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Entwined pairs and Schrödinger's equation

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Abstract

We show that a point particle moving in space–time on entwined-pair paths generates Schrödinger's equation in a static potential in the appropriate continuum limit. This provides a new realist context for the Schrödinger equation within the domain of classical stochastic processes. It also suggests that 'self-quantizing' systems may provide considerable insight into conventional quantum mechanics.

1. Introduction

Historically, there have been many interpretations of quantum mechanics and these have ranged through a spectrum of 'pictures' regarding the basic equations themselves. At one end of the spectrum, Bohr and Heisenberg viewed the mathematics of quantum mechanics as no more than an *algorithm* for calculation. In Bohr's words [1]:

> There is no quantum world. There is only an abstract quantum physical description. It is wrong to think that the task of physics is to find out how nature is. Physics concerns what we can say about nature.

At this end of the spectrum, quantum mechanics is primarily about 'epistemology,' a study of how we obtain knowledge of the world. Although Bohr's position may seem rather extreme, it is fairly close to what might be called the 'mainstream

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attitude' towards quantum mechanics. At the other end of the spectrum, and less conventional, are more 'ontological' approaches which consider the possibility that the mathematics of quantum mechanics reflects a genuine external reality. The most well-known of these approaches is due to Bohm [2]. In Bohm's approach, quantum mechanics describes an external reality which, as in classical physics, contains particles that move on real space–time trajectories. The peculiarities of quantum mechanics enter Bohm's theory through the 'quantum potential' which provides non-local and interference effects. Through the quantum potential, Bohm is able to mix the wave and particle paradigms to construct a complete ontology for quantum mechanics. It may well be an accident of history that has favoured epistemological theories over Bohm's theory. However, the mixing of paradigms in Bohm's approach does leave open the question of a physical mechanism linking the particle to its associated wavefunction. All things considered, there is little real consensus on the connection of the wavefunction to the outside world, even though there is general consensus on how we use the wavefunction.

This paper shows that in the context of a new kind of random walk, denoted entwined path random walks (EPRW) the propagator for the Schrödinger equation for a particle in a smooth bounded potential in one dimension can be obtained completely within the single-particle-continuous-trajectory paradigm, without the use of any analytic continuation, either forced or explicit.

To put this result in context with previous work, one of us (G.N.O.) has discussed simple classical statistical mechanical systems which have, as part of their description, either the Schrödinger or Dirac equations.

The focus of this program of study is the formal analytic continuation (FAC) that brings about wave-particle duality in conventional quantum mechanics. For example, if we compare the solutions of $\partial\phi/\partial t = D(\partial^2\phi/\partial x^2)$ to the solutions of $\partial\phi/\partial t = \mathrm{i}D(\partial^2\phi/\partial x^2)$ we see that the FAC (the replacement of D by $\mathrm{i}D$) changes the diffusion equation with its known ontology (Brownian motion) into Schrödinger's equation where wave-particle duality has discouraged an interpretation based completely within the single-particle paradigm. The question we address is then "Is there a context within the single-particle paradigm that yields the quantum equations as part of its description?" Thus far previous work has uncovered three distinct contexts for the equations of quantum mechanics. The first model, subsequently called the 'Spiral model' [3,4], showed that it was possible to obtain the Dirac propagator in one dimension, from a particle path that formed a spiral in space–time. The interesting feature of this model was that it showed that the FAC that is usually used to make the transition to quantum mechanics could have a *physical* origin in the presence of time-reversed portions of a classical continuous space–time trajectory. It was the existence of classical anti-particles in this model that resulted in the analog of quantum phase.

There were a number of limitations to the Spiral model. The derivation provided could produce the Dirac propagator only in a continuum limit. From the point of view of the statistical mechanics involved, this continuum limit was also a mean field calculation and ignored the statistical fluctuations expected in a random sample of Spiral paths. It became apparent that the Spiral model was sufficiently ill-conditioned that the Dirac propagator would not emerge in the continuum limit of a series of such discrete models unless these natural fluctuations could be severely suppressed. Furthermore, the Spiral model was technically difficult to work with and resisted generalization. The model did however suggest that the origin of 'quantum phase' could be the time-reversed portions of a space–time trajectory, and that one could simulate quantum propagators in ensembles of classical random walks by colouring them with just two colours, and then extracting the propagators using *projection* [5–8]. These 'colouring models' avoided the formal analytic continuation used to obtain quantum propagators and put the propagators themselves in a classical context. However, the new context was explicitly *not* quantum mechanics because the 'wave properties' were only statistical features of ensembles of particles. They were not intrinsic features of single-particle trajectories themselves. All paths in the colouring models were traversed in the $+t$-direction, so, like Feynman's path-integral, an entire sum-over-histories was required to describe a single-particle's propagator. Put another way, colouring models showed how we can extract the 'information of the Dirac propagator' from ensembles of simple classical random walks... in Bohm's terminology the focus of the colouring models is largely epistemological.

