Chapter 5: Predicate Calculus: Formulas, Models, Tableaux

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Outline

5.1 Relations and Predicates

- R: an n-ary relation on a set D

$$R \subseteq D^n = \underbrace{D \times D \times \ldots \times D}_{n \text{ times}}$$

D: **domain** of the relation R.

Observation: A unary relation R is simply a subset of D

$$R \subseteq D$$

Examples

(a) Binary relation $<$ on $\mathbb{N}$:

$$x < y \text{ if } x \text{ is a positive integer less than } y$$

$$<= \{(0,1),(0,2),\ldots,(1,2),(1,3),\ldots,(2,3),\ldots\}$$

(b) Unary relation *Prime*(x) on $\mathbb{N}$:

$$Prime = \{2,3,5,7,11,\ldots\}$$

(c) Given the graph G:

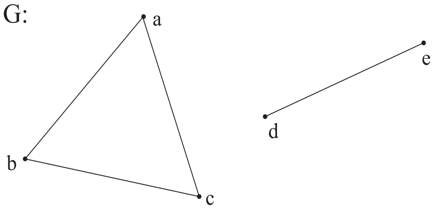


Figure: Graph G

define the binary relation r as:

$$r(x, y) \iff \text{vertex } x \text{ is connected by a path to vertex } y$$

$r = \{(a, a), (b, b), (c, c), (d, d), (e, e),$
$(a, b), (b, a), (a, c), (c, a), (b, c), (c, b), (d, e), (e, d)\}$

- We can think of an n-ary function

$$(x_1, x_2, \ldots, x_n) \mapsto f(x_1, x_2, \ldots, x_n)$$

as an $(n+1)$-ary relation R_f containing the $(n+1)$-tuples

$$(x_1, x_2, \ldots, x_n, f(x_1, x_2, \ldots, x_n))$$

R_f is called the **graph** of the function f.

- Also, we can think of an n-ary relation $R \subseteq D^n$ as a function

$$f : D^n \to \{\mathsf{T}, \mathsf{F}\}$$

$$R(d_1, d_2, \ldots, d_n) = \mathsf{T} \iff (d_1, d_2, \ldots, d_n) \in R$$

5.2 Predicate Formulas

Predicate (relation) symbols	$\mathcal{P} = \{p, q, r, \ldots\}$
Constant symbols	$\mathcal{A} = \{a, b, c, \ldots\}$
Variables	$\mathcal{V} = \{x, y, z, \ldots\}$

BNF Grammar for Predicate Formulas

$$
\begin{aligned}
\textit{argument} &::= x, \qquad \text{for any } x \in \mathcal{V} \\
\textit{argument} &::= a, \qquad \text{for any } a \in \mathcal{A} \\
\textit{argumentList} &::= \textit{argument} \\
\textit{argumentList} &::= \textit{argument}, \textit{argumentList} \\
\textit{atomicFormula} &::= p \quad | \quad p(\textit{argumentList}), \qquad \text{for any } p \in \mathcal{P}
\end{aligned}
$$

$$\begin{aligned}
\textit{formula} &::= \textit{atomicFormula} \\
\textit{formula} &::= \neg \textit{formula} \\
\textit{formula} &::= \textit{formula} \wedge \textit{formula} \\
\textit{formula} &::= \textit{formula} \vee \textit{formula} \\
\textit{formula} &::= \textit{formula} \rightarrow \textit{formula} \\
\textit{formula} &::= \textit{formula} \leftrightarrow \textit{formula} \\
\textit{formula} &::= \forall x \quad \textit{formula}, \qquad \text{for all } x \in \mathcal{V} \\
\textit{formula} &::= \exists x \quad \textit{formula}, \qquad \text{for all } x \in \mathcal{V}
\end{aligned}$$

Examples

1. $p(x, a)$ (atomic formula)
2. $p(x, a) \rightarrow q(x)$
3. $\exists x \quad p(x, a) \rightarrow \forall y \quad q(y)$
4. $\forall x \quad (p(x, a) \rightarrow q(x, y)) \rightarrow (\forall x \quad p(x, a) \rightarrow \forall x \quad q(x, y))$

Bound and Free Variables

Definition

Suppose A is a predicate formula. An occurrence of a variable x in A is a free variable of A if it is not within the scope of any quantifier $\forall x$ or $\exists x$.

