Chapter 4: Propositional Calculus: Resolution and BDDs

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Outline

4.1 Resolution

Definition

A formula is in conjunctive normal form (CNF) if it is a conjunction of disjunctions of literals.

Examples

(a) $p \wedge (\neg p \vee q \vee \neg r) \wedge (\neg q \vee q \vee r) \wedge (\neg q \vee p)$
Formula is in CNF

(b) $(\neg p \vee q \vee r) \wedge \neg(p \vee \neg r) \wedge q$
This formula is not in CNF

Theorem

Every propositional formula can be transformed into an equivalent formula in CNF.

Proof.

(Algorithm)

1. eliminate all connectives other than $\neg$, $\vee$, and $\wedge$.
2. push all negations inward using De Morgan's laws:

$$\neg(A \vee B) \equiv \neg A \wedge \neg B$$
$$\neg(A \wedge B) \equiv \neg A \vee \neg B$$

3. eliminate double negations
4. use distributivity to eliminate conjunctions within disjunctions:

$$A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$$

□

Example

Transform the formula

$$(p \to q) \to (\neg q \to \neg p)$$

into an equivalent formula in CNF.

Solution:

$$\begin{aligned}
&(p \to q) \to (\neg q \to \neg p) \\
&\equiv (\neg p \vee q) \to (\neg\neg q \vee \neg p) \\
&\equiv \neg(\neg p \vee q) \vee (\neg\neg q \vee \neg p) \\
&\equiv (\neg\neg p \wedge \neg q) \vee (\neg\neg q \vee \neg p) \\
&\equiv (p \wedge \neg q) \vee (q \vee \neg p) \\
&\equiv (p \vee q \vee \neg p) \wedge (\neg q \vee q \vee \neg p)
\end{aligned}$$

$\square$

Definition

A clause is a set of literals which is assumed (implicitly) to be a disjunction of those literals.

Example

$$\neg p \vee q \vee \neg r \qquad \leftrightsquigarrow \qquad \{\neg p, q, \neg r\}$$

- **unit clause:** clause with only one literal; e.g. $\{\neg q\}$
- clausal form of a formula: implicit conjunction of clauses.

Example

$$p \wedge (\neg p \vee q \vee \neg r) \wedge (\neg q \vee q \vee \neg r) \wedge (\neg q \vee p)$$
$$\Updownarrow$$
$$\{\{p\}, \{\neg p, q, \neg r\}, \{\neg q, q, \neg r\}, \{\neg q, p\}\}$$

- Abbreviated notation:

$$\{p, \bar{p}q\bar{r}, \bar{q}q\bar{r}, \bar{q}p\}$$

Notation:

- l-literal, l^c-complement of l
- C-clause (a set of literals)
- S-a clausal form (a set of clauses)

Properties of Clausal Forms

(1) If l appears in some clause of S, but l^c does not appear in any clause, then, if we delete all clauses in S containing l, the new clausal form S' is satisfiable if and only if S is satisfiable.

Example

Satisfiability of

$$S = \{pq\bar{r}, p\bar{q}, \bar{p}q\}$$

is equivalent to satisfiability of

$$S' = \{p\bar{q}, \bar{p}q\}$$

(2) Suppose $C = \{l\}$ is a unit clause and we obtain S' from S by deleting C and l^c from all clauses that contain it. Then, S is satisfiable if and only if S' is satisfiable.

Example

$$S = \{p, \bar{p}q\bar{r}, \bar{q}q\bar{r}, q\bar{p}\}$$

is satisfiable if and only if

$$S' = \{q\bar{r}, \bar{q}q\bar{r}, q\}$$

is satisfiable.

(3) If S contains two clauses C and C', such that $C \subseteq C'$, we can delete C' without affecting the (un)satisfiability of S.

