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The Model Companion of Width-Two Orders

ANTHONY BONATO
Department of Pure Mathematics, University of Waterloo, Waterloo, Ontario, Canada, N2L 3G1.
e-mail: acjbonato@barrow.uwaterloo.ca

DEJAN DELIĆ*
Department of Pure Mathematics, University of Waterloo, Waterloo, Ontario, Canada, N2L 3G1.
e-mail: ddelic@barrow.uwaterloo.ca

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Abstract. The class of width-two orders has a model companion. The model companion is complete, decidable, non-finitely axiomatizable, and has continuum many countable models. Generalizations of some results in (Pouzet, M., 1978, *J. Combin. Theory Ser. B* **25**) are presented in the width-n case, for $n > 2$.

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1. Introduction

In [5], Pouzet proved the existence of a countable universal model for the universal class of width-two orders (or bichains). Pouzet introduced the subclass of d-homogeneous width-two orders, and proved that the unique I-connected countable d-homogeneous width-two order, $\mathbb{Q}_2$, is in fact pseudo-homogeneous, in the sense of [2].

In this paper, we apply some of the findings of Pouzet in the width-two case to prove the existence of a model companion for the class of width-two orders. Generalizations are made of some results in [5] to the width-n case, for $n > 2$, where the picture is not as clear as the width-two case. Indeed, as proven in [8], even for $n = 3$, there is no countable universal model for the class of width-n orders. In the first part of this article, we prove that the model companion of width-two orders exists, and its theory is $\mathrm{Th}(\mathbb{Q}_2)$. Some of the properties of this model and its theory are outlined; in particular, in response to question posed by [5], we show that $\mathbb{Q}_2$ is in fact, δ-homogeneous. In the final part, we outline some generalizations of some facts presented in [5] for width-n orders, for $n > 2$. We end the paper with a problem about the companionability of width-n orders, for $n > 2$.

We assume the basic facts of model theory, as, for example, contained in [3]. In particular, we assume the machinery of existentially closed structures and model

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companions as contained in Chapter 8 of [3]. If A and B are two structures over the same language, $A \leq B$ means that A is a substructure of B. We understand an order to be a structure A over the language of one binary, reflexive, anti-symmetric, and transitive relation $\leq$, with equality.

In the width-two case, we adopt the terminology of [5] wherever possible. Further, we attempt to follow the notation of [9] for ordered sets. For lexicographic sums of orders over a chain, we use the phrase "*linear sum*."

2. The Width-Two Case

For the definitions of the terminology used in this section, see [5]. The universal class of all width-n orders is denoted by W_n. The class of existentially closed (e.c.) members of W_n is denoted W_n^{ec}. In departure from [5], define a C-path (or *comparability path*) to be an "oscillating set" of Definition I.2.1 of [5].

In the case of width-one orders, which are chains, the model companion is the class of all dense linear orderings without endpoints. In particular, W_1^{mc} is finitely axiomatizable and $\aleph_0$-categorical; the unique countable e.c. model is $\mathbb{Q}$ (which is also the countably saturated model). As proved in Corollary 2 below, the model companion of W_2 exists. However, unlike W_1, there are continuum many countable existentially closed W_2 orders, and the theory of W_2^{mc} is not finitely axiomatizable (see Corollary 2).

2.1. EXISTENTIALLY CLOSED W_2 ORDERS

Pouzet proved in [5] that an I-connected width-two order A is d-homogeneous if and only if A satisfies certain first-order schemas (H1) and (H2). For clarity, we repeat these schemas.

(H1): All hemispheres of any $a \in A$ are nonempty.

(H2): Let a and b be distinct points in A with disjoint hemispheres P and Q, respectively, on the same chain C of the chain decomposition of A. Then there is a point on C strictly between P and Q.

We extend the definition of the d-metric as follows.

DEFINITION 1. Let $A \in W_2$.

1. Define $d_A(a, b) = \infty$ if there are arbitrarily long C-paths from a to b.
2. For S, T finite subsets of A, we say that S and T are of *finite d-distance* in A if
$$d_A(a, b) < \infty, \quad \text{for all } a \in S \text{ and } b \in T.$$
S and T are of *strong infinite d-distance* in A if
$$d_A(a, b) = \infty, \quad \text{for all } a \in S \text{ and } b \in T.$$

Note that by Lemma I.2.4 of [5], if $d_A(a, b) = \infty$, then a and b cannot be in the same I-component of A.

