RYERSON UNIVERSITY
MTH 714
ASSIGNMENT #1 - SOLUTIONS

1. Any valid formula A would meet the requirements of the problem, e.g.

$$A = p \vee \neg p.$$

Namely, for any truth assignment v, if $v(A) = T$,

$$v(((A \wedge q) \to \neg p) \to ((p \to \neg q) \to A)) = T$$

2. Notice the following:

$$\neg p \equiv p \leftrightarrow false$$

Namely, when $v(p) = T$, $v(p \leftrightarrow false) = F$ and when $v(p) = F$, we have $v(p \leftrightarrow false) = T$.

Now, $\{\neg, \vee\}$ form an adequate set of connectives, and since $\neg$ can be expressed in terms of $\leftrightarrow$ and $false$, the set of connectives $\{\vee, \leftrightarrow, false\}$ is also adequate.

3. Prove that

$$\vdash ((\neg B \to \neg A) \to A) \to A$$

(a) We construct a semantic tableau for

$$(1) \quad \neg[((\neg B \to \neg A) \to A) \to A]$$

The descendent node contains the formulas

$$(2) \quad (\neg B \to \neg A) \to A, \neg A$$

Since an implication creates two branching possibilities, we get two new descendant nodes:

$$(3) \neg(\neg B \to \neg A), \neg A \quad \text{and} \quad (4) A, \neg A$$

Now, we can mark the node (4) as closed. The node (3) produces a single descendant

$$(5) \neg B, \neg\neg A, \neg A$$

which in turn creates the descendant node

$$(6) \neg B, A, \neg A$$

which is also closed.

Since all branches of the tableau lead to closed leaves, the negation of the original formula is unsatisfiable and, therefore,

$$((\neg B \rightarrow \neg A) \rightarrow A) \rightarrow A$$

is valid.

(b) In the Gentzen axiom system $\mathcal{G}$, the proof of this formula , using the tableau constructed in (a), has the following form:

1.	$B, \neg A, A$	axiom
2.	$\neg B \rightarrow \neg A, A$	α-rule 1
3.	$\neg A, A$	axiom
4.	$\neg((\neg B \rightarrow \neg A) \rightarrow A), A$	β-rule 2,3
5.	$((\neg B \rightarrow \neg A) \rightarrow A) \rightarrow A$	α-rule 4

4. (a)

1.	$\{\neg B \rightarrow \neg A, \neg B \rightarrow A\} \vdash \neg B \rightarrow \neg A$	Assumption
2.	$\{\neg B \rightarrow \neg A, \neg B \rightarrow A\} \vdash \neg B \rightarrow A$	Assumption
3.	$\{\neg B \rightarrow \neg A, \neg B \rightarrow A\} \vdash A \rightarrow B$	Contrap. Rule 1
4.	$\{\neg B \rightarrow \neg A, \neg B \rightarrow A\} \vdash \neg B \rightarrow B$	Transitivity 2,3
5.	$\{\neg B \rightarrow \neg A, \neg B \rightarrow A\} \vdash (\neg B \rightarrow B) \rightarrow B$	Theorem 3.28
6.	$\{\neg B \rightarrow \neg A, \neg B \rightarrow A\} \vdash B$	MP 4,5
7.	$\{\neg B \rightarrow \neg A\} \vdash (\neg B \rightarrow A) \rightarrow B$	Deduction Rule 6
8.	$\vdash (\neg B \rightarrow \neg A) \rightarrow ((\neg B \rightarrow A) \rightarrow B)$	Deduction Rule 7

(b) The proof of Axiom 3 from the formula

$$(\neg B \rightarrow \neg A) \rightarrow ((\neg B \rightarrow A) \rightarrow B)$$

and Axioms 1 and 2 can be constructed in the following way:

1.	$\{(\neg B \rightarrow \neg A) \rightarrow ((\neg B \rightarrow A) \rightarrow B), \neg B \rightarrow \neg A, A\} \vdash \neg B \rightarrow \neg A$	Assumptic
2.	$\{(\neg B \rightarrow \neg A) \rightarrow ((\neg B \rightarrow A) \rightarrow B), \neg B \rightarrow \neg A, A\} \vdash (\neg B \rightarrow \neg A) \rightarrow ((\neg B \rightarrow A) \rightarrow B)$	Assumptic
3.	$\{(\neg B \rightarrow \neg A) \rightarrow ((\neg B \rightarrow A) \rightarrow B), \neg B \rightarrow \neg A, A\} \vdash (\neg B \rightarrow A) \rightarrow B$	MP 1,2
4.	$\{(\neg B \rightarrow \neg A) \rightarrow ((\neg B \rightarrow A) \rightarrow B), \neg B \rightarrow \neg A, A\} \vdash A \rightarrow (\neg B \rightarrow A)$	Axiom 1
5.	$\{(\neg B \rightarrow \neg A) \rightarrow ((\neg B \rightarrow A) \rightarrow B), \neg B \rightarrow \neg A, A\} \vdash A$	Assumptic
6.	$\{(\neg B \rightarrow \neg A) \rightarrow ((\neg B \rightarrow A) \rightarrow B), \neg B \rightarrow \neg A, A\} \vdash \neg B \rightarrow A$	MP 5,4
7.	$\{(\neg B \rightarrow \neg A) \rightarrow ((\neg B \rightarrow A) \rightarrow B), \neg B \rightarrow \neg A, A\} \vdash B$	MP 6,3
8.	$\{(\neg B \rightarrow \neg A) \rightarrow ((\neg B \rightarrow A) \rightarrow B), \neg B \rightarrow \neg A\} \vdash A \rightarrow B$	Deduction
9.	$\{(\neg B \rightarrow \neg A) \rightarrow ((\neg B \rightarrow A) \rightarrow B)\} \vdash (\neg B \rightarrow \neg A) \rightarrow (A \rightarrow B)$	Deduction

(c) Since Axiom 3 can be proved from the formula

$$(\neg B \rightarrow \neg A) \rightarrow ((\neg B \rightarrow A) \rightarrow B)$$

and Axioms 1 and 2, if we replace Axiom 3 with this new formula, all valid formulas can still be derived. Namely, in the proof of any valid formula in $\mathcal{H}$, whenever we invoke Axiom 3, we can insert the proof above instead. Therefore, the proof system given by Axioms 1 and 2 along with the formula

$$(\neg B \to \neg A) \to ((\neg B \to A) \to B)$$

is still complete for propositional logic.

5. (a) Number the formulas in the set of clauses

$$F = \{pq\bar{r}, \bar{p}, pqr, p\bar{q}\}$$

as (1)-(4).
Then, one refutation is e.g.

5.	pq	Res 1,3
6.	p	Res 5,4
7.	$\square$	Res 6,2

(b) The negation of the formula

$$A = (\neg p \wedge \neg q \wedge r) \vee (\neg p \wedge \neg r) \vee (q \wedge r) \vee p$$

is equivalent to:

$$\neg A \equiv (p \vee q \vee \neg r) \wedge (p \vee r) \wedge (\neg q \vee \neg r) \wedge \neg p$$

The corresponding clausal form is:

$$\{(1)pq\bar{r}, (2)pr, (3)\overline{qr}, (4)\bar{p}\}$$

One refutation of this set of clauses is:

5.	$q\bar{r}$	Res 1,4
6.	$\bar{r}$	Res 5,3
7.	p	Res 2,6
8.	$\square$	Res 4,7