Examples

(a) $\exists y \quad p(x, y)$
x-free, y-not free

(b) $p(x, y)$
x, y-free

(c) $\forall x \exists y p(x, y)$
neither x nor y are free

(d) $\forall x p(x) \vee q(x)$
the first occurrence of x is not free while the second occurrence is

- A variable which is not free is said to be bound.
- If we write

$$A(x_1, x_2, \ldots, x_n),$$

  we mean that the free variables of the formula A are among $x_1, x_2, \ldots, x_n$.

5.3 Interpretations

- U: a set of formulas
- $\{p_1, p_2, \ldots, p_m\}$: all predicate symbols appearing in U
- $\{a_1, a_2, \ldots, a_k\}$: all constant symbols appearing in U

Definition

An interpretation I of U is a triple

$$I = (D, \{R_1, R_2, \ldots, R_m\}, \{d_1, d_2, \ldots, d_k\})$$

where

- D is a non-empty set (**domain** of I)
- R_i are n_i-ary relations on D.
- d_i are some fixed elements of D.

$$\begin{aligned} p_i &\mapsto R_i \qquad i = 1, 2, \ldots, m \\ a_j &\mapsto d_j \qquad j = 1, 2, \ldots, k \end{aligned}$$

Example

Consider the formula

$$\forall x \quad p(a, x)$$

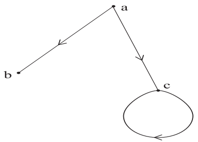
Some of its possible interpretations are:

(1) $I_1 = (\mathbb{N}, \{\leq\}, \{0\})$
"For every natural number x, $0 \leq x$."

(2) $I_2 = (\mathbb{N}, \{|\}, \{1\})$
"For every natural number x, $1|x$."

(3) $I_3 = (\{0,1\}^*, \{$ substring relation $\}, \{\epsilon\})$
"For every string x over alphabet $\{0,1\}$, empty string is a substring of x."

(4) $I_4 = (G, E, \{a\})$



"For every vertex x of G, (a, x) is an edge in G."

Definition

Suppose I is an interpretation for a predicate formula A.An assignment

$$\sigma_I : \mathcal{V} \rightarrow D$$

is a function which assigns a value in the domain D to any variable appearing in the formula A.

Truth Value of a Predicate Formula

Suppose:

- A - formula.
- I - an interpretation for A.
- σ_I - an assignment

We define $v_{\sigma_I}(A)$, the truth value of A under σ_I, inductively:

(a) If $A = p(c_1, c_2, \ldots, c_n)$ is an atomic formula, where each c_i is either a variable x_j or a constant symbol a_j, then

$$v_{\sigma_I}(A) = \text{T iff } (\sigma_I(c_1), \sigma_I(c_2), \ldots, \sigma_I(c_n)) \in R$$

(b) $v_{\sigma_I}(\neg A) = \neg v_{\sigma_I}(A)$.

(c) $v_{\sigma_I}(A_1 \wedge A_2) = v_{\sigma_I}(A_1) \wedge v_{\sigma_I}(A_2)$.

(d) $v_{\sigma_I}(A_1 \vee A_2) = v_{\sigma_I}(A_1) \vee v_{\sigma_I}(A_2)$.

[Similarly for $\rightarrow$, $\leftrightarrow$.]

(e) $v_{\sigma_I}(\forall x\ A) = \mathrm{T}$ iff $v_{\sigma_I}(A) = \mathrm{T}$ for all $x \in D$

(f) $v_{\sigma_I}(\exists x\ A) = \mathrm{T}$ iff $v_{\sigma_I}(A) = \mathrm{T}$ for some $x \in D$

Theorem

If A is a closed formula, then $v_{\sigma_I}(A)$ does not depend on σ_I. In that case, we write

$$v_I(A)$$

Theorem

Let $A' = A(x_1, x_2, \ldots, x_n)$ *be a non-closed formula and let I be an interpretation. Then:*

(a) $v_{\sigma_I}(A') = T$ *for assignment* σ_I *iff*

$$v_I(\exists x_1 \exists x_2 \ldots \exists x_n\, A') = T$$

(b) $v_{\sigma_I}(A') = T$ *for all assignments* σ_I *iff*

$$v_I(\forall x_1 \forall x_2 \ldots \forall x_n\, A') = T$$

Definition

A closed formula A is true in I, or I is a model for A, if $v_I(A) = \mathrm{T}$.

$$I \models A$$

Definition

A closed formula A is satisfiable if, for **some** interpretation I,

$$I \models A$$

A is valid if, for **all** interpretations I,

$$I \models A$$

We can also define unsatisfiable and falsifiable formulas in the usual way.