The most recent model involving entwined paths (EP) is the subject of the present paper. It combines elements of both the spiral and colouring models. Like the Spiral model, the origin of phase in the propagator is the existence of time-reversed portions of a single-particle's trajectory. However, at the level of difference equations, the colouring and EP models share the same mathematical description, but the latter does not require the mean field limit of the Spiral model. In spite of their similarity at the level of difference equations, the statistical mechanics underlying the EP model differs from that of the colouring models in two important respects. The new statistical mechanics involves entwined path random walks which are 'self-quantizing' and use only space–time geometry to provide the quantum interference effects usually produced by FAC or a colouring [9,10]. How self-quantization works will be discussed in the next section. However, from a historical perspective EPRW is (to our knowledge) the first constructive example of a 'reversible diffusion.' The connection between reversible diffusion and Schrödinger 's equation was first investigated by Fenyes [11] and Nelson [12] and subsequently studied by many authors [13,14]. The other important distinction between the EPRW model and colouring models is that EPRW replaces the *ensemble* of paths of the colouring models with *a single path*. In other words, we regard the entwined path as a coherent physical entity. This is important in that it means that the EP model *is* potentially an ontology for quantum mechanics itself. This was not the case for the colouring models which relied on ensemble averages of particle paths to provide the propagator of a single particle. If colouring models show us how to *extract* the information of the Dirac propagator from ensembles of classical random walks, EPRW's show us how Nature could *encode* that information in a single space–time path. Colouring models are primarily about epistemology—the EP model is completely about ontology.

In this sense the entwined model is a major advance over previous models. It shows that the Dirac equation in one dimension, apart from being the fundamental equation for the propagation of a 'physical' particle like an electron, is also a phenomenological equation describing 'mathematical' particles that move on entwined space–time paths. This has been verified numerically by stochastically constructing the Dirac propagator using only a single path [10]. Having such a 'hands on' model means that questions that cannot be asked directly of conventional quantum mechanics *can* be asked in the context of entwined paths. Thus we may ask: What is the origin of phase? What causes zitterbewegung? How does superposition work? What is the origin of the uncertainty principle? EPRW provides a framework in which these 'unspeakable' [22] questions may, in principle, be answered.

At this point in time we cannot say whether the ontology provided by EPRW applies beyond *the equations* of quantum mechanics. The measurement postulates and the reduction of wave-packets on measurement ('R-process') are outside the unitary propagation ('U-process') we consider. However, because we are dealing with a specific, well-defined model, we expect to be able to test whether there is any relation between the stochastic evolution of the propagator formed by EP, and the stochastic element in quantum mechanics that is typically brought in by the R-process. This will be the subject of another paper; in the present work we consider only the U-process.

Although the Schrödinger equation is more familiar than its more fundamental Dirac counterpart, the latter equation is the natural setting of the entwined paths approach. The reason for this is that, like the diffusion equation, the Schrödinger equation has no 'inner scale': the 'mean free path' of particles (Compton wavelength) is effectively zero, and the signal velocity is infinite. This is of course an idealization that is convenient when the mean free path is much smaller than the scale of interest, and the mean free speed (c) is much greater than the speed of interest. Since entwined paths are entwined on the scale of the mean free path, the non-relativistic limit has to be taken explicitly.

In this paper we demonstrate the connection between the entwined model and Schrödinger's equation in two ways. First we use a technique developed in [15] to obtain Schrödinger's equation for a particle in a static potential, as the direct continuum limit of a colouring model. This method mimics the usual transition from a symmetric random walk to the diffusion equation. However, the use of entwined paths ensures that the resulting phenomenology is reversible, and the result is Schrödinger's equation.

In the second method we take the continuum limit maintaining both a finite characteristic length (mean free path) and a finite characteristic speed. This version mimics Kac's derivation of the Telegraph equations [16,17]. Here the entwined paths again ensure reversibility and the result is a particular representation of the Dirac equation. The subsequent limit in which the mean free speed (i.e., the signal velocity) goes to infinity yields Schrödinger's equation.

2. Entwined pairs: a classical Stochastic model

Consider the following stochastic process (Fig. 1). A single particle is constrained to move in discrete time on a lattice with lattice spacing δ. The time steps are of

