Chapter 6: Predicate Calculus: Deductive Systems

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Outline

6.1 Gentzen Proof System $\mathcal{G}$

- As in propositional logic, Gentzen proof system is based on the reversal of a semantic tableau for a formula.

Example

Prove that

$$\models (\forall x\, p(x) \lor \forall x\, q(x)) \rightarrow \forall x\, (p(x) \lor q(x))$$

Solution: We will start by constructing a tableau for the negation

$$\neg[(\forall x\, p(x) \lor \forall x\, q(x)) \rightarrow \forall x\, (p(x) \lor q(x))]$$

$$\neg[(\forall x\ p(x) \lor \forall x\ q(x)) \rightarrow \forall x\ (p(x) \lor q(x))]$$

$$\forall x\ p(x) \lor \forall x\ q(x), \neg\forall x(p(x) \lor q(x))$$

$$\forall x\ p(x), \neg\forall x(p(x) \lor q(x)) \qquad \forall x\ q(x), \neg\forall x(p(x) \lor q(x))$$

$$\forall x\ p(x), \neg(p(a) \lor q(a)) \qquad \forall x\ q(x), \neg(p(b) \lor q(b))$$

$$\forall x\ p(x), \neg p(a), \neg q(a) \qquad \forall x\ q(x), \neg p(b), \neg q(b)$$

$$\forall x\ p(x), p(a), \neg p(a), \neg q(a) \qquad \forall x\ q(x), q(b), \neg p(b), \neg q(b)$$

$$\times \qquad\qquad \times$$

Deductive System $\mathcal{G}$

- Axioms: any set of formulas U containing a complementary pair of literals
- Rules: α- and β-rules are the same as in propositional logic, plus

$$\frac{U \cup \{\gamma, \gamma(a)\}}{U \cup \{\gamma\}} \qquad \frac{U \cup \{\delta(a)\}}{U \cup \{\delta\}}$$

where

γ	$\gamma(a)$
$\exists x\ A(x)$	$A(a)$
$\neg\forall x\ A(x)$	$\neg A(a)$

δ	$\delta(a)$
$\forall x\ A(x)$	$A(a)$
$\neg\exists\ A(x)$	$\neg A(a)$

where a is an arbitrary constant.

Example

The proof in $\mathcal{G}$ for

$$(\forall x\ p(x) \vee \forall x\ q(x)) \rightarrow \forall x\ (p(x) \vee q(x))$$

is

1.	$\neg\forall x\ p(x), \neg p(a), p(a), q(a)$	Axiom
2.	$\neg\forall x\ q(x), \neg q(b), p(b), q(b)$	Axiom
3.	$\neg\forall x\ p(x), p(a), q(a)$	γ-rule 1
4.	$\neg\forall x\ q(x), p(b), q(b)$	γ-rule 2
5.	$\neg\forall x\ p(x), p(a) \vee q(a)$	α-rule 3
6.	$\neg\forall x\ q(x), p(b) \vee q(b)$	α-rule 4
7.	$\neg\forall x\ p(x), \forall x(p(x) \vee q(x))$	δ-rule 5
8.	$\neg\forall x\ q(x), \forall x(p(x) \vee q(x))$	δ-rule 6
9.	$\neg[\forall x\ p(x) \vee \forall x\ q(x)], \forall x(p(x) \vee q(x))$	β-rule 8
10.	$(\forall x\ p(x) \vee \forall x\ q(x)) \rightarrow \forall x\ (p(x) \vee q(x))$	α-rule 9

Theorem

(Soundness and Completeness) Let U be a set of formulas. There is a Gentzen proof for U if and only if there is a closed semantic tableau for U.

6.2 Hilbert Proof System $\mathcal{H}$

- **Connectives:** $\neg$, $\rightarrow$
- **Quantifier:** $\forall x$

Remark

Using $\forall x$ as the only quantifier in predicate formulas is not a genuine restriction, since

$$\exists x\ A(x) \equiv \neg \forall x\ \neg A(x)$$

so $\exists$ can be expressed using $\neg$ and $\forall$.