Examples

(a)	$\forall x\, p(a,x) \to p(a,a)$	valid
(b)	$\forall x \forall y\, (p(x,y) \to p(y,x))$	not valid, satisfiable
(c)	$\forall x \exists y\, p(x,y)$	not valid, satisfiable
(d)	$\exists x \exists y\, (p(x) \land \neg p(y))$	not valid, satisfiable
(e)	$\forall x(p(x) \land q(x)) \leftrightarrow (\forall x\, p(x) \land \forall x\, q(x))$	valid
(f)	$\exists x\, (\neg p(x) \land p(x))$	unsatisfiable

5.4 Equivalence and Substitution

- Suppose A_1, A_2 are two closed formulas. If, for all interpretations I

  $$v_I(A_1) = v_I(A_2)$$

  we say that A_1 and A_2 are equivalent, and we write

  $$A_1 \equiv A_2$$

- Suppose U is a set of closed formulas, and A a closed formula

  $$U \models A$$

  means that, in all interpretations I in which all formulas from U are true, we also have

  $$v_I(A) = \mathrm{T}.$$

Examples

(a) $\forall x\, A(x) \equiv \neg\exists x\, \neg A(x)$

(b) $\exists x\, A(x) \equiv \neg\forall x \neg A(x)$

(c) $\forall x \forall y\, A(x, y) \equiv \forall y \forall x\, A(x, y)$

(d) $\exists x \exists y\, A(x, y) \equiv \exists y \exists x\, A(x, y)$

(e) $\exists x \forall y A(x, y) \not\equiv \forall y \exists x A(x, y)$
To see that these two formulas are not equivalent, consider

$$I = (\mathbb{Z}, \{\leq\}).$$

Clearly,

$$I \not\models \exists x \forall y\; x \leq y, \qquad I \models \forall y \exists x\; x \leq y$$

Theorem

(a) *$A \equiv B$ if and only if $\models A \leftrightarrow B$.*

(b) *Suppose*

$$U = \{A_1, A_2, \ldots, A_n\}$$

$U \models A$ if and only if $\models A_1 \wedge A_2 \wedge \ldots A_n \rightarrow A$.

Examples

The following are valid formulas

(a) $\exists x(A(x) \vee B(x)) \leftrightarrow \exists x\, A(x) \vee \exists x\, B(x)$

(b) $\forall x(A(x) \wedge B(x)) \leftrightarrow \forall x\, A(x) \wedge \forall x\, B(x)$

(c) $\exists x(A(x) \wedge B) \leftrightarrow \exists x\, A(x) \wedge B$, if x is not free in B.

(d) $\forall x(A(x) \vee B) \leftrightarrow \forall x\, A(x) \vee B$, if x is not free in B.

(e) $\exists x(A(x) \rightarrow B(x)) \leftrightarrow (\forall x\, A(x) \rightarrow \exists x\, B(x))$

(f) $\forall x(A(x) \rightarrow B(x)) \leftrightarrow (\exists x\, A(x) \rightarrow \forall x\, B(x))$

[For more pairs of equivalent formulas, see Fig. 5.2 in Section 5.4]

Proof.

(e)

$$\begin{aligned}\exists x(A(x) \rightarrow B(x)) &\equiv \exists x(\neg A(x) \vee B(x)) \\ &\equiv \exists x \neg A(x) \vee \exists x\, B(x) \\ &\equiv \neg \forall x\, A(x) \vee \exists x\, B(x) \\ &\equiv \forall x\, A(x) \rightarrow \exists x\, B(x)\end{aligned}$$

□

Example

Prove that

$$\exists x \forall y\ A(x, y) \rightarrow \forall y \exists x\ A(x, y)$$

is a valid formula, yet its converse is not valid.