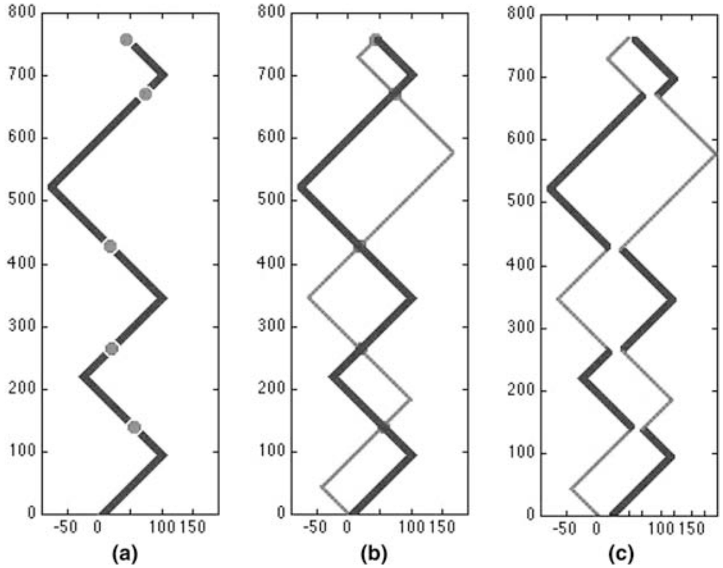

Fig. 1. Forming entwined paths in space–time: z is horizontal, t is vertical. The particle travels at constant speed but occasionally reverses direction in response to a stochastic process. (a) A stutter is introduced into the stochastic process. At every other indication from the stochastic process, a marker is dropped instead of a direction change (disks in the figure). After some specified time t_R, the process stops at the next marker. (b) Reverse direction in time but not in space. Follow the 'light-cone' paths through the markers back to the origin. (c) The entwined path formed in (b) can be regarded as two osculating paths which we call envelopes. These are separated in (c) for clarity. The geometry of the envelopes is the same as if the paths were generated by the stochastic process without a stutter. The 'colouring' denoting direction of traversal mimicks the Chessboard colouring.

length ϵ. At each step the particle moves one lattice spacing in z and one in t. For a point source, the particle starts at the origin at $(0, 0)$ and steps to the first lattice point at (δ, ϵ). The particle then steps to $(2\delta, 2\epsilon)$ with probability β or changes direction and steps to $(0, 2\epsilon)$ with probability α. Once it is moving in the $-z$-direction the particle first drops a marker with probability α and then at the second indication of the stochastic process changes direction again. This alternating sequence of changing direction and dropping markers is continued until the first marker after some specified return time t_R. At that marker the particle maintains its direction in space but reverses direction in time. Subsequent steps of the process follow the light cones of the markers back to the origin. The resulting closed loop we call an entwined path. At the origin the process begins again.

In the figure, suppose the portion of the trajectory traversed forward in time is coloured blue, with the backward portion coloured red. Whereas the colours chosen are arbitrary, the fact that we distinguish the direction of traversal in time is not. Since a particle that reverses its direction in time will be seen to be anihilated by an 'antiparticle' which is really the same particle reversing its direction in time, we shall define a 'charge' associated with the traversal of a trajectory. Forward traversals will correspond to a charge of $+1$, reversed traversals will correspond to a charge of -1. In the figure, blue portions of the trajectory correspond to a charge of $+1$ and red portions correspond to -1. Note that although each complete entwined path yields a net charge of zero at any value of t, entwined paths do separate charge and we shall see that the charge separation builds up a charge field in space–time that is oscillatory in nature. We can then imagine counting the net charge which enters a site. Regarding the entwined pair as a coherent physical entity we will have a four-component object to consider, since there are two paths to each entwined pair, and each member of the pair has two possible directions to move in space Fig. 2a.

We have generated the entwined pair in a way which allows us to see how it is equivalent to a single closed loop in space–time. However, the method of generation is tuned to counting the paths based on their outer envelopes. That is, both the left and right 'corners' in the outer envelopes are generated statistically by the same stochastic process. When we dropped markers at every other call from the random process, and then passed through the markers on the return path, we ensured that we could equally well have generated the outer envelopes by an alternating colouring of an independently generated outer envelope. For example, the left envelope in Fig. 2b may be obtained by starting on the red path at the origin and alternating blue and red sections at every second envelope corner. All left envelope paths have this colouring rule, which itself is a consequence by the geometry of entwined pairs. Note that on both envelopes, the probability of a corner is always α regardless of whether it is a right- or left-handed corner.

Regarding the left envelope, if $\phi_1'(z, t+\epsilon)$ is the net charge entering the lattice points $(z, t+\epsilon)$ from the $+z$ direction and $\phi_2'(z, t+\epsilon)$ is the net charge entering from the $-z$-direction then the difference equation expressing conservation of charge is easily found. Regarding Fig. 3 we see that for the charge on the left envelope we have:

$$\begin{aligned} \phi_1'(z, t+\epsilon) &= \beta\phi_1'(z+\delta, t) - \alpha\phi_2'(z-\delta, t), \\ \phi_2'(z, t+\epsilon) &= \beta\phi_2'(z-\delta, t) + \alpha\phi_1'(z+\delta, t). \end{aligned} \tag{1}$$

Note that as ϕ_2' scatters into ϕ_1' it changes sign so the contribution to ϕ_1' is negative.

