RYERSON UNIVERSITY
MTH 714 LAB#7 - SOLUTIONS

1. (a) Model for A_1, A_2 but not A_3: consider a set $U = \{a, b, c\}$ on which we have a binary relation $p = \{(a,a), (b,b), (c,c), (a,b), (b,a), (b,c), (c,b)\}$ (for example, one can visualize the model as a graph with three vertices a,b, and c, which has a loop at each vertex, and there are edges joining a and b and b and c, but no edge between a and c). Then, the binary relation is clearly reflexive and symmetric, but not transitive since

$$(a,b), (b,c) \in p \text{ but } (a,c) \notin p$$

(b) Model for A_1 and A_3, but not A_2: take $\mathbb{N}$ with the relation $\leq$. Clearly, $x \leq x$ is always true, and

$$\forall x \forall y \forall z (x \leq y \wedge y \leq z \rightarrow x \leq z)$$

but, e.g

$$3 \leq 5 \text{ and } 5 \not\leq 3$$

which shows that A_2 fails.

(c) Model for A_2 and A_3 but not A_1: take $U = \{a, b, c\}$ with the binary relation $p = \{(a,b), (b,a), (b,c), (c,b), (a,c), (c,a)\}$. Again, we can visualize this structure as a graph with no loops which is a triangle with vertices a, b, and c. Then, the binary relation is symmetric and transitive but not reflexive.

2. (a) Yes. The formula translates as

$$F = \exists x \exists y \exists z (x < y \wedge z < y \wedge x < z \wedge z \not< x)$$

One example of such a triple of numbers is

$$x = 1, \quad y = 3, \quad z = 2$$

(b) No. If such a triple of positive integers existed, it would mean that

$$y = x + 1, \quad y = z + 1, \quad z = x + 1, \quad x \neq z + 1.$$

However, the first two equalities yield $x = z$, but this is impossible according to the third equality.

(c) Yes. The meaning of the formula in this structure is

$$F = \exists X \exists Y \exists Z (X \subseteq Y \wedge Z \subseteq Y \wedge X \subseteq Z \wedge Z \not\subseteq X)$$

and a triple of su8ch subsets of $\mathbb{N}$ is e.g.

$$X = \{1\}, \quad Y = \{1, 2\}, \quad Z = \{1, 2\}.$$

3. F Consider

$$F = \exists x \exists y \exists z[(p(x) \wedge q(x)) \wedge (\neg p(y) \wedge q(y)) \wedge \neg q(z)]$$

where p and q are two unary relation symbols. If

$$I \models F$$

we claim that $|I| \geq 3$. The reason for this is that, if x, y, and z are three elements that witness the truth of this formula, since

$$p(x) \text{ and } \neg p(y)$$

we must have $x \neq y$. Similarly, since

$$q(y) \text{ and } \neg q(z)$$

it must be the case that $y \neq z$. Also, we cannot have $x = z$, since $q(x)$ holds while $q(z)$ does not.

4. We will show that the formula is not valid by exhibiting an interpretation I in which the formula is false. Since the formula is in the form of an implication, we are looking for an interpretation I, which is a set with a binary relation on it, such that

$$I \models \forall x \forall y \forall z[p(x,x) \wedge (p(x,z) \rightarrow (p(x,y) \vee p(y,z)))]$$
$$I \models \neg \exists y \forall z p(y,z)$$

The latter is equivalent to

$$I \models \forall y \exists z \neg p(y,z)$$

Consider the structure

$$I = (\mathbb{N}, \{\leq\})$$

Then,

$$I \models \forall x \forall y \forall z[x \leq x \wedge (x \leq z \rightarrow (x \leq y \vee y \leq z))]$$
$$I \models \neg \exists y \forall z(y \leq z)$$

The former is true, since it always the case that $x \leq x$, and if $x \leq z$ and $x \not\leq y$, then $y < x \leq z$ which yields $y \leq z$.

The latter formula is also true, since $\mathbb{N}$ has the minimum element $y = 1$.