Example

$$S = \{p, \bar{p}q\bar{r}, \bar{q}q\bar{r}, q\bar{p}\}$$

is satisfiable if and only if

$$S' = \{p, \bar{q}q\bar{r}, q\bar{p}\}$$

is satisfiable.

(4) If a clause C in S contains a pair of complementary literals l, l^c, then C can be deleted from S without affecting its (un)satisfiability.

Example

$$S = \{p, \bar{p}q\bar{r}, \bar{q}q\bar{r}, q\bar{p}\}$$

is satisfiable if and only if

$$S' = \{p, \bar{p}q\bar{r}, q\bar{p}\}$$

is such.

Definition

The empty clause will be denoted $\Box$. The empty set of clauses (i.e. the empty clausal form) will be denoted $\emptyset$.

Caution: We have to be careful not to confuse the empty clause with the empty clausal form.

For example,

$$S = \{p\bar{q}, \bar{p}qr, \Box\}$$

is a nonempty clausal form ($S \neq \emptyset$) which does contain the empty clause.

Resolution Rule

Suppose C_1, C_2 are clauses such that $l \in C_1$, $l^c \in C_2$. The clauses C_1 and C_2 are said to be **clashing clauses** and they clash on the complementary literals l, l^c.

C, the resolvent of C_1, C_2 is the clause

$$Res(C_1, C_2) = (C_1 - \{l\}) \cup (C_2 - \{l^c\})$$

C_1 and C_2 are called the **parent clauses** of C.

C_1 C_2

$$C = (C_1 - \{l\}) \cup (C_2 - \{l^c\})$$

Example

The clauses

$$C_1 = \bar{p}q\bar{r}, \qquad C_2 = \bar{q}p$$

clash on $p, \bar{p}$.

$$Res(C_1, C_2) = q\bar{r} \cup \bar{q} = q\bar{r}\bar{q}$$

C_1, C_2 also clash on $q, \bar{q}$, so, another way to find a resolvent for these two clauses is

$$Res(C_1, C_2) = \bar{p}\bar{r} \cup p = \bar{p}\bar{r}p$$

Theorem

Resolvent C is satisfiable if and only if the parent clauses C_1, C_2 are simultaneously satisfiable.

Proof.

($\Longleftarrow$) Suppose C_1 and C_2 are simultaneously satisfiable, and let v be a truth-assignment which makes all formulas in C_1 and C_2 true. Let l, l^c be the pair of clashing literals used in resolving C_1 and C_2.

Then, either

- $v(l) = \mathrm{T}$, $v(l^c) = \mathrm{F}$; or
- $v(l) = \mathrm{F}$, $v(l^c) = \mathrm{T}$

If $v(l) = \mathrm{T}$, then C_2 can be satisfied only if $v(l') = \mathrm{T}$, for some literal l' different from l^c.
Since l' still appears in $Res(C_1, C_2)$, the resolvent clause will be satisfied by v. The other possibility is handled analogously.

($\Longrightarrow$) Suppose the resolvent C is satisfiable. Then, for some truth-assignment v and some literal $l' \in C$, we have

$$v(l') = \mathsf{T}$$

By resolution, this l' was originally either in C_1 or in C_2 (or, maybe, both). Then, it is not difficult to see that it is possible to extend this assignment v to the deleted literals l and l^c so that both clauses are satisfied by v. $\square$

Resolution Algorithm

Input: S - a set of clauses
Output: "S is satisfiable" or "S is not satisfiable"

1. Set $S_0 := S$.
2. Suppose S_i has already been constructed.
3. To construct S_{i+1}, choose a pair of clashing literals and clauses C_1, C_2 in S (if there are any) and derive

$$C := Res(C_1, C_2)$$
$$S_{i+1} := S_i \cup \{C\}$$

4. If $C = \square$, output "S is not satisfiable"; if $S_{i+1} = S_i$, output "S is satisfiable".
5. Otherwise,set $i := i + 1$ and go back to Step 2.

