Chapter 3: Propositional Calculus: Deductive Systems

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Outline

3.1 Deductive (Proof) System

- Deductive system:
  1. (finite) set of **axioms**
  2. (finite) set of **rules of inference**

- Proof in a deductive system: a finite sequence of formulas such that each formula in the sequence is either:
  (a) an axiom; or
  (b) derived from previous formulas in the sequence using a rule of inference.

- The last formula A in the sequence is called a theorem

$$\vdash A$$

In this course, we will study two proof systems for propositional logic:

1. Gentzen system $\mathcal{G}$
2. Hilbert system $\mathcal{H}$

3.2 Gentzen System $\mathcal{G}$

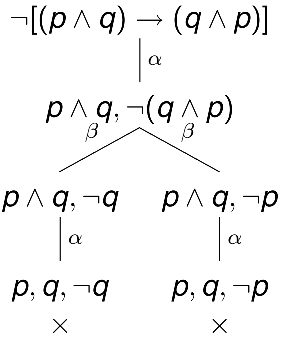
- this proof system is based on the reversal of semantic tableaux.
- Main Idea: in order to prove that A is valid, we are trying to show that $\neg A$ is unsatisfiable, i.e. that its semantic tableau is closed. After that, we write the proof in $\mathcal{G}$ by traversing the tableau from the bottom to the top, changing every formula in every node to its negation.

Example

Prove that

$$\vdash (p \wedge q) \rightarrow (q \wedge p)$$

(1) We first construct a tableau for $\neg[(p \wedge q) \rightarrow (q \wedge p)]$:



The corresponding proof in $\mathcal{G}$:

1.	$\neg p, \neg q, q$	Axiom
2.	$\neg p, \neg q, p$	Axiom
3.	$\neg(p \wedge q), q$	α-rule applied to 1
4.	$\neg(p \wedge q), p$	α-rule applied to 2
5.	$\neg(p \wedge q), q \wedge p$	β-rule applied to 3,4
6.	$(p \wedge q) \rightarrow (q \wedge p)$	α-rule applied to 5

$\square$

Gentzen Proof System $\mathcal{G}$

- Axioms: all sets of formulas containing a pair of complementary literals
- Rules of Inference:
  1. α-rules: $\dfrac{\vdash U_1 \cup \{\alpha_1, \alpha_2\}}{\vdash U_1 \cup \{\alpha\}}$
  2. β-rules: $\dfrac{\vdash U_1 \cup \{\beta_1\}, \quad \vdash U_2 \cup \{\beta_2\}}{\vdash U_1 \cup U_2 \cup \{\beta\}}$

α	α_1	α_2
$\neg\neg A$	A	
$\neg(A_1 \wedge A_2)$	$\neg A_1$	$\neg A_2$
$A_1 \vee A_2$	A_1	A_2
$A_1 \rightarrow A_2$	$\neg A_1$	A_2
$\neg(A_1 \leftrightarrow A_2)$	$\neg(A_1 \rightarrow A_2)$	$\neg(A_2 \rightarrow A_1)$

β	β_1	β_2
$B_1 \wedge B_2$	B_1	B_2
$\neg(B_1 \vee B_2)$	$\neg B_1$	$\neg B_2$
$\neg(B_1 \rightarrow B_2)$	B_1	$\neg B_2$
$B_1 \leftrightarrow B_2$	$B_1 \rightarrow B_2$	$B_2 \rightarrow B_1$

Theorem

Suppose U is a set of formulas and $\overline{U}$ is the set of complements of formulas from U. Then

$$\vdash U$$

in system $\mathcal{G}$ if and only if there is a closed semantic tableau for $\overline{U}$.

Corollary

$\vdash A$ in system $\mathcal{G}$ if and only if there is a closed semantic tableau for $\neg A$.

