Linear Combination

Definition 1 *Given a set of vectors* $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ *in a vector space* V, *any vector of the form*

$$\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_k\mathbf{v}_k$$

for some scalars $a_1, a_2, \ldots, a_k$, *is called a* linear combination *of* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$.

Example 2

1. Let $\mathbf{v}_1 = (1, 2, 3), \mathbf{v}_2 = (1, 0, 2)$.

   (a) Express $\mathbf{u} = (-1, 2, -1)$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$,

   We must find scalars a_1 and a_2 such that $\mathbf{u} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2$.

   Thus

$$\begin{array}{rcrcr} a_1 & + & a_2 & = & -1 \\ 2a_1 & + & 0a_2 & = & 2 \\ 3a_1 & + & 2a_2 & = & -1 \end{array}$$

   This is 3 equations in the 2 unknowns a_1, a_2. Solving for a_1, a_2:

$$\left(\begin{array}{cc|c} 1 & 1 & -1 \\ 2 & 0 & 2 \\ 3 & 2 & -1 \end{array}\right) \quad \begin{array}{rcl} R_2 & \to & R_2 - 2R_1 \\ R_3 & \to & R_3 - 3R_1 \end{array}$$

$$\left(\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & -2 & 4 \\ 0 & -1 & 2 \end{array}\right)$$

   So $a_2 = -2$ and $a_1 = 1$.

Note that the components of $\mathbf{v}_1$ are the coefficients of a_1 and the components of $\mathbf{v}_2$ are the coefficients of a_2, so the initial coefficient matrix looks like $\left(\begin{array}{cc|c} & & \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{u} \\ & & \end{array}\right)$

(b) Express $\mathbf{u} = (-1, 2, 0)$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

We proceed as above, augmenting with the new vector.

$$\left(\begin{array}{cc|c} 1 & 1 & -1 \\ 2 & 0 & 2 \\ 3 & 2 & 0 \end{array}\right) \quad \begin{array}{rcl} R_2 & \rightarrow & R_2 - 2R_1 \\ R_3 & \rightarrow & R_3 - 3R_1 \end{array}$$

$$\left(\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & -2 & 4 \\ 0 & -1 & 3 \end{array}\right)$$

This system has no solution, so $\mathbf{u}$ cannot be expressed as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. i.e. $\mathbf{u}$ does not lie in the plane generated by $\mathbf{v}_1$ and $\mathbf{v}_2$.

2. Let $\mathbf{v}_1 = (1, 2)$, $\mathbf{v}_2 = (0, 1)$, $\mathbf{v}_3 = (1, 1)$.

   Express $(1, 0)$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$.

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 0 \end{array}\right) \quad R_2 \to R_2 - 2R_1$$
$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -1 \end{array}\right)$$

   Let $a_3 = t$, $a_2 = -1 + t$, $a_3 = 1 - t$.

This system has multiple solutions. In this case there are multiple possibilities for the a_i. Note that $\mathbf{v}_3 = \mathbf{v}_1 - \mathbf{v}_2$, which means that $a_3\mathbf{v}_3$ can be replaced by $a_3(\mathbf{v}_1 - \mathbf{v}_2)$, so $\mathbf{v}_3$ is redundant.

Span

Definition 3 *Given a set of vectors* $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ *in a vector space* V*, the set of all vectors which are a linear combination of* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ *is called the* span *of* $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$*. i.e.*

$$\textit{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} = \\ \{\mathbf{v} \in V \mid \mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_k\mathbf{v}_k\}$$

Definition 4 *Given a set of vectors* $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ *in a vector space* V*,* S *is said to* span V *if span*$(S) = V$

In the first case the word *span* is being used as a noun, span$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is an object.

In the second case the word *span* is being used as a verb, we ask whether $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ san the space V.

Example 5

1. Find span$\{\mathbf{v}_1, \mathbf{v}_2\}$, where $\mathbf{v}_1 = (1,2,3)$ and $\mathbf{v}_2 = (1,0,2)$.

   span$\{\mathbf{v}_1, \mathbf{v}_2\}$ is the set of all vectors $(x,y,z) \in \mathbb{R}^3$ such that $(x,y,z) = a_1(1,2,3) + a_2(1,0,2)$. We wish to know for what values of (x,y,z) does this system of equations have solutions for a_1 and a_2.

$$\left(\begin{array}{cc|c} 1 & 1 & x \\ 2 & 0 & y \\ 3 & 2 & z \end{array}\right) \qquad \begin{array}{rcl} R_2 & \to & R_2 - 2R_1 \\ R_3 & \to & R_3 - 3R_1 \end{array}$$

$$\left(\begin{array}{cc|c} 1 & 1 & x \\ 0 & -2 & y-2x \\ 0 & -1 & z-3x \end{array}\right) \qquad R_2 \to \tfrac{-1}{2} R_2$$

$$\left(\begin{array}{cc|c} 1 & 1 & x \\ 0 & 1 & x - \frac{1}{2}y \\ 0 & -1 & z-3x \end{array}\right) \qquad R_3 \to R_3 + R_2$$

$$\left(\begin{array}{cc|c} 1 & 1 & x \\ 0 & 1 & x - \frac{1}{2}y \\ 0 & 0 & z - 2x - \frac{1}{2}y \end{array}\right)$$

   So solutions when $4x + y - 2z = 0$. Thus span$\{\mathbf{v}_1, \mathbf{v}_2\}$ is the plane $4x + y - 2z = 0$.