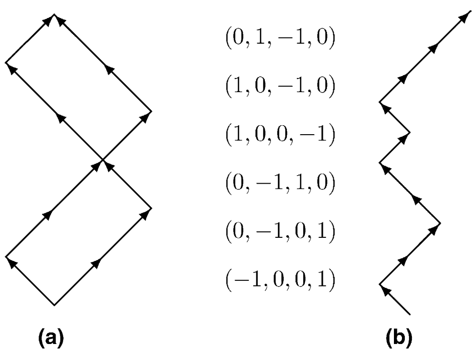

Fig. 2. (a) A sample entwined path with the corresponding velocity 4-vector. Note the change in sign of the two envelopes when the paths cross. (b) A left envelope path. Note that the left envelope path would change colour at every left-hand corner.

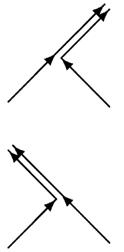

Fig. 3. ϕ_1' and ϕ_2' on the left envelope receive contributions from the previous time step. Note that ϕ_1' changes colour at a left-hand corner. ϕ_2' does not change colour at a right-hand corner.

Similarly, for the right envelope, if $\phi_3'(z, t+\epsilon)$ is the net number entering the lattice points $(z, t+\epsilon)$ from the $+z$-direction and $\phi_4'(z, t+\epsilon)$ is the net number entering from the $-z$-direction we may write:

$$\begin{aligned} \phi_3'(z, t+\epsilon) &= \beta\phi_3'(z+\delta, t) + \alpha\phi_4'(z-\delta, t), \\ \phi_4'(z, t+\epsilon) &= \beta\phi_4'(z-\delta, t) - \alpha\phi_3'(z+\delta, t). \end{aligned} \tag{2}$$

Note the alternating signs in the 'scattering' terms in these equations. These arise because of the fact that every other 'corner' in an envelope is actually an exchange with an antiparticle. The two envelopes naturally partition the states into two block diagonal systems coupled only by their initial conditions. Versions of these equations have been obtained and numerically verified for the underlying stochastic process in [10].

3. Entwined pairs and diffusive scaling

Symmetric binary random walks have a well-known scaling that allows a description in the continuum limit. This scaling is termed diffusive scaling and corresponds to

$$(\delta)^2/(2\epsilon) \to D \quad \text{as } \epsilon \to 0, \tag{3}$$

where $D > 0$ is a diffusion constant. The reason that this scaling 'works' is simply because the *mean-square* end-to-end length of a symmetric binary random walk increases as the number of steps n, so (3) is the natural scaling for such walks. Notice also that this scaling is the basis of the uncertainty principle in this model. δ is a measure of the uncertainty in the position of a particle when we compare the lattice model to the continuum. δ/ϵ is the speed of the particle on the lattice, which would be proportional to the uncertainty in the momentum. (3) then states that the product of the uncertainties in the position and momentum of a particle is asymptotically constant. The origin of the uncertainty principle here is simply the geometry of the underlying paths.

In the entwined-pair model above, if we choose α and β to be asymptotically $1/2$ then the underlying envelopes are asymptotically symmetric binary random walks. However, the entwining of paths makes each envelope with its orthogonal twin a reversible path. The result is that a phenomenology that would otherwise be described by the diffusion equation is instead described by the Schrödinger equation, reflecting the change from an entropy-dominated to an entropyless system. The new system is entropyless because each time-reversed path essentially undoes the disorder from its original partner.

To proceed, we scale system (1) diffusively. That is, we shall let α and β approach $1/2$ as the lattice spacing gets small. The potential will then enter through the limiting process in just how the terms α and β approach $1/2$.

Consider the left envelope densities ϕ'_1 and ϕ'_2. We will write (1) in matrix form using shift operators $E_z^{\pm}$ and E_t where $E_z^{\pm}\phi'(z,t) = \phi'(z \pm \delta, t)$ and $E_t\phi'(z,t) = \phi'(z, t+\epsilon)$. Writing $\Phi' = \binom{\phi'_1}{\phi'_2}$, (1) may be written as

$$E_t\Phi'(z,t) = \begin{pmatrix} \beta E_z^+ & -\alpha E_z^- \\ \alpha E_z^+ & \beta E_z^- \end{pmatrix} \Phi'(z,t). \tag{4}$$

Now suppose the particle chooses its next step according to a canonical ensemble in which a smooth bounded potential $v(z)\epsilon$ acts like an energy. That is, suppose the particle associates a relative energy of $+v(z)\epsilon$ to continue the direction of travel, and an energy of $-v(z)\epsilon$ for a direction reversal. For large positive values of $v(x)$ reversals are favoured. This means that the reversal probability is

$$\alpha = \frac{e^{v(z)\epsilon}}{e^{-v(z)\epsilon} + e^{v(z)\epsilon}} \tag{5}$$

so that

$$\alpha = \frac{1}{2}(1 + v(z)\epsilon) + O(\epsilon^2), \quad \beta = \frac{1}{2}(1 - v(z)\epsilon) + O(\epsilon^2). \tag{6}$$

Note that the effect of the field is to alter the local mean free path of the particle. If $v(z)$ is negative the particle tends to stay moving in the same direction for longer periods. Conversely, if $v(z)$ is positive the particle changes direction more frequently.

