Homogeneous Equations

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Theorem 1 Given a system of \( m \) equations in \( n \) unknowns, let \( B \) be the \( m \times (n + 1) \) augmented matrix. Recall \( r \) is the number of leading ones in the REF of \( B \), also the number of parameters in a solution is \( n - r \).

- If \( r = n \), there is a unique solution (no parameters in the solution).
- If \( r > n \) (so \( r = n + 1 \)) the system is inconsistent (no solution).
- If \( r < n \), either the system is inconsistent (no solution) or an \( n - r \)-parameter solution.
  - In this case, the difference is determined only by the values of the constants (the \( b_i \)).

1 Homogeneous Systems

Given a system of \( m \) equations in \( n \) unknowns

\[
\begin{align*}
  a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
  a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
  \vdots & \quad \vdots \\
  a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m
\end{align*}
\]

If all of the constant terms are zero, i.e. \( b_i = 0 \) for \( i = 1, \ldots m \) the corresponding system of equations is called a homogeneous system of equations.

Example 2

\[
\begin{align*}
  x_1 + 2x_2 - 3x_3 + x_4 &= 0 \\
  x_2 + x_3 - 3x_4 &= 0 \\
  x_1 + x_2 + x_3 + x_4 &= 0
\end{align*}
\]

A homogeneous system of equations always has the solution

\[
 x_1 = x_2 = \ldots = x_n = 0
\]

This is called the Trivial Solution.

Since a homogeneous system always has a solution (the trivial solution), it can never be inconsistent. Thus a homogeneous system of equations always either has a unique solution or an infinite number of solutions.

Theorem 3 If \( n > m \) then a homogeneous system of equations has infinitely many solutions.

Example 4
1. 

\[
\begin{align*}
  x_1 + x_2 + x_3 &= 0 \\
  x_1 + 2x_2 + x_3 &= 0 \\
  x_1 + x_2 + 2x_3 &= 0 \\
\end{align*}
\]

\[
\begin{pmatrix}
  1 & 1 & 1 & 0 \\
  1 & 2 & 1 & 0 \\
  1 & 1 & 2 & 0 \\
\end{pmatrix}
\]

\[R_2 \rightarrow R_2 - R_1\]

\[R_3 \rightarrow R_3 - R_1\]

Write back:

\[
\begin{align*}
  x_1 + x_2 + x_3 &= 0 \\
  x_2 + x_3 &= 0 \\
  x_3 &= 0 \\
\end{align*}
\]

So the trivial solution \((x_1, x_2, x_3) = (0, 0, 0)\) is the only solution.

2. 

\[
\begin{align*}
  x_1 + x_2 + x_3 &= 0 \\
  x_1 + 2x_2 + x_3 &= 0 \\
  2x_1 + 3x_2 + 2x_3 &= 0 \\
\end{align*}
\]

\[
\begin{pmatrix}
  1 & 1 & 1 & 0 \\
  1 & 2 & 1 & 0 \\
  2 & 3 & 2 & 0 \\
\end{pmatrix}
\]

\[R_2 \rightarrow R_2 - R_1\]

\[R_3 \rightarrow R_3 - 2R_1\]

\[
\begin{pmatrix}
  1 & 1 & 1 & 0 \\
  0 & 1 & 1 & 0 \\
  0 & 1 & 1 & 0 \\
\end{pmatrix}
\]

\[R_3 \rightarrow R_3 - R_2\]

Write back:

\[
\begin{align*}
  x_1 + x_2 + x_3 &= 0 \\
  x_2 + x_3 &= 0 \\
  0 &= 0 \\
\end{align*}
\]

Which has the 1-parameter solution:

Let \(t \in \mathbb{R}, x_3 = t, x_2 = -t, x_1 = 0\).

Or \((x_1, x_2, x_3) = (0, -t, t)\).