2. Show that $\mathbf{i} = \mathbf{e}_1 = (1, 0)$ and $\mathbf{j} = \mathbf{e}_2 = (0, 1)$ span $\mathbb{R}^2$.

   We are being asked to show that any vector in $\mathbb{R}^2$ can be written as a linear combination of $\mathbf{i}$ and $\mathbf{j}$.

   $(x, y) = a(1, 0) + b(0, 1)$ has solution
   $a = x$, $b = y$ for every $(x, y) \in \mathbb{R}^2$.

3. Show that $\mathbf{v}_1 = (1, 1)$ and $\mathbf{v}_2 = (2, 1)$ span $\mathbb{R}^2$.

   We are being asked to show that any vector in $\mathbb{R}^2$ can be written as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.
   Consider $(a, b) \in \mathbb{R}^2$ and $(a, b) = s(1, 1) + t(2, 1)$.

$$\left(\begin{array}{cc|c} 1 & 2 & a \\ 1 & 1 & b \end{array}\right) \qquad R_2 \to R_2 - R_1$$
$$\left(\begin{array}{cc|c} 1 & 2 & a \\ 0 & -1 & b - a \end{array}\right) \qquad R_2 \to -R_2$$
$$\left(\begin{array}{cc|c} 1 & 2 & a \\ 0 & 1 & a - b \end{array}\right) \qquad R_1 \to R_1 - 2R_2$$
$$\left(\begin{array}{cc|c} 1 & 0 & -a + 2b \\ 0 & 1 & a - b \end{array}\right)$$

   Which has the solution $s = 2b - a$ and $t = a - b$ for every $(a, b) \in R^2$.

   Note that these two vectors span $\mathbb{R}^2$, that is every vector in $\mathbb{R}^2$ can be expressed as a linear combination of them, but they are not orthogonal.

4. Show that $\mathbf{v}_1 = (1, 1)$, $\mathbf{v}_2 = (2, 1)$ and $\mathbf{v}_3 = (3, 2)$ span $\mathbb{R}^2$.

   Since $\mathbf{v}_1$ and $\mathbf{v}_2$ span $\mathbb{R}^2$, any set containing them will as well. We will get infinite solutions for any $(a, b) \in \mathbb{R}^2$.

In general

1. Any set of vectors in $\mathbb{R}^2$ which contains two non colinear vectors will span $\mathbb{R}^2$.

2. Any set of vectors in $\mathbb{R}^3$ which contains three non coplanar vectors will span $\mathbb{R}^3$.

3. Two non-colinear vectors in $\mathbb{R}^3$ will span a plane in $\mathbb{R}^3$.

Want to get the smallest spanning set possible.

Linear Independence

Definition 6 *Given a set of vectors* $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$*, in a vector space* V*, they are said to be* linearly independent *if the equation*

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k = \mathbf{0}$$

has only the trivial solution

If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ *are* **not** *linearly independent they are called* linearly dependent*.*

Note $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly dependent if and only if some $\mathbf{v}_i$ can be expressed as a linear combination of the rest.

Example 7

1. Determine whether $\mathbf{v}_1 = (1,2,3)$ and $\mathbf{v}_2 = (1,0,2)$ are linearly dependent or independent.

   Consider the Homogeneous system

   $$c_1(1,2,3) + c_2(1,0,2) = (0,0,0)$$

   $$\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 0 & 0 \\ 3 & 2 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

   Only solution is the trivial solution
   $a_1 = a_2 = 0$, so linearly independent.

2. Determine whether $\mathbf{v}_1 = (1,1,0)$ and $\mathbf{v}_2 = (1,0,1)$ and $\mathbf{v}_3 = (3,1,2)$ are linearly dependent.

   Want to find solutions to the system of equations

   $$c_1(1,1,0) + c_2(1,0,1) + c_3(3,1,2) = (0,0,0)$$

   Which is equivalent to solving

   $$\begin{pmatrix} 1 & 1 & 3 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

   $$\begin{pmatrix} 1 & 1 & 3 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

Example 8

Determine whether $\mathbf{v}_1 = (1, 1, 1)$, $\mathbf{v}_2 = (2, 2, 2)$ and $\mathbf{v}_3 = (1, 0, 1)$ are linearly dependent or independent.

$$2(1, 1, 1) - (2, 2, 2) = (0, 0, 0)$$

So linearly dependent.

Theorem 9 *Given two vectors in a vector space V, they are linearly dependent if and only if they are multiples of one another, i.e. $\mathbf{v}_1 = c\mathbf{v}_2$ for some scalar c.*

Proof:

$$a\mathbf{v}_1 + b\mathbf{v}_2 = \mathbf{0} \Leftrightarrow \mathbf{v}_2 = \left(\frac{-a}{b}\right)\mathbf{v}_1$$

Example 10

Determine whether $\mathbf{v}_1 = (1, 1, 3)$ and $\mathbf{v}_2 = (1, 3, 1)$, $\mathbf{v}_3 = (3, 1, 1)$ and $\mathbf{v}_4 = (3, 3, 3)$ are linearly dependent.

Must solve $A\mathbf{x} = \mathbf{0}$, where $A = \left(\begin{array}{cccc} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \end{array} \right)$

$$\left(\begin{array}{cccc|c} 1 & 1 & 3 & 3 & 0 \\ 1 & 3 & 1 & 3 & 0 \\ 3 & 1 & 1 & 3 & 0 \end{array} \right)$$

Since the number of columns is greater then the number of rows, we can see immediately that this system will have infinite solutions.

Theorem 11 *Given m vectors in $\mathbb{R}^n$, if $m > n$ they are linearly dependent.*

Theorem 12 *A linearly independent set in $\mathbb{R}^n$ has at most n vectors.*