Now we wish to approximate solutions of (4) for small δ by solutions of a partial differential equation. To do this we expand the shift operators

$$E_z^{\pm 1} = 1 \pm \delta\frac{\partial}{\partial z} + \frac{1}{2}\delta^2\frac{\partial^2}{\partial z^2} + O(\delta^3) \tag{7}$$

and

$$E_t = 1 + \epsilon\frac{\partial}{\partial t} + O(\epsilon^2). \tag{8}$$

The matrix in (4) may then be written as

$$\begin{pmatrix} \beta E_z^+ & -\alpha E_z^- \\ \alpha E_z^+ & \beta E_z^- \end{pmatrix} = \frac{1}{2}(1 - v(z)\epsilon)\left(I + \sigma_z\delta\frac{\partial}{\partial z} + I\frac{1}{2}\delta^2\frac{\partial^2}{\partial z^2}\right) \tag{9}$$

$$+ \frac{1}{2}(1 + v(z)\epsilon)\left(\sigma_q + \sigma_x\delta\frac{\partial}{\partial z} + \sigma_q\frac{1}{2}\delta^2\frac{\partial^2}{\partial z^2}\right) + O(\delta^3), \tag{10}$$

where σ_x and σ_z are the usual Pauli matrices, σ_q is $-i\,\sigma_y$ and I is the 2×2 identity matrix.

Keeping only the lowest order terms this is

$$\begin{pmatrix} \beta E_z^+ & -\alpha E_z^- \\ \alpha E_z^+ & \beta E_z^- \end{pmatrix} = \frac{1}{2}(I + \sigma_q) + \frac{1}{2}(\sigma_z + \sigma_x)\delta\frac{\partial}{\partial z} + \cdots + \frac{1}{2}(I + \sigma_q)\left(\delta^2\frac{\partial^2}{\partial z^2}\right) - \frac{1}{2}(I - \sigma_q)v(z)\epsilon. \tag{11}$$

The term in (11) that is independent of both δ and ϵ is an unnormalized finite rotation. If we are going to match solutions of the difference equation to that of a differential equation we must correct the normalization and the finite rotation. To correct the normalization we write $\Phi(z,t) = (\sqrt{2})^{t/\epsilon}\Phi'(z,t)$, and to avoid the finite rotation we note that $((1/\sqrt{2})(I + \sigma_q))^8 = I$. As long as we restrict our comparison of the difference equation and the differential equation so that steps in the t-direction are integer multiples of 8ϵ, we will avoid the finite rotations, which are in any case an artifact of the diffusive (non-relativistic) limit. The difference equation we are considering is then

$$E_t^8\Phi(z,t) = 2^4\begin{pmatrix} \beta E_+ & -\alpha E_- \\ \alpha E_+ & \beta E_- \end{pmatrix}^8 \Phi(z,t). \tag{12}$$

This is the original difference equation, transformed to remove a decaying exponential, and viewed eight steps at a time. Expanding Eq. (12), keeping lowest order terms and using (3), we get, after some algebra

$$\frac{\partial}{\partial t}\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} 0 & D\frac{\partial^2}{\partial z^2} - v(z) \\ -D\frac{\partial^2}{\partial z^2} + v(z) & 0 \end{bmatrix}\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} + \mathrm{O}(\delta), \tag{13}$$

where the ϕ are real. We may express this in complex form as

$$\mathrm{i}\frac{\partial}{\partial t}(\phi_2 + \mathrm{i}\phi_1) = \left(-D\frac{\partial^2}{\partial z^2} + v(z)\right)(\phi_2 + \mathrm{i}\phi_1) + \mathrm{O}(\delta). \tag{14}$$

Thus solutions of Schrödinger's equation (14) approximate solutions of the entwined-pair difference equation (12) for small δ. Note that ϕ_1 and ϕ_2 here are real functions which are themselves just limits of ensemble averages of the net charge accumulated via entwined paths. They are not components of wave functions in the sense of quantum mechanics, since there has been no FAC or quantization involved. The ϕ are strictly real classical objects representing an expected net flow of charge in the $-z$- and $+z$-directions, respectively.

We also note that the potential energy $v(z)$ that affected the particle's mean free path in (5) enters the Schrödinger equation as a potential term in the Hamiltonian. Although $v(z)$ does not favour either direction explicitly in the actual walk, as the particle moves it tends to reverse direction more frequently when moving in a region of high potential energy, and less frequently when moving in a region of low potential. The net effect at the level of the Schrödinger equation is that regions of maximum $v(z)$ tend to repel the particle and regions of minimum $v(z)$ tend to attract it.

