# RYERSON UNIVERSITY
# MTH 714 LAB#8 - SOLUTIONS

1. We will show that the formula is not valid by exhibiting an interpretation $I$ in which the formula is false. Since the formula is in the form of an implication, we are looking for an interpretation $I$, which is a set with a binary relation on it, such that

$$I \models \forall x \forall y \forall z[p(x,x) \wedge (p(x,z) \rightarrow (p(x,y) \vee p(y,z)))]$$
$$I \models \neg \exists y \forall z p(y,z)$$

The latter is equivalent to

$$I \models \forall y \exists z \neg p(y,z)$$

Consider the structure

$$I = (\mathbb{N}, \{\leq\})$$

Then,

$$I \models \forall x \forall y \forall z[x \leq x \wedge (x \leq z \rightarrow (x \leq y \vee y \leq z))]$$
$$I \models \neg \exists y \forall z (y \leq z)$$

The former is true, since it always the case that $x \leq x$, and if $x \leq z$ and $x \not\leq y$, then $y < x \leq z$ which yields $y \leq z$.

The latter formula is also true, since $\mathbb{N}$ has the minimum element $y = 1$.

2. It suffices to show that the negation of the formula

$$\neg[\forall x(p(x) \vee q(x)) \rightarrow (\forall x p(x) \vee \forall x q(x))]$$

is satisfiable. In fact, we will also find the smallest model for this negation, i.e. the smallest structure in which the original formula fails to be true.

- From $\neg[\forall x(p(x) \vee q(x)) \rightarrow (\forall x p(x) \vee \forall x q(x))]$, we get the descendent

$$\forall x(p(x) \vee q(x)),\ \neg(\forall x p(x) \vee \forall x q(x)).$$

- As the next level of the tableau, we get:

$$\forall x(p(x) \vee q(x)),\ \exists x \neg p(x),\ \exists x \neg q(x)$$

- After introducing two new constant symbols $a$ and $b$:

$$\forall x(p(x) \vee q(x)),\ \neg p(a),\ \neg q(b)$$

- We can now generate the following instances of the universal formula:

$$\forall x(p(x) \vee q(x)),\ p(a) \vee q(a),\ p(b) \vee q(b),\ \neg p(a),\ \neg q(b)$$

- After this, it can be shown that, after branching off different possibilities for $p(a) \vee q(a)$ and $p(b) \vee q(b)$, we will never be able to terminate all the branches and mark them as closed.

In fact, by examining what happens after the tree starts branching, one can show by following one of the branches that the smallest model for the negation has the universe $\{a, b\}$ and that the relations are defined as follows:

$$p(a), \quad \neg q(a), \quad \neg p(b), \quad q(b)$$

which will falsify the original formula.

3.

| | | |
|---|---|---|
| 1. | $\{\exists x(p(x) \to q(x)), \forall x p(x)\} \vdash \exists x(p(x) \to q(x))$ | Assumption |
| 2. | $\{\exists x(p(x) \to q(x)), \forall x p(x)\} \vdash p(a) \to q(a)$ | C-Rule 1 |
| 3. | $\{\exists x(p(x) \to q(x)), \forall x p(x)\} \vdash \forall x p(x)$ | Assumption |
| 4. | $\{\exists x(p(x) \to q(x)), \forall x p(x)\} \vdash \forall x p(x) \to p(a)$ | Axiom 4 |
| 5. | $\{\exists x(p(x) \to q(x)), \forall x p(x)\} \vdash p(a)$ | MP 3,4 |
| 6. | $\{\exists x(p(x) \to q(x)), \forall x p(x)\} \vdash q(a)$ | MP 5,2 |
| 7. | $\{\exists x(p(x) \to q(x)), \forall x p(x)\} \vdash \exists q(x)$ | Existential Rule 6 |
| 8. | $\{\exists x(p(x) \to q(x))\} \vdash \forall x p(x) \to \exists x q(x)$ | Deduction Rule 7 |
| 9. | $\vdash \exists x(p(x) \to q(x)) \to (\forall x p(x) \to \exists x q(x))$ | Deduction Rule 8 |

4. What is wrong with the following "proof" of

$$\{\exists x A(x), \exists x B(x)\} \vdash \exists x(A(x) \wedge B(x))?$$

| | | |
|---|---|---|
| 1. | $\{\exists x A(x), \exists x B(x)\} \vdash \exists x A(x)$ | Assumption |
| 2. | $\{\exists x A(x), \exists x B(x)\} \vdash A(c)$ | C-Rule 1 |
| 3. | $\{\exists x A(x), \exists x B(x)\} \vdash \exists x B(x)$ | Assumption |
| 4. | $\{\exists x A(x), \exists x B(x)\} \vdash B(c)$ | C-Rule 3 |
| 5. | $\{\exists x A(x), \exists x B(x)\} \vdash A(c) \wedge B(c)$ | Conjunction Rule 2,4 |
| 6. | $\{\exists x A(x), \exists x B(x)\} \vdash \exists x(A(x) \wedge B(x))$ | Existential Rule 5 |

**Solution:** The problem lies in the fact that we introduced the same constant symbol $c$ as the witness of both existential formulas $\exists x A(x)$ and $\exists x B(x)$.