Theorem

(Soundness and Completeness)

$$\models A \text{ if and only if } \vdash_{\mathcal{G}} A$$

Example

Prove

$$\vdash_{\mathcal{G}} (A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A)$$

A semantic tableau for $\neg[(A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A)]$:

$$\begin{array}{c}
\neg[(A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A)] \\
\mid \alpha \\
A \rightarrow B, \neg(\neg B \rightarrow \neg A) \\
\mid \alpha \\
A \rightarrow B, \neg B, \neg\neg A \\
\mid \alpha \\
A \rightarrow B, \neg B, A \\
\beta \swarrow \quad \searrow \beta \\
\neg A, \neg B, A \qquad B, \neg B, A \\
\times \qquad\qquad\qquad \times
\end{array}$$

Proof in $\mathcal{G}$:

1.	$A, B, \neg A$	Axiom
2.	$\neg B, B, \neg A$	Axiom
3.	$\neg(A \rightarrow B), B, \neg A$	β-rule 1,2
4.	$\neg(A \rightarrow B), \neg B \rightarrow \neg A$	α-rule 3
5.	$(A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A)$	α-rule 4

3.3 Hilbert System $\mathcal{H}$

- Recall, first, that every propositional formula is equivalent to one using $\neg$ and $\rightarrow$ as its only connectives.

Axioms:

1. $\vdash A \rightarrow (B \rightarrow A)$
2. $\vdash (A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$
3. $\vdash (\neg B \rightarrow \neg A) \rightarrow (A \rightarrow B)$

Rule of Inference (Modus Ponens):

$$\text{MP:} \qquad \frac{\vdash A, \quad \vdash A \rightarrow B}{\vdash B}$$

Theorem

$$\vdash_{\mathcal{H}} A \rightarrow A$$

Proof.

1.	$\vdash A \rightarrow ((A \rightarrow A) \rightarrow A)$	Axiom 1
2.	$\vdash [A \rightarrow ((A \rightarrow A) \rightarrow A)]$ $\rightarrow [((A \rightarrow (A \rightarrow A)) \rightarrow (A \rightarrow A))]$	Axiom 2
3.	$\vdash (A \rightarrow (A \rightarrow A)) \rightarrow (A \rightarrow A)$	MP 1,2
4.	$\vdash A \rightarrow (A \rightarrow A)$	Axiom 1
5.	$\vdash A \rightarrow A$	MP 3,4

□

- We can simplify proofs in system $\mathcal{H}$ by using “shortcuts”; namely, if we have proved a certain theorem or a rule, we can use it in later proofs.

Definition

$U \vdash A$ will mean the following: A can be proved from axioms **and** additional assumptions U, using Modus Ponens.

Deduction Rule

$$\frac{U \cup \{A\} \vdash B}{U \vdash A \to B}$$

Proof.

Proof is by induction on the length of the proof of

$$U \cup \{A\} \vdash B$$

□

Contrapositive Rule

$$\frac{U \vdash \neg B \rightarrow \neg A}{U \vdash A \rightarrow B}$$

Proof.

Suppose

$$1. \qquad U \vdash \neg B \rightarrow \neg A$$

Then,

2.	$U \vdash (\neg B \rightarrow \neg A) \rightarrow (A \rightarrow B)$	Ax.3
3.	$U \vdash A \rightarrow B$	MP 1,2

□

Theorem

$$\vdash (A \to B) \to [(B \to C) \to (A \to C)]$$

Proof.

1.	$\{A \to B, B \to C, A\} \vdash A$	Assumption
2.	$\{A \to B, B \to C, A\} \vdash A \to B$	Assumption
3.	$\{A \to B, B \to C, A\} \vdash B$	MP 1,2
4.	$\{A \to B, B \to C, A\} \vdash B \to C$	Assumption
5.	$\{A \to B, B \to C, A\} \vdash C$	MP 3,4
6.	$\{A \to B, B \to C\} \vdash A \to C$	Ded. Rule 5
7.	$\{A \to B\} \vdash (B \to C) \to (A \to C)$	Ded. Rule 6
8.	$\vdash (A \to B) \to [(B \to C) \to (A \to C)]$	Ded. Rule 7

□

- We have just proved:

Transitivity Rule

$$\frac{U \vdash A \to B, \quad U \vdash B \to C}{U \vdash A \to C}$$

Theorem

$$\vdash [A \to (B \to C)] \to [B \to (A \to C)]$$

Proof.

1.	$\{A \to (B \to C), B, A\} \vdash A$	Assumption
2.	$\{A \to (B \to C), B, A\} \vdash A \to (B \to C)$	Assumption
3.	$\{A \to (B \to C), B, A\} \vdash B \to C$	MP 1,2
4.	$\{A \to (B \to C), B, A\} \vdash B$	Assumption
5.	$\{A \to (B \to C), B, A\} \vdash C$	MP 4,3
6.	$\{A \to (B \to C), B\} \vdash A \to C$	Ded. Rule 5
7.	$\{A \to (B \to C)\} \vdash B \to (A \to C)$	Ded. Rule 6
8.	$\vdash [A \to (B \to C)] \to [B \to (A \to C)]$	Ded. Rule 7

□

- This proves

Exchange of Antecedent Rule

$$\frac{U \vdash A \rightarrow (B \rightarrow C)}{U \vdash B \rightarrow (A \rightarrow C)}$$

Theorem

$$\vdash \neg A \rightarrow (A \rightarrow B)$$

Proof.