Solution:

Let I be an interpretation. Suppose

$$I \models \exists x \forall y\ A(x, y).$$

Then, for some $a \in D$

$$I \models \forall y\ A(a, y)$$

So,

$$I \models \forall y (\exists x\ A(x, y))$$

which proves that, for every I,

$$I \models \exists x \forall y\ A(x, y) \rightarrow \forall y \exists x\ A(x, y)$$

$I = (\mathbb{Z}, \{\leq\})$ shows that the implication cannot be reversed if we want the formula to be valid. $\square$

Example

We will try to show that

$$\forall x(p(x) \rightarrow q(x)) \rightarrow (\forall x\ p(x) \rightarrow \forall x\ q(x))$$

is a valid formula

We consider its negation

$$\neg[\forall x(p(x) \rightarrow q(x)) \rightarrow (\forall x\ p(x) \rightarrow \forall x\ q(x))]$$

and try to show that it is unsatisfiable.

$$\neg[\forall x(p(x) \rightarrow q(x)) \rightarrow (\forall x\, p(x) \rightarrow \forall x\, q(x))]$$

$$\forall x(p(x) \rightarrow q(x)),\ \neg(\forall x\, p(x) \rightarrow \forall x\, q(x))$$

$$\forall x(p(x) \rightarrow q(x)),\ \forall x\, p(x),\ \neg\forall x\, q(x)$$

$$\forall x(p(x) \rightarrow q(x)),\ \forall x\, p(x),\ \neg q(a)$$

$$\forall x(p(x) \rightarrow q(x)),\ p(a),\ \neg q(a)$$

$$p(a) \rightarrow q(a),\ p(a),\ \neg q(a)$$

$$\neg p(a), p(a), \neg q(a) \qquad q(a), p(a), \neg q(a)$$

$$\times \qquad\qquad\qquad \times$$

Example

Now, we consider the formula

$$\forall x(p(x) \vee q(x)) \to (\forall x\ p(x) \vee \forall x\ q(x))$$

which is not valid, but is satisfiable.

$$\neg[\forall x(p(x) \vee q(x)) \rightarrow (\forall x\, p(x) \vee \forall x\, q(x))]$$

$$\forall x(p(x) \vee q(x)), \neg(\forall x\, p(x) \vee \forall x\, q(x))$$

$$\forall x(p(x) \vee q(x)), \exists x\, \neg p(x), \exists x\, \neg q(x)$$

$$\forall x(p(x) \vee q(x)), \neg p(a), \exists x \neg q(x)$$

$$p(a) \vee q(a), \neg p(a), \exists x \neg q(x)$$

$$p(a), \neg p(a), \exists x \neg q(x) \qquad q(a), \neg p(a), \exists \neg q(x)$$

$$\times$$

$$q(a), \neg p(a), \neg q(a)$$

$$\times$$

Question: What went wrong?

- We used the same constant a twice to eliminate two distinct existential quantifiers.
- We were forced to use the same constant since, once we eliminated the universal quantifier in

$$\forall x(p(x) \vee q(x))$$

  we replaced it with a and were forced to work with that constant exclusively from that point on.

Solution: We will not delete universal quantifiers from nodes of the tableau; instead, we introduce some instance of that variable but keep writing the universal quantifier. E.g.

$$\forall x\ p(x)$$

$$|$$

$$\forall x\ p(x), p(a)$$

Using these guidelines, if we construct a correct tableau for he formula from the previous example (exercise!), we notice that one branch ends with the open leaf

$$p(a), \neg q(a), \neg p(b), q(b)$$

In fact, this leaf gives us a model for this satisfiable formula; the domain is

$$D = \{a, b\}$$

and the unary relations are subsets

$$p = \{a\}, \qquad q = \{b\}$$

[This is what we will define as an Herbrand model for this formula in Chapter 7.]

Example

Consider the formulas

$$A_1 = \forall x \exists y\ p(x, y)$$
$$A_2 = \forall x \neg p(x, x)$$
$$A_3 = \forall x \forall y \forall z (p(x, y) \wedge p(y, z) \rightarrow p(x, z))$$

Check whether

$$A = A_1 \wedge A_2 \wedge A_3$$

is a satisfiable formula and, if so, find one model for A.