Deductive System $\mathcal{H}$

- Axioms: The three axioms for the Hilbert system in propositional logic, plus

Axiom 4. $\vdash \forall x\, A(x) \rightarrow A(a)$

Axiom 5. $\vdash \forall x(A \rightarrow B(x)) \rightarrow (A \rightarrow \forall x\, B(x))$, assuming x is not free in A.

- Rules of Inference: Modus Ponens, plus

(Generalization:) $$\frac{\vdash A(a)}{\vdash \forall x\, A(x)}$$

- There is a problem with the Generalization Rule if it is not being used judiciously; consider the following derivation in the set $\mathbb{N}$ with the unary predicate $\mathit{even}(x)$:

1.	$\mathit{even}(2) \vdash \mathit{even}(2)$	Assumption
2.	$\mathit{even}(2) \vdash \forall x\ \mathit{even}(x)$	Gen. Rule 1

- We derived a wrong conclusion that every natural number is even, starting from the true assumption that 2 is even. What went wrong?
- **Answer:** We should not be able to generalize based on a constant included in the assumptions. Namely, assumptions may contain very specific facts and not simply general logical truths.

Generalization Rule:

$$\frac{U \vdash A(a)}{U \vdash \forall x\, A(x)}$$

provided a **does not** appear in U.

Deduction Rule:

$$\frac{U \cup \{A\} \vdash B}{U \vdash A \to B}$$

Theorem

(Soundness and Completeness) Hilbert proof system $\mathcal{H}$ for predicate logic is sound and complete.

Specification Rule (Axiom 4):

$$\frac{U \vdash \forall x\ A(x)}{U \vdash A(a)}$$

for any constant a.

Theorem

$$\vdash A(a) \rightarrow \exists x\, A(x)$$

Proof.

1.	$\vdash \forall x\, \neg A(x) \rightarrow \neg A(a)$	Axiom 4
2.	$\vdash A(a) \rightarrow \neg \forall x\, \neg A(x)$	Contrap. Rule
3.	$\vdash A(a) \rightarrow \exists x\, A(x)$	Def. of $\exists$

$\square$

Theorem

$$\vdash \forall x\, A(x) \rightarrow \exists x\, A(x)$$

Proof.

1.	$\forall x\, A(x) \vdash \forall x\, A(x)$	Assumption
2.	$\forall x\, A(x) \vdash \forall x\, A(x) \rightarrow A(a)$	Axiom 4
3.	$\forall x\, A(x) \vdash A(a)$	MP 1,2
4.	$\forall x\, A(x) \vdash A(a) \rightarrow \exists x\, A(x)$	Proved earlier
5.	$\forall x\, A(x) \vdash \exists x\, A(x)$	MP 3,4
6.	$\vdash \forall x\, A(x) \rightarrow \exists x\, A(x)$	Ded. Rule 5

□

Theorem

$$\vdash \forall x(A(x) \to B(x)) \to (\forall x\, A(x) \to \forall x\, B(x))$$

Proof.

1.	$\forall x(A(x) \to B(x)), \forall x\, A(x) \vdash \forall x\, A(x)$	Assumption
2.	$\forall x(A(x) \to B(x)), \forall x\, A(x) \vdash A(a)$	Specif. Rule 1
3.	$\forall x(A(x) \to B(x)), \forall x\, A(x) \vdash \forall x(A(x) \to B(x))$	Assumption
4.	$\forall x(A(x) \to B(x)), \forall x\, A(x) \vdash A(a) \to B(a)$	Specif. Rule 3
5.	$\forall x(A(x) \to B(x)), \forall x\, A(x) \vdash B(a)$	MP 2,4
6.	$\forall x(A(x) \to B(x)), \forall x\, A(x) \vdash \forall x\, B(x)$	Gen. Rule 5
7.	$\forall x(A(x) \to B(x)) \vdash \forall x\, A(x) \to \forall x\, B(x)$	Ded. Rule 6
8.	$\vdash \forall x(A(x) \to B(x)) \to (\forall x\, A(x) \to \forall x\, B(x))$	Ded. Rule 7

□

- We have just proved a more general version of the Generalization Rule:

Generalization Rule: $$\frac{\vdash A(x) \rightarrow B(x)}{\vdash \forall x\, A(x) \rightarrow \forall x\, B(x)}$$

Theorem

$$\vdash \exists x \forall y\, A(x,y) \rightarrow \forall y \exists x\, A(x,y)$$

Proof.