The remaining two densities satisfy the time-reversed version of (13) and are remnants of the four component description of entwined pairs. The two PDE's are connected by their initial conditions and their solutions, in complex form, are conjugates of each other, so their product, integrated over all space is a time-independent positive constant. This fact potentially opens the door to the Born postulate which would associate a probability density with the product of the two wavefunctions. However, at this point we have no justification for invoking such a postulate. In future work we shall examine the microscopic dynamics to see if the postulate is implied by any reasonable measurement scheme.

Although in the above calculation we were forced to look at the densities only every eight steps because of the finite rotation at each step, there is an alternative approach. We could define a new set of densities which rotated by $\pi/4$ with every step as the ensemble of walks actually does. This would allow us to avoid the 'stroboscope' approach above, and we could take the continuum limit as an approximation of a single step. In the next section we avoid the problem entirely by taking the continuum limit with a finite signal velocity. This in turn changes the finite rotation at each step into an infinitesimal rotation which admits a continuum limit directly.

4. Entwined pairs with fixed signal velocity

In the previous section the asymptotic scaling was diffusive. In the absence of a potential field, at each lattice scale (δ, ϵ), the probability that the envelope walk turns left or right is exactly $1/2$. This symmetry reflects the paradigm that the walk is actually random and symmetric on all scales below a given detector resolution, whatever the current scale. As detector resolution increases, more and more detail is revealed of the random walk which is statistically a self-similar fractal (of dimension two) on all scales. There is no inner characteristic scale in this picture — the mean free path of the particle is 0 in the continuum limit. Similarly the mean free time is also zero, and the diffusive scaling implies an infinite signal velocity in the continuum limit. All of these features are evident in the resulting phenomenologies (the diffusion equation and the Schrödinger equation) for symmetric random walks and entwined pairs alike.

In real physical diffusive systems, inner characteristic lengths, times, and velocities are all finite, and are determined by the density, composition, and temperature of the surrounding fluid. Three such measures are l, τ, and c. l is a mean free path and is roughly the average distance the diffusing particle moves before being scattered by particles from the surrounding medium. τ is roughly the expected time between scattering, c is the ratio of these two and is roughly the speed of sound in the system. For real physical systems then the scaling relation (3) is a convenient mathematical fiction which allows us to replace a sequence of difference equations with a limiting partial differential equation. The PDE itself is then only a useful description on scales where the scaling relation (3) is valid. For example, the diffusion equation is only useful on space scales greater than l and time scales greater than τ. Below these scales, a diffusing particle moves on a piecewise smooth path rather than a Fractal trajectory, and the differential equation is no longer a sensible description.

In quantum mechanics, there is a formal parallel to this in the transition from Schrödinger dynamics to the relativistic equations. The analog of the characteristic speed of sound is of course the speed of light. The analog of the mean free path is the Compton length. The analog of the mean free time is the Compton time. All of this is well-known, and easily seen, particularly in the path-integral formulation of quantum mechanics. However, the reason for mentioning it here is that, for example, the relation between λ_C, the Compton length, and l the mean free path is purely formal. l is physically a measurable feature of a real classical system and mathematically an ensemble average. In contrast, in quantum mechanics λ_C is a characteristic length at the level of the wavefunction equations only. It is a physical parameter which is not directly measurable as a distance between collisions, neither is it an ensemble average over any known microscopic dynamic. The analogy between the two sets of parameters, quantum and classical, is interesting but formal.

In this section we shall see that entwined pairs generate the Dirac and Schrödinger equations in such a way that the analogy between classical and 'quantum' characteristic lengths is no longer formal. λ_C in the context of entwined pairs is precisely a mean free path generated as an ensemble average. The other characteristic constants are either prescribed constants, ensemble averages, or, in the case of the 'diffusion constant' $\hbar/(2m)$ an assumption on how space and time scale, as is the case for the usual diffusion constant D.

Since Eqs. (1) and (2) are very similar and are coupled only by the initial conditions, we shall work with the first system only. We use the scaling

$$\begin{aligned} &\alpha = a\epsilon, \\ &\beta = (1-\alpha), \\ &\frac{\delta}{\epsilon} = c \end{aligned} \tag{15}$$

where a and c are both fixed parameters. a is the inverse of the mean free time of the system and c is the fixed mean free speed. Substitution of these into (1) gives

$$\begin{aligned} \phi_1'(z, t+\epsilon) &= (1-a\epsilon)\phi_1'(z+\delta, t) - a\epsilon\phi_2'(z-\delta, t), \\ \phi_2'(z, t+\epsilon) &= a\epsilon\phi_1'(z+\delta, t) + (1-a\epsilon)\phi_2'(z-\delta, t). \end{aligned} \tag{16}$$

Notice that as $\epsilon \to 0$, the likelihood of a direction change goes down as ϵ. Small steps are correlated in the same direction, giving a finite mean free path.

