RYERSON UNIVERSITY
MTH 714 LAB#10 - SOLUTIONS

1.

$$
\begin{aligned}
&\forall x(p(x) \to \exists y q(y)) \\
&\equiv \forall x(\neg p(x) \vee \exists y q(y)) \\
&\equiv \forall x \exists y(\neg p(x) \vee q(y)) \\
&\approx \forall x(\neg p(x) \vee q(f(x)))
\end{aligned}
$$

$$
\begin{aligned}
&\forall x \forall y(\exists z p(z) \wedge \exists u(q(x,u) \to \exists v q(y,v))) \\
&\equiv \forall x \forall y(\exists z p(z) \wedge \exists u(\neg q(x,u) \vee \exists v q(y,v))) \\
&\equiv \forall x \forall y \exists z \exists u \exists v(p(z) \wedge (\neg q(x,u) \vee q(y,v))) \\
&\approx \forall x \forall y(p(f(x,y) \wedge (\neg q(x, g(x,y)) \vee q(y, h(x,y))))
\end{aligned}
$$

$$
\begin{aligned}
&\exists x(\neg \exists y p(y) \to \exists z(q(z) \to r(x))) \\
&\equiv \exists x(\neg\neg \exists y p(y) \vee \exists z(\neg q(z) \vee r(x))) \\
&\equiv \exists x(\exists y p(y) \vee \exists z(\neg q(z) \vee r(x))) \\
&\equiv \exists x \exists y \exists z(p(y) \vee (\neg q(z) \vee r(x))) \\
&\approx p(b) \vee \neg q(c) \vee r(a)
\end{aligned}
$$

2. ($\Leftarrow$) Clearly, if $\forall x_1 \ldots \forall x_n A(x_1, \ldots, x_n)$ is satisfiable in a model with only one element it is satisfiable in general.

   ($\Rightarrow$) Now, suppose $\forall x_1 \ldots \forall x_n A(x_1, \ldots, x_n)$ is true in *some* interpretation

$$I = (D, \{r_1, \ldots, r_k\}, \{a_1, \ldots, a_m\})$$

   where r_i are relations on D interpreting predicate symbols and a_j are elements which interpret constant symbols in the formula. We want to show that the formula is true in some interpretation whose domain has only one element.

   Fix one element of D and denote it b. Since $\forall x_1 \ldots \forall x_n A(x_1, \ldots, x_n)$ expresses the fact that some property holds for *all* elements $x_1, \ldots, x_n$ of D, in particular, it will be true when all $x_i = b$ and we interpret all constant symbols as that same element b. So, our formula will be true in the smaller interpretation I' whose universe is $\{b\}$.

3.

$$\begin{aligned} \sigma_1\sigma_2 &= \{y \leftarrow b, z \leftarrow f(g(a)), w \leftarrow c\} \\ \sigma_2\sigma_1 &= \{x \leftarrow g(a), z \leftarrow f(x), w \leftarrow c\} \end{aligned}$$

4.

$$\begin{aligned} E\theta &= p(f(g(y)), f(u), g(u), f(y)) \\ (E\theta)\sigma &= p(f(g(f(a))), f(y), g(y), f(f(a))) \\ \theta\sigma &= \{x \leftarrow f(g(f(a))), z \leftarrow f(f(a)), u \leftarrow y\} \\ E(\theta\sigma) &= p(f(g(f(a))), f(y), g(y), f(f(a))) \end{aligned}$$

5.

- $p(a, x, f(g(y)))$ and $p(y, f(z), f(z))$ can be unified by

$$\{y \leftarrow a, x \leftarrow f(g(y)), z \leftarrow g(y)\}$$

- $p(x, g(f(a)), f(x))$ and $p(f(a), y, y)$ cannot be unified since from the second and third equation we have $g(f(a)) = f(x)$ (both terms equal y).
- $p(x, g(f(a)), f(x))$ and $p(f(y), z, y)$ cannot be unified since after we substitute $f(y)$ for x in the system (based on the first equation), the third equation becomes $f(f(y)) = y$.
- $p(a, x, f(g(y)))$ and $p(z, h(z, u), f(u))$ can be unified using

$$\{x \leftarrow h(a, g(y)), z \leftarrow a, u \leftarrow g(y)\}$$