RYERSON UNIVERSITY
MTH 714 LAB#6 - SOLUTIONS

1. Every formula is equivalent to one in CNF, so it suffices to show that every CNF formula is equivalent to a *complete* CNF formula.

   Assume A is a formula in CNF. If A is not complete to start with, there is a disjunction which does not include all variables that appear elsewhere in A, e.g.

   $$p_1 \lor p_2 \lor \ldots \lor p_k$$

   and there is another variable q which is also in A. Now, we have the following chain of equivalences

   $$\begin{aligned} & p_1 \lor p_2 \lor \ldots \lor p_k \\ & \equiv p_1 \lor p_2 \lor \ldots \lor p_k \lor F \\ & \equiv p_1 \lor p_2 \lor \ldots \lor p_k \lor (q \land \neg q) \\ & \equiv (p_1 \lor p_2 \lor \ldots \lor p_k \lor q) \land (p_1 \lor p_2 \lor \ldots \lor p_k \lor \neg q) \text{ (distributivity)} \end{aligned}$$

   Therefore, we can introduce a variable into an already existing disjunction term at the cost of "doubling" it, since we also get another term containing the negation of the variable. So, the process of converting a CNF into a complete CNF will double the number of terms in the formula and, in general, the number of terms increases exponentially after this procedure is applied.

2. (a) $\{p\overline{q}, q\overline{r}, rs, p\overline{s}\} \approx \{q\overline{r}, rs\} \approx \{q\overline{r}\}$
   (b) $\{pqr, \overline{q}, p\overline{r}s, qs, p\overline{s}\} \approx \{pr, p\overline{r}s, s, p\overline{s}\} \approx \{pr, p\} \approx \{p\}$
   (c) $\{pqrs, \overline{q}rs, \overline{p}rs, qs, \overline{p}s\} \approx \{\overline{q}rs, \overline{p}rs, qs, \overline{p}s\}$
   (d) $\{\overline{p}q, qrs, \overline{pq}rs, \overline{r}, q\} \approx \{\overline{p}rs, \overline{r}\} \approx \{\overline{r}\}$

3. If the starting clauses are enumerated as (1)-(4), a refutation (derivation of the empty clause) by eliminating the literals in the order $\{p, q, r\}$ is as follows:

   (5) $\overline{q}r$ (resolve clauses 1 and 2)
   (6) r (resolve clauses 3 and 5)
   (7) $\square$ (resolve clauses 4 and 6)

   If we eliminate literals in the reverse order $\{r, q, p\}$, we have a different refutation of the given set of clauses:

   (5) $\overline{pq}$ (resolve clauses 1 and 4)

(6) p (resolve clauses 2 and 4)

(7) $\overline{q}$ (resolve clauses 5 and 6)

(8) q (resolve clauses 3 and 4)

(9) $\square$ (resolve clauses 7 and 8)

4. The corresponding clausal form is

$$\{p, \overline{p}qr, \overline{pqr}, \overline{p}st, \overline{ps}\overline{t}, \overline{s}q, rt, \overline{t}s\}$$

Assume these clauses are enumerated (1)-(8).

Then, we have the following refutation of the set of clauses:

(9) $\overline{ps}r$ (from clauses 5,7)

(10) $\overline{p}s$ (from clauses 4,8)

(11) $\overline{pqs}$ (from clauses 3,9)

(12) $\overline{ps}$ (from clauses 11,6)

(13) $\overline{p}$ (from clauses 12,10)

(14) $\square$ (from clauses 13,1)

5. Suppose a set of clauses

$$S = \{C_1, C_2, \ldots, C_k\}$$

contains no positive clauses. This means that every clause C_i contains at least one negative literal:

$$C_i = \ldots \vee \neg p_i \vee \ldots$$

For every C_i choose this p_i which appears negated in it. We construct the truth evaluation which makes S satisfiable in the following way:

$$v(p_1) = v(p_2) = \ldots = v(p_k) = F$$

Then, $v(C_i) = T$, for every $i = 1, \ldots, k$, and S is satisfiable.

6. Suppose C_1 and C_2 are two clashing Horn clauses. First, notice the following: every Horn clause has one of the following two forms

$$\{\neg p_1, \neg p_2, \ldots, \neg p_k\}$$

or

$$\{\neg p_1, \neg p_2, \ldots, \neg p_k, r\}$$

since it can contain at most one positive literal.

If C_1 and C_2 clash, they cannot both have the first form, since two clauses clash over a pair consisting of a variable and its negation.

If one clause has a positive literal and the other one does not, we have e.g.

$$C_1 = \{\neg p_1, \neg p_2, \dots, \neg p_k\}$$
$$C_2 = \{\neg q_1, \neg q_2, \dots, \neg q_m, r\}$$

Since C_1 and C_2 clash, r must be the same as one of the variables $p_1, p_2, \dots, p_k$, say p_1. Then, resolving C_1 and C_2 gives us

$$Res(C_1, C_2) = \{\neg p_2, \dots, \neg p_k, \neg q_1, \neg q_2, \dots, \neg q_m\}$$

which is also a Horn clause.

If both clauses contain one positive literal, e.g.

$$C_1 = \{\neg p_1, \neg p_2, \dots, \neg p_k, r\}$$
$$C_2 = \{\neg q_1, \neg q_2, \dots, \neg q_m, s\}$$

Again, one positive literal from one clause has to clash with a negative literal from the other clause, e.g. r and q_1, so we get:

$$Res(C_1, C_2) = \{\neg p_1, \neg p_2, \dots, \neg p_k, \neg q_2, \dots, \neg q_m, s\}$$

Again, we have a Horn clause as the resolvent.