Solution: We will first construct a semantic tableau for the formula:

$$\forall x \exists y\ p(x, y), A_2, A3$$

$$\forall x \exists y\ p(x, y), \exists y(a_1, y), A_2, A_3$$

$$\forall x \exists y\ p(x, y), p(a_1, a_2), A_2, A_3$$

$$\forall x \exists y\ p(x, y), \exists y\ p(a_2, y), p(a_1, a_2), A_2, A_3$$

$$\forall x \exists y\ p(x, y), p(a_2, a_3), p(a_1, a_2), A_2, A_3$$

$$\vdots$$

We see that the tableau does not terminate; namely, every time we drop the universal or an existential quantifier, we can introduce a new constant symbol a_i, to get an infinite sequence of constants:

$$a_1, a_2, \ldots, a_n, \ldots$$

The formula does have an obvious infinite model:

$$I = (\mathbb{N}, \{<\})$$

Furthermore, one can prove, using the formulas A_2 and A_3 (see the proof of Theorem 5.24 in the textbook) that **every** model of

$$A = A_1 \wedge A_2 \wedge A_3$$

must be infinite. So, the tableau construction effectively produces a "generic" infinite model for A. □

- One stark difference in comparison with semantic tableaux for propositional logic is (as seen in the previous example) that a tableau of a predicate formula may not terminate.
- The reason for this anomaly is that, in propositional logic, nodes of a tableau simplify in terms of the formula complexity. In predicate logic, this is not the case, since we can never eliminate universal quantifiers.

Algorithm for Semantic Tableaux

- Two new types of rules:

γ	$\gamma(a)$
$\forall x\ A(x)$	$A(a)$
$\neg\exists x\ A(x)$	$\neg A(a)$

δ	$\delta(a)$
$\exists x\ A(x)$	$A(a)$
$\neg\forall x\ A(x)$	$\neg A(a)$

- **Literal:** closed atomic formula $p(a_1, a_2, \ldots, a_n)$ or the negation of such a formula.

Input: A - a predicate formula

Output: Semantic tableau $\mathcal{T}$ for A; all branches are either infinite, or finite with leaves marked $\times$ (closed) or $\odot$ (open).

(1) Initially, $\mathcal{T}$ is a single node, labeled $\{A\}$.

(2) We build the tableau inductively by choosing an unmarked leaf l, labeled $U(l)$, and applying one of the following rules:

- If $U(l)$ is a set of literals and γ-formulas containing a pair of complementary literals $\{p(a_1, a_, \ldots, a_n), \neg p(a_1, a_2, \ldots, a_n)\}$, mark it as closed ($\times$)
- If $U(l)$ is not a set of literals, choose a formula A in $U(l)$ which is not a literal:
  - α- and β-rules are applied just as in propositional logic.
  - If A is a γ-formula, add a new node l', a child of l, and label it

    $$U(l') = U(l) \cup \{\gamma(a)\}$$

    where a is a constant appearing in $U(l)$. If $U(l)$ consists of literals and γ-formulas only, mark it $\times$ or $\odot$, depending on whether there is a set of complementary literals.
  - If A is a δ-formula, create a new node l' as a child of l and label it

    $$U(l') = (U(l) - \{A\}) \cup \{\delta(a)\}$$

    where a is some constant that does not appear in $U(l)$.

Definition

A branch in $\mathcal{T}$ is closed if it terminates in a leaf marked $\times$. Otherwise, it is open.

Theorem

(Soundness) Suppose A is a predicate formula and $\mathcal{T}$ its semantic tableau. If $\mathcal{T}$ closes, then A is unsatisfiable.

Theorem

(Completeness) Suppose A is a valid formula. Then, the systematic semantic tableau for A terminates and is closed.

- Systematic tableau: a tableau in which every node is labeled

$$W(l) = (U(l), C(l))$$

  where $U(l)$ is a list of formulas and $C(l)$ is the list of all constant symbols appearing in $U(l)$.
- In a systematic tableau, if using a γ-rule, we do the following: suppose $\{\gamma_1, \ldots, \gamma_m\}$ are all γ-formulas in $U(l)$ and

$$C(l) = \{a_1, \ldots, a_k\}$$

  The new node l' will be labeled

$$(U(l) \cup \{\gamma_i(a_j)\}, C(l))$$

  In other words, we create all possible instances of formulas γ_i where the variable is replaced by all possible constants a_j.

5.7 Finite and Infinite Models

Theorem

(Löwenheim) If a formula is satisfiable, then it is satisfiable in a countable model.

Theorem

(Löwenheim - Skolem) If a countable set of predicate formulas is satisfiable, then it is satisfiable in a countable model.