1.	$\{\neg A\} \vdash \neg A \rightarrow (\neg B \rightarrow \neg A)$	Axiom 1
2.	$\{\neg A\} \vdash \neg A$	Assumption
3.	$\{\neg A\} \vdash \neg B \rightarrow \neg A$	MP 2,1
4.	$\{\neg A\} \vdash (\neg B \rightarrow \neg A) \rightarrow (A \rightarrow B)$	Axiom 3
5.	$\{\neg A\} \vdash A \rightarrow B$	MP 3,4
6.	$\vdash \neg A \rightarrow (A \rightarrow B)$	Ded. Rule 5

□

One consequence of the preceding theorem is the following:

Corollary

$$\vdash A \rightarrow (\neg A \rightarrow B)$$

Proof.

By the exchange of antecedent rule, applied to

$$\vdash \neg A \rightarrow (A \rightarrow B)$$

$\square$

Double Negation Rule

$$\frac{U \vdash \neg\neg A}{U \vdash A}$$

Proof.

We need to show: $\vdash \neg\neg A \rightarrow A$

1.	$\{\neg\neg A\} \vdash \neg\neg A \rightarrow (\neg\neg\neg\neg A \rightarrow \neg\neg A)$	Axiom 1
2.	$\{\neg\neg A\} \vdash \neg\neg A$	Assumption
3.	$\{\neg\neg A\} \vdash \neg\neg\neg\neg A \rightarrow \neg\neg A$	MP 2,1
4.	$\{\neg\neg A\} \vdash \neg A \rightarrow \neg\neg\neg A$	Contrap. Rule 3
5.	$\{\neg\neg A\} \vdash \neg\neg A \rightarrow A$	Contrap. Rule 4
6.	$\{\neg\neg A\} \vdash A$	MP 2,5
7.	$\vdash \neg\neg A \rightarrow A$	Ded. Rule 6

□

One can prove similarly:

1. $\vdash (A \to B) \to (\neg B \to \neg A)$
2. $\vdash A \to \neg\neg A$

- Notation:

$$\begin{aligned} \textit{false} &= \text{ any contradictory formula, e.g. } \neg(p \to p) \\ \textit{true} &= \text{ any valid formula, e.g. } p \to p \end{aligned}$$

Reduction to Contradiction Rule

$$\frac{U \vdash \neg A \rightarrow \mathit{false}}{U \vdash A}$$

- So, we need to prove in $\mathcal{H}$:

$$\vdash (\neg A \rightarrow \mathit{false}) \rightarrow A$$

Proof.

1.	$\{\neg A \rightarrow \mathit{false}\} \vdash \neg A \rightarrow \mathit{false}$	Assumption
2.	$\{\neg A \rightarrow \mathit{false}\} \vdash \neg \mathit{false} \rightarrow \neg\neg A$	Contrap. Rule 1
3.	$\{\neg A \rightarrow \mathit{false}\} \vdash \neg\neg A \rightarrow A$	Double. Neg. Rule
4.	$\{\neg A \rightarrow \mathit{false}\} \vdash \neg \mathit{false} \rightarrow A$	Transitivity 2,3
5.	$\{\neg A \rightarrow \mathit{false}\} \vdash p \rightarrow p$	Proved earlier
6.	$\{\neg A \rightarrow \mathit{false}\} \vdash \neg\neg(p \rightarrow p)$	Double Neg. 5
7.	$\{\neg A \rightarrow \mathit{false}\} \vdash A$	MP 6,4
8.	$\vdash (\neg A \rightarrow \mathit{false}) \rightarrow A$	Ded. Rule 7

□

We can now introduce the remaining logical connectives $\wedge$, $\vee$, $\leftrightarrow$ into our proof system $\mathcal{H}$ as abbreviations for certain equivalent formulas that use $\neg$ and $\rightarrow$ only.

