RYERSON UNIVERSITY
MTH 714
LAB #4 - SOLUTIONS

1.

1.	$A, B, \neg A$	axiom
2.	$\neg B, B, \neg A$	axiom
3.	$\neg(A \to B), B, \neg A$	β 1,2
4.	$\neg(A \to), \neg\neg B, \neg A$	α 3
5.	$\neg(A \to B), (\neg B \to \neg A)$	α 4
6.	$(A \to B) \to (\neg B \to \neg A)$	α 5

1.	$A, \neg A, B$	axiom
2.	$A, \neg B, B$	axiom
3.	$A, \neg(\neg A \to B), B$	β 1,2
4.	$\neg B, \neg A, B$	axiom
5.	$\neg B, \neg B, B$	axiom
6.	$\neg B, \neg(\neg A \to B), B$	β 4,5
7.	$\neg(A \to B), \neg(\neg A \to B), B$	β 3,6
8.	$\neg(A \to B), (\neg A \to B) \to B$	α 7
9.	$(A \to B) \to ((\neg A \to B) \to B)$	α 8

1.	$\neg A, B, A$	axiom
2.	$A \to B, A$	α 1
3.	$\neg A, A$	axiom
4.	$\neg((A \to B) \to A), A$	β 2,3
5.	$((A \to B) \to A) \to A$	α 4

2.

1.	$\vdash (A \to B) \to (\neg B \to \neg A)$	Theorem 3.24
2.	$\vdash A \to B$	assumption
3.	$\vdash \neg B \to \neg A$	MP 2,1
4.	$\vdash \neg B$	assumption
5.	$\vdash \neg A$	MP 4,3

3. This is Theorem 3.22

4. This is Theorem 3.32

5. (a) This is a direct consequence of the Compactness Theorem which states the following: a countable set of formulas S is satisfiable if and only if every finite subset of S is satisfiable.

(b) Suppose that, for *some* formula A, neither $S \cup \{A\}$ nor $S \cup \{\neg A\}$ are consistent sets of formulas. By a theorem from class (Theorem 3.41), we must have both

$$S \vdash \neg A$$

and

$$S \vdash \neg\neg A \quad \text{(and also } S \vdash A\text{)}$$

However, by definition, this means that S is inconsistent, which contradicts the assumption about the set S.

Therefore, for *every* formula A, at least one of the larger sets of formulas $S \cup \{A\}$ or $S \cup \{\neg A\}$ must be consistent.

(c) We can assume that there is an algorithm (procedure) which systematically lists all well-formed propositional formulas, one at a time. Let the output of that procedure be the infinite list

$$A_1, A_2, \ldots, A_n, \ldots$$

In other words, all formulas of propositional logic can be systematically generated and indexed by positive integers. Starting with the set S, we will construct a larger set U in the following way:

Step 1. Set $i := 0$ and let $U_0 := S$

Step 2. Check if the formula A_i from the enumeration is already in U_i; if yes, go to Step 3. If not, determine if $U_i \cup \{A_i\}$ is consistent: if yes, set $U_{i+1} := U_i \cup \{A_i\}$; otherwise, set $U_{i+1} := U_i \cup \{\neg A_i\}$.

Step 3. Set $i := i + 1$. Go to Step 2.

Now, by part (b), after each pass through Step 2, U_i remains a consistent set of formulas. On the other hand, if we let this algorithm run forever, U will contain, for every propositional formula A, either A or its negation. Therefore, $U = \bigcup_{i=0}^{\infty} U_i$ will be a maximally consistent set, since it is consistent and it cannot be extended in any way, since every formula or its negation is already in U.

[Remark: This is a very common construction in logic in order to construct maximal sets with a given property; we construct them inductively, by checking at each step whether a formula from the list of all possible formulas can be added to it or not.]