1 Components and Projections

Given two vectors \( u \) and \( v \), we can ask how far we will go in the direction of \( v \) when we travel along \( u \). The distance we travel in the direction of \( v \), while traversing \( u \) is called the component of \( u \ with respect to \ v \) and is denoted \( \text{comp}_v u \). The vector parallel to \( v \), with magnitude \( \text{comp}_v u \), in the direction of \( v \) is called the projection of \( u \ onto \ v \) and is denoted \( \text{proj}_v u \).

\[
\text{So, } \text{comp}_v u = ||\text{proj}_v u||
\]

Note \( \text{proj}_v u \) is a vector and \( \text{comp}_v u \) is a scalar. From the picture \( \text{comp}_v u = ||u|| \cos \theta \)

We wish to find a formula for the projection of \( u \) onto \( v \).

Consider \( u \cdot v = ||u|| ||v|| \cos \theta \)

Thus \( ||u|| \cos \theta = \frac{u \cdot v}{||v||} \)

So \( \text{comp}_v u = \frac{u \cdot v}{||v||} \)

The unit vector in the same direction as \( v \) is given by \( \frac{v}{||v||} \). So

\[
\text{proj}_v u = \left( \frac{u \cdot v}{||v||^2} \right) v
\]

Example 1

1. Find the projection of \( u = i + 2j \) onto \( v = i + j \).

\[
u \cdot v = 1 + 2 = 3, \quad ||v||^2 = \left( \sqrt{2} \right)^2 = 2\]

\[
\text{proj}_v u = \left( \frac{u \cdot v}{||v||^2} \right) v = \frac{3}{2}(i + j) = \frac{3}{2}i + \frac{3}{2}j
\]
2. Find $\text{proj}_v u$, where $u = (1, 2, 1)$ and $v = (1, 1, 2)$

$$u \cdot v = 1 + 2 + 2 = 5, \quad ||v||^2 = \left(\sqrt{1^2 + 1^2 + 2^2}\right)^2 = 6$$

So, $\text{proj}_v u = \frac{5}{6}(1, 1, 2)$

3. Find the component of $u = i + j$ in the direction of $v = 3i + 4j$.

$$u \cdot v = 3 + 4 = 7, \quad ||v|| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

$$\text{comp}_v u = \frac{u \cdot v}{||v||} = \frac{7}{5}$$

4. Find the components of $u = i + 3j - 2k$ in the directions $i$, $j$ and $k$.

$$u \cdot i = 1, \quad u \cdot j = 3, \quad u \cdot k = -2, \quad ||i|| = ||j|| = ||k|| = 1$$

So

$$\text{comp}_i u = 1, \quad \text{comp}_j u = 3, \quad \text{comp}_k u = -2.$$ 

So the use of the term *component* is justified in this context.

Indeed, coordinate axes are arbitrarily chosen and are subject to change.

If $u$ is a new coordinate vector given in terms of the old set then $\text{comp}_u w$ gives the component of the vector $w$ in the new coordinate system.

**Example 2**

If coordinates in the plane are rotated by 45°, the vector $i$ is mapped to $u = \frac{1}{\sqrt{2}} i + \frac{1}{\sqrt{2}} j$, and the vector $j$ is mapped to $v = -\frac{1}{\sqrt{2}} i + \frac{1}{\sqrt{2}} j$. Find the components of $w = 2i - 5j$ with respect to the new coordinate vectors $u$ and $v$. i.e. Express $w$ in terms of $u$ and $v$.

$$w \cdot u = \frac{-3}{\sqrt{2}}, \quad w \cdot v = \frac{-7}{\sqrt{2}}, \quad ||u|| = ||v|| = 1$$

So

$$\text{comp}_u w = \frac{-3}{\sqrt{2}}, \quad \text{comp}_v w = \frac{-7}{\sqrt{2}}$$

and

$$w = \frac{-3}{\sqrt{2}} u + \frac{-7}{\sqrt{2}} v$$
2 Orthogonal Projections

Given a non-zero vector \( \mathbf{v} \), we may represent any vector \( \mathbf{u} \) as a sum of a vector, \( \mathbf{u}_\parallel \) parallel to \( \mathbf{v} \) and a vector \( \mathbf{u}_\perp \) perpendicular to \( \mathbf{v} \).

So, \[
\mathbf{u} = \mathbf{u}_\parallel + \mathbf{u}_\perp.
\]

Now, \[
\mathbf{u}_\parallel = \text{proj}_\mathbf{v} \mathbf{u},
\]

and so \[
\mathbf{u}_\perp = \mathbf{u} - \text{proj}_\mathbf{v} \mathbf{u}.
\]

Example 3

Express \( \mathbf{u} = 2\mathbf{i} + 4\mathbf{j} + 2\mathbf{k} \) as a sum of vectors parallel and perpendicular to \( \mathbf{v} = \mathbf{i} + 2\mathbf{j} - \mathbf{k} \).

\[
\mathbf{u} \cdot \mathbf{v} = 2 + 8 - 2 = 8, \quad ||\mathbf{v}||^2 = (\sqrt{1^2 + 2^2 + 1^2})^2 = 6
\]

\[
\mathbf{u}_\parallel = \text{proj}_\mathbf{v} \mathbf{u} = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{v}||^2} \right) \mathbf{v} = \frac{4}{3}(\mathbf{i} + 2\mathbf{j} - \mathbf{k})
\]

\[
\mathbf{u}_\perp = \mathbf{u} - \text{proj}_\mathbf{v} \mathbf{u} = (2\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}) - \frac{4}{3}(\mathbf{i} + 2\mathbf{j} - \mathbf{k})
\]

\[
= \left( 2 - \frac{4}{3} \right) \mathbf{i} + \left( 4 - \frac{8}{3} \right) \mathbf{j} + \left( 2 + \frac{4}{3} \right) \mathbf{k}
\]

\[
= \frac{2}{3}\mathbf{i} + \frac{4}{3}\mathbf{j} + \frac{10}{3}\mathbf{k}
\]

\[
= \frac{2}{3}(\mathbf{i} + 2\mathbf{j} + 5\mathbf{k})
\]

Check

\[
\mathbf{u}_\parallel \cdot \mathbf{u}_\perp = \left( \frac{2}{3}(\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}) \right) \cdot \left( \frac{2}{3}(\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \right)
\]

\[
= \frac{8}{9} ((\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}) \cdot (\mathbf{i} + 2\mathbf{j} - \mathbf{k}))
\]

\[
= \frac{8}{9} (1 + 4 - 5)
\]

\[
= 0
\]

So \( \mathbf{u}_\parallel \) and \( \mathbf{u}_\perp \) are orthogonal.