RYERSON UNIVERSITY
MTH 714 LAB#11 - SOLUTIONS

1. (a) Unsatisfiable; after unification, the first clause becomes

$$\neg p(f(x), f(x)) \vee \neg p(f(x), f(x)) \quad (\equiv \neg p(f(x), f(x)))$$

which resolves with the second clause to give us $\Box$.

(b) Unsatisfiable; the derivation of the empty clause is

(1) $\neg p(f(f(x)), f(y)) \vee \neg p(f(y), f(f(x)))$ (first clause after unification; before this we had to rename the variables in the first clause as e.g. x', y')

(2) $\neg p(f(y), f(f(x)))$ (resolution - second clause and 1)

(3) $\neg p(f(z), f(f(x))$ (rename variables in 2)

(4) $\Box$ (resolve second clause and line 3, using unifier $\{z \leftarrow f(x), y \leftarrow f(x)\}$)

(c) Unsatisfiable; a derivation of the empty clause is:

(a) $\neg p(f(x), y) \vee \neg p(y, f(f(f(x))))$ (resolve the first and the second clause by substituting $f(x)$ for x and $f(f(f(x)))$ for z in the first clause)

(b) $\neg p(f(f((x)), f(f(f(x))))$ (resolve the second clause with the previous line, by substituting $f(f(x))$ for y)

(c) $\Box$ (resolve the third clause in the set with the previous line using substitution $f(x)$ for x in the third clause from the original set)

2. We will give the solution for the validity of:

$$\forall x(A(x) \to B(x)) \to (\exists x A(x) \to \exists x B(x))$$

(i) First, consider the negation of the formula:

$$\neg[\forall x(A(x) \to B(x)) \to (\exists y A(y) \to \exists z B(z))]$$

(ii) We convert the formula from (i) into a PCNF:

$$\begin{aligned}
\neg[\forall x(A(x) \to B(x)) \to (\exists y A(y) \to \exists z B(z))] &\equiv \forall x(A(x) \to B(x)) \wedge \neg(\exists y A(y) \to \exists z B(z)) \\
&\equiv \forall x(\neg A(x) \vee B(x)) \wedge \neg(\neg \exists y A(y) \vee \exists z B(z)) \\
&\equiv \forall x(\neg A(x) \vee B(x)) \wedge (\exists y A(y) \wedge \forall z \neg B(z)) \\
&\equiv \forall x \exists y \forall z[(\neg A(x) \vee B(x)) \wedge A(y) \wedge \neg B(z)] \\
&\approx \forall x[(\neg A(x) \vee B(x)) \wedge A(f(x)) \wedge \neg B(z)]
\end{aligned}$$

(iii) Finally, we try to refute the set of clauses

$$\{\neg A(x) \vee B(x), A(f(x)), \neg B(z)\}$$

First, unify $\neg A(x) \vee B(x)$ and $A(f(y))$ using

$$\{x \leftarrow f(y)\}$$

to get as the resolvent

$$B(f(y))$$

Next, we unify $B(f(y))$ and $B(z)$ using the unifier

$$\{z \leftarrow f(y)\}$$

and apply resolution to get the empty clause $\Box$.

So,

$$\neg[\forall x(A(x) \rightarrow B(x)) \rightarrow (\exists x A(x) \rightarrow \exists x B(x))]$$

is unsatisfiable, which means that its complement is a valid formula of predicate logic.

3. We use resolution to determine whether

$$C = \forall x \exists y \forall z[p(f(x), y) \vee p(y, f(z))]$$

is a logical consequence of the formulas

$$\begin{aligned} A &= \forall x \exists y[p(x, f(y)) \rightarrow p(y, f(x))] \\ B &= \exists x \forall y \exists z[\neg p(x, f(y)) \rightarrow \neg p(y, f(z))] \end{aligned}$$

Note that C is a logical consequence of A and B if and only if the formula $A \wedge B \rightarrow C$ is valid. This, in turn, is equivalent to checking whether $\neg(A \wedge B \rightarrow C) \equiv A \wedge B \wedge \neg C$ is unsatisfiable.

We skolemize A, B, and $\neg C$ to get the following set of universal formulas:

$$\begin{aligned} A' &= \forall x[\neg p(x, f(g(x)) \vee p(g(x), f(x))] \\ B' &= \forall y[p(a, f(y)) \vee \neg p(y, f(h(y)))] \\ C' &= \forall y[\neg p(f(b), y) \wedge \neg p(y, f(i(y)))] \end{aligned}$$

where f, g, h, i are new unary function symbols, and a, b are new constant symbols.

So, we need to check if the following set of clauses is unsatisfiable:

$$S = \{\neg p(x, f(g(x)) \vee p(g(x), f(x)), p(a, f(y)) \vee \neg p(y, f(h(y))), \neg p(f(b), y), \neg p(y, f(i(y)))\}$$

Each clause in S contains a negative literal. In general, if both premises of a resolution rule contain a negative literal, so does the conclusion. Thus, we can only derive clauses with negative literals from S (by resolution), but not the empty clause (a contradiction). Therefore, S is satisfiable and C cannot be a logical consequence of A and B.