# RYERSON UNIVERSITY
# MTH 714
# LAB #2 - SOLUTIONS

1. Determine whether the following formulas are valid (tautologies) or not:
   (a) Not valid: $v(p) = T$, $v(q) = F$ makes the formula false.
   (b) Valid.
   (c) Valid.
   (d) Not valid: $v(q) = F$ and any combination of opposite values for $p$ and $r$.

2. Determine whether the following pairs of formulas are equivalent:
   (a) Equivalent.
   (b) Equivalent.
   (c) Equivalent.
   (d) Equivalent.
   (e) Equivalent.

3. (a) We will use structural induction on $A$.

   BASE CASE: If $A$ is an atom $p$, there is nothing to prove since the formula has no dual.

   INDUCTIVE HYPOTHESIS Suppose $B$ and $C$ are formulas for which the Duality Theorem holds. Notice the following: if the inductive hypothesis holds for some formula $B$, then it also holds for its dual $B'$ since $B$ and $B'$ have the same number of connectives. Also, for every formula $B$, $(B')' = B$.

   If $A = \neg B$, then $A' = \neg B'$. By inductive hypothesis, $\neg B'$ is valid if and only if $B$ is valid. By the remark above, the hypothesis also holds for $B'$.

$$\begin{aligned} \neg A' = \neg(\neg B') \text{ is valid} &\Leftrightarrow B' \text{ is valid} \\ &\Leftrightarrow \neg(B')' \text{ is valid (ind.hyp.)} \\ &\Leftrightarrow \neg B = A \text{ is valid} \end{aligned}$$

   If $A = B \wedge C$, then $A' = B' \vee C'$. We are assuming that $\neg B'$ is valid if and only if $B$ is valid and the same for $C$. Then,

$$\begin{aligned} A = B \wedge C \text{ is valid} &\Leftrightarrow B, C \text{ are valid} \\ &\Leftrightarrow \neg B', \neg C' \text{ are valid} \\ &\Leftrightarrow \neg B' \wedge \neg C' \text{ is valid} \\ &\Leftrightarrow \neg(B' \vee C') \text{ is valid} \\ &\Leftrightarrow \neg A' \text{ is valid} \end{aligned}$$

If $A = B \vee C$ with the same assumptions on $B$ and $C$ as above, then $A' = B' \wedge C'$, and

$$\begin{aligned} \neg A' = \neg(B' \wedge C') \text{ is valid} \quad &\Leftrightarrow \neg B' \vee \neg C' \text{ is valid} \\ &\Leftrightarrow B \vee C \text{ is valid (ind.hyp.)} \\ &\Leftrightarrow A \text{ is valid} \end{aligned}$$

The inductive proof is now complete.

(b) Suppose $A \to B$ is valid. This also means that $\neg A \vee B$ is valid. By part (a), $\neg(\neg A \vee B)' = \neg(\neg A' \wedge B') \equiv A' \vee \neg B'$ will be valid, too. But

$$A' \vee \neg B' \equiv B' \to A'$$

and this is precisely what is claimed in (b).

4. (a) We prove by induction on the structure of $A$: if $A$ is a formula containing $\to$ and $\vee$ as its only connectives, then $v(A) = T$ for every assignment $v$ which assigns $T$ to every atom.

   If $A = p$, the statement is trivial. So, suppose that the hypothesis applies to two formulas $B$ and $C$.

   If $A = B \vee C$, then $v(B) = v(C) = T$ for any assignment that makes all atoms true, so $v(B \vee C) = T$. Also, if $A = B \to C$, $v(A) = T \to T = T$.

   Therefore, if $A$ is a formula that uses $\to$ and $\vee$ as its only connectives, $v(A) = T$ whenever $v$ is an assignment making all atoms in $A$ true.

   (b) The set of connectives $\{\to, \vee\}$ cannot generate all Boolean operators, for the following reason: the negation operator $\neg$ cannot be expressed using $\to$ and $\vee$ only. If there was a formula $A(p)$ which involves $p$ and the two connectives only, we cannot have

$$\neg p \equiv A(p)$$

   since when $v(p) = T$, $v(A) = T$, according to (a).

5.

$$\begin{aligned} p \wedge q \quad &\equiv (p \to q) \leftrightarrow p \\ p \to q \quad &\equiv p \leftrightarrow p \wedge q \\ p \leftrightarrow q \quad &\equiv (p \to q) \wedge (q \to p) \end{aligned}$$

6. Suppose $\{\circ\}$ is adequate. Let's consider the possible values for $p \circ p$: since $\neg$ can be expressed as a formula using $\circ$ as its only operator, the table for $p \circ p$ must be:

| $p$ | $p \circ p$ |
|---|---|
| T | F |
| F | T |

Indeed, if e.g. T$\circ$ T = T, it would be impossible to express $\neg$ using $\circ$, since $\neg$ T = F. Similarly, we must have F $\circ$ F = T.

Now, the table for $p \circ q$ must have the following form:

| $p$ | $q$ | $p \circ q$ |
|---|---|---|
| T | T | F |
| T | F | |
| F | T | |
| F | F | T |

Now, the remaining two entries in the table must be either:

(i) T,T; or

(ii) T,F; or

(iii) F,T; or

(iv) F,F

The cases (ii) and (iii) are impossible; if (ii) was true, $p \circ q \equiv p$, so $\circ$ is not binary. Similarly, if (iii) was true, then $p \circ q \equiv \neg p$ and $\circ$ wouldn't be binary either. So, the remaining two fields in the truth table are either both true or both false. If they are both true, we have

$$\circ = \uparrow,$$

and if they are both false, we have

$$\circ = \downarrow .$$