$A \wedge B$	means	$\neg(A \rightarrow \neg B)$
$A \vee B$	means	$\neg A \rightarrow B$
$A \leftrightarrow B$	means	$(A \rightarrow B) \wedge (B \rightarrow A)$
		(or: $\neg((A \rightarrow B) \rightarrow \neg(B \rightarrow A))$)

Example

Prove

$$\vdash A \rightarrow (B \rightarrow A \wedge B)$$

Solution:

1.	$\{A, B\} \vdash (A \rightarrow \neg B) \rightarrow (A \rightarrow \neg B)$	Proved earlier
2.	$\{A, B\} \vdash A \rightarrow ((A \rightarrow \neg B) \rightarrow \neg B)$	Exch. Antec. 1
3.	$\{A, B\} \vdash A$	Assumption
4.	$\{A, B\} \vdash (A \rightarrow \neg B) \rightarrow \neg B$	MP 3,2
5.	$\{A, B\} \vdash \neg\neg B \rightarrow \neg(A \rightarrow \neg B)$	Contrap. Rule 4
6.	$\{A, B\} \vdash B \rightarrow \neg\neg B$	Double Neg.
7.	$\{A, B\} \vdash B \rightarrow \neg(A \rightarrow \neg B)$	Transitivity 6,5

8.	$\{A, B\} \vdash B$	Assumption
9.	$\{A, B\} \vdash \neg(A \rightarrow \neg B)$	MP 8,7
10.	$\{A\} \vdash B \rightarrow \neg(A \rightarrow \neg B)$	Ded. Rule 9
11.	$\vdash A \rightarrow (B \rightarrow \neg(A \rightarrow \neg B))$	Ded. Rule 10

$\square$

Example

Prove

$$\vdash A \vee B \leftrightarrow B \vee A$$

Solution: It suffices to show

$$\vdash A \vee B \rightarrow B \vee A, \text{ and}$$
$$\vdash B \vee A \rightarrow A \vee B$$

1.	$\{\neg A \rightarrow B, \neg B\} \vdash \neg A \rightarrow B$	Assumption
2.	$\{\neg A \rightarrow B, \neg B\} \vdash \neg B \rightarrow \neg\neg A$	Contrap. Rule 1
3.	$\{\neg A \rightarrow B, \neg B\} \vdash \neg B$	Assumption
4.	$\{\neg A \rightarrow B, \neg B\} \vdash \neg\neg A$	MP 3,2
5.	$\{\neg A \rightarrow B, \neg B\} \vdash \neg\neg A \rightarrow A$	Double Neg.
6.	$\{\neg A \rightarrow B, \neg B\} \vdash A$	MP 4,5
7.	$\{\neg A \rightarrow B\} \vdash \neg B \rightarrow A$	Ded. Rule 6
8.	$\vdash (\neg A \rightarrow B) \rightarrow (\neg B \rightarrow A)$	Ded. Rule 7

The other implication has an analogous proof. $\square$

3.4 Soundness and Completeness; Consistency

Theorem

Hilbert system $\mathcal{H}$ is sound; i.e.

$$\text{if } \vdash A \text{ then } \models A$$

Proof.

By induction on the length n of the proof $\vdash A$.

- If $n = 1$, A is an axiom, and every axiom is a valid formula

- If $n > 1$, then A is derived from two previous lines of the proof using Modus Ponens:

$$\begin{array}{lll} \vdots & & \\ \text{i.} & \vdash B & \\ \vdots & & \\ \text{n-1.} & \vdash B \to A & \\ \text{n.} & \vdash A & \text{MP } i, n-1 \end{array}$$

By inductive hypothesis:

$$\models B, \qquad \models B \to A$$

so A must be valid, too. □

Theorem

Hilbert system $\mathcal{H}$ is complete; i.e.

$$\textit{if } \models A \textit{ then } \vdash A$$

Definition

A set of formulas is inconsistent if, for **some** formula A,

$$U \vdash A \text{ and } U \vdash \neg A$$

Theorem

*A set of formulas U is inconsistent if, and only if, **for all** formulas A,*

$$U \vdash A$$

Proof.

($\Longrightarrow$) Suppose U is an inconsistent set of formulas. Then, for some formula A,

$$U \vdash A, \qquad U \vdash \neg A$$

We have proved earlier that, for **any** formula B

$$U \vdash \neg A \rightarrow (A \rightarrow B)$$

(reduction to contradiction)

1.	$U \vdash \neg A \rightarrow (A \rightarrow B)$	Contrad. Rule
2.	$U \vdash \neg A$	given
3.	$U \vdash A \rightarrow B$	MP 1,2
4.	$U \vdash A$	given
5.	$U \vdash B$	MP 4,3

So, **all** formulas B are logical consequences of U.