Using the expansions of the shift operators (7) and (8) and truncating these to first order give:

$$\begin{aligned} \left(1+\epsilon\frac{\partial}{\partial t}\right)\phi_1' &= (1-a\epsilon)\left(1+\delta\frac{\partial}{\partial z}\right)\phi_1' - a\epsilon\left(1-\delta\frac{\partial}{\partial z}\right)\phi_2' + O(\delta^2), \\ \left(1+\delta\frac{\partial}{\partial t}\right)\phi_2' &= a\epsilon\left(1-\delta\frac{\partial}{\partial z}\right)\phi_1' + (1-a\epsilon)\left(1+\delta\frac{\partial}{\partial z}\right)\phi_2' + O(\delta^2). \end{aligned} \tag{17}$$

Matching first order terms and using $\delta = \epsilon c$ gives:

$$\begin{aligned}\frac{\partial \phi_1'}{\partial t} &= c\frac{\partial \phi_1'}{\partial z} - a\phi_2',\\ \frac{\partial \phi_2'}{\partial t} &= -c\frac{\partial \phi_2'}{\partial z} + a\phi_1'.\end{aligned} \tag{18}$$

Examining (18) we can see that the ϕ are real, but oscillatory. The oscillatory character arises through the two different signs in the scattering terms on the right of the equation. The alternating signs in these terms are a result of the entwining of the paths and the resulting 'colouring' of the envelopes (see Fig. 1). This is the origin of 'phase' in this system. Note that if we start with initial conditions such that the ϕ' are constant in space, the spatial derivatives are zero and the system reduces to

$$\begin{aligned}\frac{\partial \phi_1'}{\partial t} &= -a\phi_2',\\ \frac{\partial \phi_2'}{\partial t} &= a\phi_1'.\end{aligned} \tag{19}$$

A suggestive solution of (19) is

$$\begin{aligned}\phi_1' &= A\cos(at),\\ \phi_2' &= A\sin(at).\end{aligned} \tag{20}$$

The trigonometric functions signal the implicit presence of phase in the system. Notice that were we only counting paths without the return path present, the sign of the scattering terms in (18) would both be positive and the solutions of (19) would be hyperbolic, not trigonometric.

With the appropriate numerical constant for a (i.e., $a = mc^2/\hbar$), Eq. (18) is a form of the Dirac equation in one dimension which admits real solutions. Note in passing that zitterbewegung in this model is built in by entwined paths. The eigenvalues of the 'velocity operator' in Eq. (18) is $\pm c$. This is simply because entwined paths move on the 'light cones' of the lattice. Since in this context the constant c is the speed of the particle on the lattice, which has been fixed in the continuum limit, we should be able to recover the 'non-relativistic' limit by letting $c \to \infty$. To do this we shall re-write Eq. (18) in a more convenient form. First we change variables to:

$$\begin{aligned}\psi_1 &= \mathrm{i}\phi_1'\mathrm{e}^{-\mathrm{i}at},\\ \psi_2 &= \phi_2'\mathrm{e}^{-\mathrm{i}at}.\end{aligned} \tag{21}$$

The i's in (21) are not FAC's. They simply give a convenient linear combination of the two densities ϕ'. Eq. (18) then becomes

$$\frac{\partial \psi_+}{\partial t} = c\frac{\partial \psi_-}{\partial z}, \tag{22}$$

$$\frac{\partial \psi_-}{\partial t} = c\frac{\partial \psi_+}{\partial z} + 2\mathrm{i}a\psi_-. \tag{23}$$

Now eliminate one of the ψ. For example we may eliminate ψ_+ by differentiating (22) with respect to z and (23) with respect to t and combine to get

$$\frac{1}{c^2}\frac{\partial^2\psi_-}{\partial t^2} = \left(\frac{2ia}{c^2}\right)\frac{\partial\psi_-}{\partial t} + \frac{\partial^2\psi_-}{\partial z^2}, \tag{24}$$

ψ_+ may be shown to satisfy the same equation.

In this form, contact with the Schrödinger equation is easily made. The choice of a which identifies (18) with the usual Dirac equation is $a = mc^2/\hbar$. So a depends on c. If we substitute this into (24) we get

$$i\frac{\partial\psi_-}{\partial t} = -\left(\frac{\hbar}{2m}\right)\left(\frac{\partial^2\psi_-}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2\psi_-}{\partial t^2}\right), \tag{25}$$

where now the only term that depends on c is the last one. Here, as expected, $\lim_{c\to\infty}$ gives the free particle Schrödinger equation. Furthermore, if we take m to the left-hand side of (25) we see that in the limit as $m \to 0$, ψ_- obeys the wave equation. This makes sense from the original model in that when $m = 0$, the particles never scatter, so they stay on the same light-cone and hence they obey the wave equation. By writing the solutions of (25) in the form $\psi_- = \exp(imc^2t)\chi$ it is seen that (25) is equivalent to the Klein–Gordon equation for a particle of mass m.

