RYERSON UNIVERSITY
MTH 714
LAB #3 - SOLUTIONS

1. Suppose A is a formula containing the variable p and the connectives $\wedge$ and $\vee$ only. We will show that, for any truth assignment v such that $v(p) = T$, $v(A) = T$ as well.

   We can prove this using structural induction. For the base case, assume $A = p$. Obviously, if $v(p) = T$, we must have $v(A) = T$. Now, suppose $B = A_1 \wedge A_2$, and that the inductive hypothesis applies to A_1 and A_2. In other words, if $v(p) = T$, we must have $v(A_1) = v(A_2) = T$. So, assume $v(p) = T$ and let's look at $v(B)$:

   $$v(B) = v(A_1) \wedge v(A_2) = T \wedge T = T.$$

   Similarly, we can show that if $B = A_1 \vee A_2$, if $v(p) = T$, we have $v(B) = T$.

   Now, it should be clear that $\neg p$ is not equivalent to a formula using $\wedge$ and $\vee$ only, since $\neg p$ does not have the property proved in the preceding paragraph.

2. Take $U = \{p\}$ and let $B = \neg p$. Obviously, U is satisfiable (any assignment in which $v(p) = T$ will do), while $U \cup \{B\} = \{p, \neg p\}$ cannot be satisfiable.

3. Theorem 2.38: ($\Rightarrow$) Suppose

   $$\{A_1, \ldots, A_n\} \models A$$

   Now, let v be any assignment. If at least one of the formulas $A_1, \ldots, A_n$ is false in this assignment, we have

   $$v(A_1 \wedge A_2 \wedge \ldots \wedge A_n \to A) = F \to v(A) = T$$

   On the other hand, if v is an assignment in which all formulas $A_1, \ldots, A_n$ are true, the definition of logical consequence yields that $v(A) = T$ as well. So,

   $$v(A_1 \wedge A_2 \wedge \ldots \wedge A_n \to A) = T \wedge T \wedge \ldots \wedge T \to T = T.$$

   In either case, the implication is true, so

   $$\models A_1 \wedge A_2 \wedge \ldots \wedge A_n \to A$$

   ($\Leftarrow$) If $A_1 \wedge A_2 \wedge \ldots \wedge A_n \to A$ is a valid formula, then, whenever all formulas $A_1, \ldots, A_n$ are true, the conclusion A is also true, which means that

   $$\{A_1, \ldots, A_n\} \models A.$$

Theorem 2.39: adding an additional formula to the set U can only reduce the number of interpretations in which U is true, which can only reduce the number of interpretations in which A can be true.

Theorem 2.40: Since any interpretation satisfies a valid formula, the interpretations in which U is true are the same interpretations in which $U - \{B\}$ is true.

4. (a) Satisfiable. One interpretation in which the set of formulas is true is e.g.

$$v(p) = T, \quad v(q) = T, \quad v(r) = F$$

(b) Satisfiable. One interpretation which witnesses that is e.g.

$$v(p) = F, \quad v(q) = T, \quad v(r) = T.$$

5. (a) Not valid. The formula is false e.g. for $v(p) = T$, $v(q) = F$.
   (b) Valid.
   (c) Valid.
   (d) Not valid. The formula is false e.g. for $v(p) = T$, $v(q) = F$, $v(r) = F$.