Linear Combination

Definition 1 Given a set of vectors \( \{ v_1, v_2, \ldots, v_k \} \) in a vector space \( V \), any vector of the form

\[
v = a_1 v_1 + a_2 v_2 + \ldots + a_k v_k
\]

for some scalars \( a_1, a_2, \ldots, a_k \), is called a linear combination of \( v_1, v_2, \ldots, v_k \).
Example 2

1. Let $v_1 = (1, 2, 3), v_2 = (1, 0, 2)$.

(a) Express $u = (-1, 2, -1)$ as a linear combination of $v_1$ and $v_2$.

We must find scalars $a_1$ and $a_2$ such that $u = a_1v_1 + a_2v_2$.

Thus

$$
\begin{align*}
    a_1 + a_2 &= -1 \\
    2a_1 + 0a_2 &= 2 \\
    3a_1 + 2a_2 &= -1
\end{align*}
$$

This is 3 equations in the 2 unknowns $a_1$, $a_2$. Solving for $a_1$, $a_2$:

$$
\begin{pmatrix}
1 & 1 & \vdash & -1 \\
2 & 0 & \vdash & 2 \\
3 & 2 & \vdash & -1
\end{pmatrix}
\quad
\begin{align*}
    R_2 & \rightarrow R_2 - 2R_1 \\
    R_3 & \rightarrow R_3 - 3R_1
\end{align*}
$$

So $a_2 = -2$ and $a_1 = 1$. 
Note that the components of \( v_1 \) are the coefficients of \( a_1 \) and the components of \( v_2 \) are the coefficients of \( a_2 \), so the initial coefficient matrix looks like
\[
\begin{pmatrix}
v_1 & v_2 & u
\end{pmatrix}
\]

(b) Express \( u = (-1, 2, 0) \) as a linear combination of \( v_1 \) and \( v_2 \).

We proceed as above, augmenting with the new vector.

\[
\begin{pmatrix}
1 & 1 & -1 \\
2 & 0 & 2 \\
3 & 2 & 0 \\
1 & 1 & -1 \\
0 & -2 & 4 \\
0 & -1 & 3
\end{pmatrix}
\]

\[
R_2 \rightarrow R_2 - 2R_1 \\
R_3 \rightarrow R_3 - 3R_1
\]

This system has no solution, so \( u \) cannot be expressed as a linear combination of \( v_1 \) and \( v_2 \). i.e. \( u \) does not lie in the plane generated by \( v_1 \) and \( v_2 \).
2. Let \( v_1 = (1, 2), \ v_2 = (0, 1), \ v_3 = (1, 1). \)

Express \((1, 0)\) as a linear combination of \(v_1, v_2\) and \(v_3\).

\[
\begin{pmatrix} 1 & 0 & 1 & | & 1 \\ 2 & 1 & 1 & | & 0 \\ 1 & 0 & 1 & | & 1 \\ 0 & 1 & -1 & | & -1 \end{pmatrix}
\]

Let \(a_3 = t, \ a_2 = -1 + t, \ a_3 = 1 - t.\)

This system has multiple solutions. In this case there are multiple possibilities for the \(a_i\). Note that \(v_3 = v_1 - v_2\), which means that \(a_3v_3\) can be replaced by \(a_3(v_1 - v_2)\), so \(v_3\) is redundant.
## Span

**Definition 3** Given a set of vectors \( \{v_1, v_2, \ldots, v_k\} \) in a vector space \( V \), the set of all vectors which are a linear combination of \( v_1, v_2, \ldots, v_k \) is called the span of \( \{v_1, v_2, \ldots, v_k\} \). i.e.

\[
\text{span}\{v_1, v_2, \ldots, v_k\} = \{v \in V | v = a_1v_1 + a_2v_2 + \ldots + a_kv_k\}
\]

**Definition 4** Given a set of vectors \( S = \{v_1, v_2, \ldots, v_k\} \) in a vector space \( V \), \( S \) is said to span \( V \) if \( \text{span}(S) = V \)

In the first case the word \text{span} is being used as a noun, \( \text{span}\{v_1, v_2, \ldots, v_k\} \) is an object.

In the second case the word \text{span} is being used as a verb, we ask whether \( \{v_1, v_2, \ldots, v_k\} \) span the space \( V \).
Example 5

1. Find $\text{span}\{v_1, v_2\}$, where $v_1 = (1, 2, 3)$ and $v_2 = (1, 0, 2)$.

$\text{span}\{v_1, v_2\}$ is the set of all vectors $(x, y, z) \in \mathbb{R}^3$ such that $(x, y, z) = a_1(1, 2, 3) + a_2(1, 0, 2)$. We wish to know for what values of $(x, y, z)$ does this system of equations have solutions for $a_1$ and $a_2$.

\[
\begin{pmatrix}
1 & 1 & x \\
2 & 0 & y \\
3 & 2 & z
\end{pmatrix}
\begin{pmatrix}
1 & 1 \\
0 & 2 \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
\]

\[
R_2 \rightarrow R_2 - 2R_1 \\
R_3 \rightarrow R_3 - 3R_1
\]

\[
\begin{pmatrix}
1 & 1 & x \\
0 & -2 & y - 2x \\
0 & -1 & z - 3x
\end{pmatrix}
\begin{pmatrix}
1 & 1 \\
0 & 1 \\
0 & -1
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
\]

\[
R_2 \rightarrow -\frac{1}{2}R_2 \\
R_3 \rightarrow R_3 + R_2
\]

So solutions when $4x + y - 2z = 0$. Thus $\text{span}\{v_1, v_2\}$ is the plane $4x + y - 2z = 0$. 
2. Show that $i = e_1 = (1, 0)$ and $j = e_2 = (0, 1)$ span $\mathbb{R}^2$.