1.	$\vdash A(a,b) \rightarrow \exists x\, A(x,b)$	Proved earlier
2.	$\vdash \forall y\, A(a,y) \rightarrow \forall y \exists x\, A(x,y)$	Gen. Rule 1
3.	$\vdash \neg\forall y \exists x\, A(x,y) \rightarrow \neg\forall y\, A(a,y)$	Axiom 4
4.	$\vdash \forall x[\neg\forall y \exists x\, A(x,y) \rightarrow \neg\forall y\, A(x,y)]$	Gen. Rule 3
5.	$\vdash \neg\forall y \exists x\, A(x,y) \rightarrow \forall x \neg\forall y\, A(x,y)$	Axiom 5
6.	$\vdash \neg\forall x \neg\forall y\, A(x,y) \rightarrow \forall y \exists x\, A(x,y)$	Contrap. Rule 5
7.	$\vdash \exists x \forall y\, A(x,y) \rightarrow \forall y \exists x\, A(x,y)$	Def. of $\exists$ 6

□

Theorem

$$\vdash \forall x(A \rightarrow B(x)) \leftrightarrow (A \rightarrow \forall x\, B(x))$$

Proof.

1.	$A \rightarrow \forall x\, B(x) \vdash A \rightarrow \forall x\, B(x)$	Assumption
2.	$A \rightarrow \forall x\, B(x) \vdash \forall x\, B(x) \rightarrow B(a)$	Axiom 4
3.	$A \rightarrow \forall x\, B(x) \vdash A \rightarrow B(a)$	Transitivity 1,2
4.	$A \rightarrow \forall x\, B(x) \vdash \forall x(A \rightarrow B(x))$	Gen. Rule 3
5.	$\vdash (A \rightarrow \forall x\, B(x)) \rightarrow \forall x(A \rightarrow B(x))$	Ded. Rule 4
6.	$\vdash \forall x(A \rightarrow B(x)) \rightarrow (A \rightarrow \forall x\, B(x))$	Axiom 5
7.	$\vdash \forall x(A \rightarrow B(x)) \leftrightarrow (A \rightarrow \forall x\, B(x))$	Def. of $\leftrightarrow$ 5,6

□

One can also prove

Theorem

$$\vdash \forall x(A(x) \to B) \leftrightarrow (\exists x\, A(x) \to B)$$

Proof.

Exercise; see Theorem 6.20 in the textbook. $\square$

C-Rule

Suppose U is a set of formulas, and a a constant symbol which does not appear in any formula from U or in $\exists x\ A(x)$:

$$\frac{U \vdash \exists x\ A(x)}{U \vdash A(a)}$$

Theorem

If $U \vdash A$ using the C-rule, then $U \vdash A$ can be proved without sing the C-rule, with the proviso that nowhere in the proof are we using Generalization Rule on a formula which involves the new constant symbol a.

Theorem

$$\vdash \exists x \forall y\, A(x, y) \to \forall y \exists x\, A(x, y)$$

Proof.

1.	$\exists x \forall y\, A(x, y) \vdash \exists x \forall y\, A(x, y)$	Assumption
2.	$\exists x \forall y\, A(x, y) \vdash \forall y\, A(a, y)$	C-rule 1
3.	$\exists x \forall y\, A(x, y) \vdash A(a, b)$	Specif. Rule 2
4.	$\exists x \forall y\, A(x, y) \vdash \exists x\, A(x, b)$	Proved earlier
5.	$\exists x \forall y\, A(x, y) \vdash \forall y \exists x\, A(x, y)$	Gen. Rule 4
6.	$\vdash \exists x \forall y\, A(x, y) \to \forall y \exists x\, A(x, y)$	Ded. Rule 5

□