Example

Determine whether

$$S = \{\bar{p}q, \bar{q}\bar{r}s, p, r, \bar{s}\}$$

is satisfiable.

Solution:

1. $S_0 = \{\bar{p}q, \bar{q}\bar{r}s, p, r, \bar{s}\}$
2. $C_1 = \bar{p}q, C_2 = p, C = q, \quad S_1 = \{\bar{p}q, \bar{q}\bar{r}s, p, r, \bar{s}, q\}$
3. $C_1 = \bar{q}\bar{r}s, C_2 = q, C = \bar{r}s, \quad S_2 = \{\bar{p}q, \bar{q}\bar{r}s, p, r, \bar{s}, q, \bar{r}s\}$
4. $C_1 = r, C_2 = \bar{r}s, C = s, \quad S_3 = \{\bar{p}q, \bar{q}\bar{r}s, p, r, \bar{s}, q, \bar{r}s, s\}$
5. $C_1 = \bar{s}, C_2 = s, C = \Box$

S is not satisfiable.

In the preceding example, we can use facts about sets of clauses (1)-(4), mentioned earlier, in order to keep the sets S_i shorter; the drawback is that this approach requires a large number of checks before reducing the set S_i to a simplified set S_i' in each step.

1. $S_0 = \{\bar{p}q, \bar{q}\bar{r}s, p, r, \bar{s}\}$
2. $C_1 = \bar{p}q, C_2 = p, C = q, \quad S_1 = \{\bar{p}q, \bar{q}\bar{r}s, p, r, \bar{s}, q\}$ which can be reduced to $S_1' = \{\bar{q}\bar{r}s, p, r, \bar{s}, q\}$
3. $C_1 = \bar{q}\bar{r}s, C_2 = q, C = \bar{r}s, \quad S_2 = \{\bar{q}\bar{r}s, p, r, \bar{s}, q, \bar{r}s\}$ which can be reduced to $S_2' = \{p, r, \bar{s}, q, \bar{r}s\}$
4. $C_1 = r, C_2 = \bar{r}s, C = s, \quad S_3 = \{p, r, \bar{s}, q, \bar{r}s, s\}$ which can be reduced to $S_3' = \{p, r, \bar{s}, q, s\}$
5. $C_1 = \bar{s}, C_2 = s, C = \Box$

Example

Show that

$$(p \to q) \to (\neg q \to \neg p)$$

is a valid formula.

Solution: We will show that

$$\neg[(p \to q) \to (\neg q \to \neg p)]$$

is not satisfiable

(1) Transform the formula into CNF:

$$\begin{aligned}
\neg[(p \to q) \to (\neg q \to \neg p)] &\equiv (p \to q) \wedge \neg(\neg q \to \neg p) \\
&\equiv (\neg p \vee q) \wedge \neg q \wedge \neg\neg p \\
&\equiv (\neg p \vee q) \wedge \neg q \wedge p
\end{aligned}$$

(2) Show, using resolution, that

$$S = \{\bar{p}q, \bar{q}, p\}$$

1. $S_0 = \{\bar{p}q, \bar{q}, p\}$
2. $C_1 = \bar{p}q, \quad C_2 = \bar{q}, \quad C = \bar{p}, \qquad S_1 = \{\bar{p}q, \bar{q}, p, \bar{p}\}$
3. $C_1 = p, \quad C_2 = \bar{p}, \quad C = \square$

Definition

A derivation of $\square$ from S is called a refutation of S.

Soundness and Completeness

Theorem

If the set of a clauses labeling the leaves of a resolution tree is satisfiable, then the clause at the root is satisfiable.

Proof.

This is a simple consequence of a theorem proved earlier. □

Theorem

(*Soundness*) *If the empty clause □ is derived from a set of clauses, then the set of clauses is unsatisfiable.*

Theorem

(*Completeness*) *If a set of clauses is unsatisfiable, then the empty clause □ can be derived from it using resolution algorithm.*