($\Longleftarrow$) Suppose that every formula A is a consequence of U. Then, for any formula B, we have both

$$U \vdash B \text{ and } U \vdash \neg B$$

which shows that U is inconsistent. $\square$

- So, if there is a propositional formula in the proof system which is not valid, the proof system will be consistent.

Theorem
$U \vdash A$ if and only if $U \cup \{\neg A\}$ is inconsistent.

Proof. ($\Longrightarrow$) Suppose $U \vdash A$. Since

$$U \cup \{\neg A\} \vdash \neg A$$
$$U \cup \{\neg A\} \vdash A$$

the set $U \cup \{\neg A\}$ is inconsistent.

($\Longleftarrow$) Suppose $U \cup \{\neg A\}$ is an inconsistent set of formulas. Then, since any formula can be derived from $U \cup \{\neg A\}$,

1.	$U \cup \{\neg A\} \vdash \mathit{false}$	given
2.	$U \vdash \neg A \rightarrow \mathit{false}$	Ded. Rule 1
3.	$U \vdash \neg \mathit{false} \rightarrow \neg\neg A$	Contrap. Rule 2
4.	$U \vdash \neg \mathit{false}$	Proved earlier
5.	$U \vdash \neg\neg A$	MP 4,3
6.	$U \vdash A$	Double Neg. 5

$\square$

- All of these facts also apply to the case when U is an **infinite** set of formulas.

- Note the following: an infinite set of formulas is consistent if and only if every finite subset is consistent.

Compactness Theorem

Theorem

(Compactness Theorem) An infinite set of propositional formulas U is satisfiable if and only if every finite subset of U is satisfiable.

Example

(Graph Colorability Problem) We say that a (possibly, infinite) graph G is *n-colorable*, if every vertex of G can be assigned one of the n different colors

$$\{c_1, c_2, \ldots, c_n\}$$

in such a way that no two vertices joined by an edge are assigned the same color.
Given an infinite graph G and some positive integer $n > 1$, show that, if every finite induced subgraph of G is n-colorable, then so is G.

Solution: We will try to capture the n-colorability property using the language of propositional logic.
Suppose $G = (V, E)$, where

$$V = \{v_1, v_2, \dots, v_m, \dots\}.$$

We need to express two properties:

1. Every vertex v_i is assigned exactly one of the colors c_j ($j = 1, \dots, n$).
2. If $(v_k, v_l) \in E$ is an edge of the graph G, then the colors assigned to v_k and v_l have to be different.

First, we introduce infinitely many propositional atoms

$$p_{i,j}, \qquad i = 1, 2, \ldots \quad j = 1, 2, \ldots, n$$

whose meaning will be the following:
" The variable $p_{i,j}$ is true if the vertex v_i is assigned the color c_j in a coloring of G."
Then, our two requirements can be coded as follows:

1. For every $i = 1, 2, \ldots$, we include the formula

$$(p_{i,1} \wedge \neg p_{i,2} \wedge \ldots \wedge \neg p_{i,n}) \vee \ldots \vee (\neg p_{i,1} \wedge \neg p_{i,2} \wedge \ldots \wedge \neg p_{i,n-1} \wedge p_{i,n})$$

2. For every edge $(v_k, v_l) \in E$, we include the formula

$$\neg(p_{k,1} \wedge p_{l,1}) \wedge \neg(p_{k,2} \wedge p_{l,2}) \wedge \ldots \wedge \neg(p_{k,n} \wedge p_{l,n})$$

Let U denote the infinite set of formulas obtained in this way. Clearly, G is n-colorable if and only if U is satisfiable.

To show that U is satisfiable, we will use the Compactness Theorem. So, it suffices to show that every finite subset of U is satisfiable.

Let U_0 be a finite subset of U. Obviously, the formulas in U_0 can mention only finitely many vertices of G.

Let G_0 be the induced subgraph of G whose vertices are those mentioned by U_0. Then, G_0 is a finite induced subgraph of G and is n-colorable, by the assumption made about G.

So, the set of formulas U_0 is satisfiable, which is precisely what we were trying to show.

Therefore, U is satisfiable as an infinite set, so G is an n-colorable graph. $\square$.