Note that the above route to the Schrödinger equation did not include a potential; we arrived only at the free particle equation. To include a potential in this case would require us to put field interactions into the entwined model that are not real. This may be done, but would be difficult to interpret in terms of the original stochastic model. Instead, in a subsequent paper we shall show how to include a field, itself generated by a classical stochastic process, which will allow us to avoid this second FAC.

The two approaches above represent two different methods of taking a continuum limit. In the first approach we took a single continuum limit in which, as the lattice spacing went to zero, the speed of the particle on the lattice increased without bound. This can be seen from the diffusive scaling in (3) which can also be written as

$$(\delta)/(\epsilon) \to 2D/\delta \quad \text{as } \epsilon, \delta \to 0, \tag{26}$$

where δ/ϵ is just the hopping speed of the particle on the lattice. This is the appropriate scaling for symmetric random walks and reflects their intrinsic fractal dimension. In the context of Schrödinger's equation it is this scaling that supports the uncertainty principle. Note that if we call the lattice speed as a function of the lattice scale $v(\delta)$, (26) may be suggestively written as

$$v(\delta) \times \delta \approx 2D. \tag{27}$$

The apparent speed of a particle increases as the scale of measurement goes down in such a way that the product of the length scale and speed is asymptotically constant. This in turn would put a lower bound on the product of the uncertainties in the spatial resolution and speed which would depend only on D.

In the second route to Schrödinger's equation we acknowledged that there should be a finite signal velocity c. This meant that the above scaling could only hold down to some characteristic length which in our model was $1/a$. Below that scale, particle speeds had to be constant and as a result the continuum limit was taken with the scaling (15). This scaling yields trajectories in the continuum which look like the paths in Fig. 1, except there is no lattice, and the lengths of the line segments between corners is governed by a Poisson process, with an expected length of $1/a$. When the continuum limit is taken in this fashion, the ultimate particle speed c remains a parameter in the resulting PDE. By expressing the PDE in a convenient form we then let $c \to \infty$ to recover 'the non-relativistic' form of the equation. The conceptual link with the first continuum limit is that our broken line paths look like ordinary symmetric random walks if the scale of measurement in space and time has a natural 'speed' much less than the ultimate speed c. By sending $c \to \infty$ you ensure that this is always the case, and the resulting PDE is the same for both cases.

5. Discussion

In the above, we showed that in 1 + 1 dimensions, the Schrödinger equation arises as the continuum limit of a classical stochastic model by taking the limit of a discrete system in two separate ways: (a) by assuming that the stochastic behaviour of entwined paths had no inner scale, and (b) by first taking the continuum limit at fixed signal velocity, which is then sent to infinity. The first method allows the insertion of a potential. The second shows how relativistic corrections can be brought in.

In qualitative terms, the above calculations replaced the Brownian motion underlying the diffusion equation by the Brownian motion of entwined paths. This replacement has then changed the macroscopic phenomenology from the diffusion equation to the Schrödinger equation. The 'anti-particle' current of the entwined paths provides the interference effects and reversibility characteristic of Schrödinger's equation.

The idea that Schrödinger 's equation somehow involves time-reversed 'fluids' has been the source of many alternative approaches to quantum mechanics [12,13,18–20]. Our work can be understood as a modification of Brownian/Poisson motion that provides a microscopic basis (and resulting statistical mechanics) for a time-symmetric 'diffusion.' It thus provides a new context for the Schrödinger equation as a legitimate phenomenological equation for entwined paths. This is in marked contrast to quantum mechanics where the Schrödinger equation is the fundamental equation of the theory.

In terms of conventional single-particle quantum mechanics in one dimension, the above model imitates it exactly, up to the measurement postulates, in a realistic context. In terms of the Penrose [21] partition of quantum mechanics into 'U' (unitary)-processes and 'R' (reduction)-processes, entwined pairs provide a microscopic model for the U-process.

It is tempting to classify the model as a 'hidden variables model.' If we graft the measurement postulates onto the 'wavefunctions' generated by entwined paths, then we would indeed have a type of (non-local) hidden variables theory. It would differ from earlier reversible diffusion models such as stochastic quantum mechanics primarily in the fact that it is constructive. Our time-reversed paths tell us exactly how to construct a reversible diffusion. However, we do not advocate postulating responses to measurement at this point. Just as we do not need measurement postulates for diffusive systems, we should not need such postulates for entwined paths. Instead, the task ahead is to see if any reasonable measurement schemes verify or contradict the postulates of quantum mechanics within the new context of entwined paths. By such tests we stand to gain insight into how wave-particle duality might, or might not, be produced in Nature.

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