We are being asked to show that any vector in $\mathbb{R}^2$ can be written as a linear combination of $i$ and $j$.

$$(x, y) = a(1, 0) + b(0, 1)$$ has solution $a = x$, $b = y$ for every $(x, y) \in \mathbb{R}^2$. 
3. Show that \( v_1 = (1, 1) \) and \( v_2 = (2, 1) \) span \( \mathbb{R}^2 \).

We are being asked to show that any vector in \( \mathbb{R}^2 \) can be written as a linear combination of \( v_1 \) and \( v_2 \).

Consider \((a, b) \in \mathbb{R}^2\) and \((a, b) = s(1, 1) + t(2, 1)\).

\[
\begin{pmatrix}
1 & 2 & | & a \\
1 & 1 & | & b \\
0 & 1 & | & a - b \\
1 & 0 & | & -a + 2b \\
0 & 1 & | & a - b
\end{pmatrix}
\]

Which has the solution \( s = 2b - a \) and \( t = a - b \) for every \((a, b) \in \mathbb{R}^2\).

Note that these two vectors span \( \mathbb{R}^2 \), that is every vector in \( \mathbb{R}^2 \) can be expressed as a linear combination of them, but they are not orthogonal.
4. Show that \( v_1 = (1, 1) \), \( v_2 = (2, 1) \) and \( v_3 = (3, 2) \) span \( \mathbb{R}^2 \).

Since \( v_1 \) and \( v_2 \) span \( \mathbb{R}^2 \), any set containing them will as well. We will get infinite solutions for any \((a, b) \in \mathbb{R}^2\).

In general

1. Any set of vectors in \( \mathbb{R}^2 \) which contains two non colinear vectors will span \( \mathbb{R}^2 \).

2. Any set of vectors in \( \mathbb{R}^3 \) which contains three non coplanar vectors will span \( \mathbb{R}^3 \).

3. Two non-colinear vectors in \( \mathbb{R}^3 \) will span a plane in \( \mathbb{R}^3 \).

Want to get the smallest spanning set possible.
Linear Independence

Definition 6 Given a set of vectors \( \{v_1, v_2, \ldots, v_k\} \), in a vector space \( V \), they are said to be linearly independent if the equation

\[
c_1v_1 + c_2v_2 + \ldots + c_kv_k = 0
\]

has only the trivial solution.

If \( \{v_1, v_2, \ldots, v_k\} \) are not linearly independent they are called linearly dependent.

Note \( \{v_1, v_2, \ldots, v_k\} \) is linearly dependent if and only if some \( v_i \) can be expressed as a linear combination of the rest.
Example 7

1. Determine whether \( \mathbf{v}_1 = (1, 2, 3) \) and \( \mathbf{v}_2 = (1, 0, 2) \) are linearly dependent or independent.

Consider the Homogeneous system

\[
c_1(1, 2, 3) + c_2(1, 0, 2) = (0, 0, 0)
\]

\[
\begin{pmatrix}
1 & 1 & 0 \\
2 & 0 & 0 \\
3 & 2 & 0
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

Only solution is the trivial solution \( a_1 = a_2 = 0 \), so linearly independent.
2. Determine whether \( \mathbf{v}_1 = (1, 1, 0) \) and \( \mathbf{v}_2 = (1, 0, 1) \) and \( \mathbf{v}_3 = (3, 1, 2) \) are linearly dependent.

Want to find solutions to the system of equations

\[
c_1(1, 1, 0) + c_2(1, 0, 1) + c_3(3, 1, 2) = (0, 0, 0)
\]

Which is equivalent to solving

\[
\begin{pmatrix}
1 & 1 & 3 \\
1 & 0 & 1 \\
0 & 1 & 2
\end{pmatrix}
\begin{pmatrix}
c_1 \\
c_2 \\
c_3
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 1 & 3 \\
1 & 0 & 1 \\
0 & 1 & 2
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 1 & 3 \\
0 & 1 & 2 \\
0 & 0 & 0
\end{pmatrix}
\]
Example 8

Determine whether $v_1 = (1,1,1)$, $v_2 = (2,2,2)$ and $v_3 = (1,0,1)$ are linearly dependent or independent.

$$2(1,1,1) - (2,2,2) = (0,0,0)$$

So linearly dependent.

Theorem 9 Given two vectors in a vector space $V$, they are linearly dependent if and only if they are multiples of one another, i.e. $v_1 = cv_2$ for some scalar $c$.

Proof:

$$av_1 + bv_2 = 0 \Leftrightarrow v_2 = \left(\frac{-a}{b}\right)v_1$$
Example 10

Determine whether $v_1 = (1, 1, 3)$ and $v_2 = (1, 3, 1)$, $v_3 = (3, 1, 1)$ and $v_4 = (3, 3, 3)$ are linearly dependent.

Must solve $Ax = 0$, where $A = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 \end{pmatrix}$

\[
\begin{pmatrix}
1 & 1 & 3 & 3 & 0 \\
1 & 3 & 1 & 3 & 0 \\
3 & 1 & 1 & 3 & 0
\end{pmatrix}
\]

Since the number of columns is greater than the number of rows, we can see immediately that this system will have infinite solutions.

**Theorem 11** Given $m$ vectors in $\mathbb{R}^n$, if $m > n$ they are linearly dependent.

**Theorem 12** A linearly independent set in $\mathbb{R}^n$ has at most $n$ vectors.