\int_a^b f\mathrm{d}\phi = [f(b)\phi(b) - f(a)\phi(a)] - \int_a^b \phi\mathrm{d}f.$$

3. Definition of an LS integral

3.1. Let ϕ be an increasing, right-continuous function on $(a,b]$. Then there exists a unique measure μ on $\Big((a,b],\mathscr{B}\big((a,b]\big)\Big)$ such that

$$\mu\big((s,t]\big) = \phi(t) - \phi(s) \quad \text{whenever } a < s < t \leq b.$$

For a Borel measurable function f on $(a,b]$, write

$$\int_{(a,b]} f\mathrm{d}\phi := \int_{(a,b]} f\mathrm{d}\mu$$

(exists or not), and call it the Lebesgue-Stieljes integral of f with respect to ϕ on $(a,b]$.

3.2. If ϕ is increasing and absolutely continuous on $(a,b]$[1], then for a Borel measurable function f on $(a,b]$,

$$\int_{(a,b]} f\mathrm{d}\phi = \int_{(a,b]} f\phi'\mathrm{d}x;$$

the integrals on two sides either both exist or both do not exist.

4. Connecting RS and LS integrals

4.1. Let ϕ be an increasing, right-continuous function on $[a,b]$, and f be a bounded Borel measurable function on $[a,b]$. If $\int_a^b f\mathrm{d}\phi$ exists, then

$$\int_{(a,b]} f\mathrm{d}\phi = \int_a^b f\mathrm{d}\phi.$$

[1] i.e., for any $\varepsilon > 0$, there exists $\delta > 0$ such that $\sum_{i=1}^n \big(\phi(t_i) - \phi(s_i)\big) < \varepsilon$ whenever $a < s_i < t_i \leq b$, $i = 1,\dots,n$, (s_i,t_i)'s are disjoint, and $\sum_{i=1}^n (t_i - s_i) < \delta$.

4.2. The value of the previous theorem is that if f has good properties but ϕ does not then one can invoke 2.2 to reduce $\int_a^b f\mathrm{d}\phi$ to $\int_a^b \phi\mathrm{d}f$.

Example 1. Let X be a non-negative random variable over $(\Omega, \mathscr{F}, \mathbb{P})$ such that $\mathbb{E}[X^p] < \infty$. For $t \in \mathbb{R}$, put

$$\phi(t) = \mathbb{P}(X \le t), \quad \omega(t) = 1 - \phi(t) = \mathbb{P}(X > t).$$

Clearly, ϕ is increasing and right-continuous on $\mathbb{R}$. Moreover,

$$n^p\omega(n) \le \mathbb{E}[X^p \mathbb{1}_{\{X>n\}}] \longrightarrow 0.$$

Therefore,

$$\begin{aligned}
\mathbb{E}[X^p] &= \int_{(0,\infty)} t^p \mathrm{d}\phi(t) \\
&= \lim_{n\to\infty} \int_{(0,n]} t^p \mathrm{d}\phi(t) \\
&= \lim_{n\to\infty} \int_0^n t^p \mathrm{d}\phi(t) && \text{by 2.1 and 4.1} \\
&= \lim_{n\to\infty} \left[t^p\phi(t)\Big|_0^n - \int_0^n \phi(t)\mathrm{d}t^p \right] && \text{by 2.2} \\
&= \lim_{n\to\infty} \left[n^p\phi(n) - \int_{(0,n]} \phi(t)\mathrm{d}t^p \right] && \text{by 4.1} \\
&= \lim_{n\to\infty} \left[n^p\phi(n) - \int_{(0,n]} \phi(t)pt^{p-1}\mathrm{d}t \right] && \text{by 3.2} \\
&= \lim_{n\to\infty} \left[n^p\phi(n) - \int_0^n \phi(t)pt^{p-1}\mathrm{d}t \right] && \text{by 4.1} \\
&= \lim_{n\to\infty} \left[n^p\big(\phi(n) - 1\big) + p\int_0^n (1-\phi(t))t^{p-1}\mathrm{d}t \right] \\
&= p\int_0^\infty \omega(t)t^{p-1}\mathrm{d}t.
\end{aligned}$$

5. Appendix: more on lower and upper sums

5.1. Let f be bounded and ϕ be increasing on $[a, b]$. It may happen that $\lim_{|\Gamma|\to 0} L_\Gamma$ does not exist. If it does exist, then $\lim_{|\Gamma|\to 0} L_\Gamma = \sup_\Gamma L_\Gamma$. Similar facts hold for U_Γ.

5.2. Let f be bounded, and let ϕ be increasing and continuous on $[a, b]$. Then $\lim_{|\Gamma|\to 0} L_\Gamma$ and $\lim_{|\Gamma|\to 0} U_\Gamma$ both exist, so that $\lim_{|\Gamma|\to 0} L_\Gamma = \sup_\Gamma L_\Gamma$ and $\lim_{|\Gamma|\to 0} U_\Gamma = \inf_\Gamma U_\Gamma$. Therefore, if, in addition, $\sup_\Gamma L_\Gamma = \inf_\Gamma U_\Gamma$, then $\int_a^b f\mathrm{d}\phi$ exists.

5.3. Let f be bounded and ϕ be increasing on $[a, b]$. Even when $\sup_\Gamma L_\Gamma = \inf_\Gamma U_\Gamma$, it may happen that $\int_a^b f\mathrm{d}\phi$ does not exist.