Theorem

(Compactness Theorem) Let U be a countable set of formulas. If all finite subsets of U are satisfiable, then so is U.

5.8 Undecidability of the Predicate Logic

- Turing machines can be viewed as devices which compute functions on natural numbers; i.e. given a Turing machine T, we can associate to it a function

$$f_T : \mathbb{N} \to \mathbb{N}$$

so that $f_T(n) = m$ if T halts with the tape consisting of m 1's when started on the tape with the input of n consecutive 1's. If T never halts on the input of n consecutive 1's, then $f_T(n)$ is undefined.

Theorem

(Church) It is undecidable whether a Turing machine, started on a blank tape, will halt.

- In other words, it is undecidable, given a Turing machine T, whether $f_T(0)$ is defined.

Two-Register Machines

Definition

Two-register machine (or, a Minsky machine) M consists of a pair of registers (x, y) which can store natural numbers, and a program $P = \{L_0, L_1, \ldots, L_n\}$, which is a sequential list of instructions. L_n is always the command "halt", and for $0 \leq i < n$, L_i has one of the two forms

1. $r := r + 1$, for $r \in \{x, y\}$
2. if $r = 0$ then go to L_j else $r := r - 1$, for $r \in \{x, y\}$, $0 \leq j \leq n$.

- **Execution** of M: sequence of states

$$s_k = (L_i, x, y)$$

where L_i is the current instruction during the execution, and x,y are current contents of the two registers.

- **Initial state:**

$$s_0 = (L_0, m, 0), \qquad \text{for some } m$$

- If

$$s_k = (L_n, x, y), \text{ for some } k$$

then M halts and

$$y = f(m)$$

is computed by M.

Theorem

For every Turing machine T that computes $f : \mathbb{N} \to \mathbb{N}$*, a two-register machine M can be constructed which computes the same function.*

Corollary

It is undecidable whether, given a two-register machine M, whether $f_M(0)$ *exists or not.*

Theorem

(Church) Validity in predicate calculus is undecidable.

Sketch of the Proof.

To each two-register machine M, we associate a predicate formula S_M such that

$$M \text{ halts started at } (L_0, 0, 0) \quad \Longleftrightarrow \quad \models S_M$$

We use the language:

- Binary relations: $p_i(x, y)$ ($i = 0, 1, \ldots, n$)
- Unary function: $s(x)$
- Constant symbol: a

Intended interpretation:

- $p_i(x, y)$: M is at the state (L_i, x, y)
- $s(x)$: successor function $s(x) = x + 1$
- a: $a = 0$

L_i	S_i
$x := x + 1$	$\forall x \forall y (p_i(x, y) \rightarrow p_{i+1}(s(x), y))$
$y := y + 1$	$\forall x \forall y (p_i(x, y) \rightarrow p_{i+1}(x, s(y)))$
if $x = 0$ then goto L_j else $x := x - 1$	$\forall y (p_i(a, y) \rightarrow p_j(a, y))$ $\wedge \forall x \forall y (p_i(s(x), y) \rightarrow p_{i+1}(x, y))$
if $y = 0$ then goto L_j else $y := y - 1$	$\forall x (p_i(x, a) \rightarrow p_j(x, a))$ $\wedge \forall x \forall y (p_i(x, s(y)) \rightarrow p_{i+1}(x, y))$

Finally, define

$$S_M = (S_0 \wedge S_1 \wedge \ldots \wedge S_n \wedge p_0(a, a)) \rightarrow \exists z_1 \exists z_2 \; p_n(z_1, z_2)$$

S_M says the following: if a machine with the program

$$P = \{L_0, L_1, \ldots, L_n\}$$

is started at the initial state $(L_0, 0, 0)$, then the computation will halt with the values at the registers being (z_1, z_2), for some natural numbers z_1, z_2.

Since the Halting Problem for two-register machines is undecidable, it is impossible to verify algorithmically whether

$$\models S_M$$

or not. $\square$

Church’s Theorem is also true for some restricted classes of predicate logic:

1. Formulas containing only a finite number of binary predicate symbols, one unary function symbol, and one constant symbol.
2. Formulas written as Prolog programs.
3. Formulas with no function symbols.

[Skip ’Solvable Cases of the Decision Problem’ in Section 5.8]