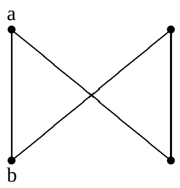

Figure 1.

The converse is clearly false; for example see Figure 1.
We want a condition implying the converse. Namely, if a and b are in different I-components, then $d_A(a, b) = \infty$.

For each $n \geq 2$, consider the following conditions:

(I_n^+): For every $a \in A$, there exists $b > a$, and an increasing sequence $a_0, a_1, a_2, \ldots, a_{n-2}$ of length $n - 2$ with $a = a_0$ and $b = a_{n-2}$, and I-paths of length $i + 2$ connecting a to a_i.

(I_n^-): For every $a \in A$, there exists $b < a$, and a decreasing sequence $a_0, a_1, a_2, \ldots, a_{n-2}$ of length $n - 2$ with $a = a_0$ and $b = a_{n-2}$, and I-paths of length $i + 2$ connecting a to a_i.

Remark 1. Note that any such sequence $a_0, a_1, a_2, \ldots, a_{n-2}$ that satisfies the conditions of either (I_n^+) or (I_n^-) is a C-path.

Remark 2. Define $\mathrm{I} = \{(\mathrm{I}_n^+) : n \geq 2\} \cup \{(\mathrm{I}_n^-) : n \geq 2\}$. Each sentence in (I) is first-order.

Remark 3. $\mathbb{Q}_2$ satisfies (I), so all d-homogeneous width-two orders do as well (since they are all elementarily equivalent as noted by [5]).

Remark 4. One consequence of (I) is that in every I-connected component, given a point a in that component, there are arbitrarily long C-paths (both increasing and decreasing) starting at a in that I-component. Hence, if a and b are in different I-component, then $d_A(a, b) = \infty$.

We need the following lemma, which follows by the definition of pseudo-homogeneity (see [2]), and by a back-and-forth argument. See also [1] and [4].

LEMMA 1 (amalgamating into). *Let $A \in W_2$ so that A is pseudo-homogeneous. Let $A_0 \leq A$ with A_0 finite. Assume that $A_0 \leq A_1 \leq A$, with A_1 finite regular. Assume that $A_1 \leq B$, where B is finite regular. Then there exists $C \leq A$ so that C is finite regular and $A_1 \leq C$, and an isomorphism β: $B \to C$ so that $\beta \restriction A_1$ is the identity on A_1 (and hence, $\beta \restriction A_0$ is the identity on A_0).*

Proposition 1 demonstrates that W_2^{ec} is indeed first-order.

PROPOSITION 1. *Let* $A \in W_2$. *Then* $A \in W_2^{\mathrm{ec}}$ *if and only if* A *satisfies* (H1), (H2), *and* (I).

Proof. For the forward direction, let $B \in W_2^{\mathrm{ec}}$. B has a countable elementary e.c. substructure A. It is enough to show that A satisfies (H1), (H2), and (I).

Let $A = \sum_{i \in \gamma} A_i$ be the linear sum decomposition of A into indecomposables, and hence, into I-connected components, where γ is a countable linear ordering. Then, by Theorem II.4.1 of [5], A embeds into the countable universal width-two order, $D = \mathbb{Q}_2 \cdot \mathbb{Q}$. Note that D satisfied (H1) and (H2) (this follows by Remark II.3.6 of [5] since $D \equiv \mathbb{Q}_2$).

Fix $a \in A$.

To establish (H1), we must show that $C_A(a, 1) \neq \emptyset$, and that $C_A^+(a, n) \neq \emptyset$ and $C_A^-(a, n) \neq \emptyset$, for all $n > 1$.

Note that

$$D \models \exists z(z \parallel a). \tag{2.1}$$

As A is e.c., it follows by (2.1) that $A \models \exists z(z \parallel a)$ so that $C_A(a, 1) \neq \emptyset$.

It must be shown that $C_A^+(a, n) \neq \emptyset$ for $n > 1$. The proof for $C_A^-(a, n) \neq \emptyset$ will follow in a similar manner and so is omitted. Fix $n > 1$. As D satisfies (H1), there exists $b \in D$ so that $a < b$ and $d_D(a, b) = n$. From the definition of D choose b in the same I-component of D as a, with $\delta_D(a, b) = n$.

To show $C_A^+(a, n) \neq \emptyset$, we list the following easily verified observations.

1. In D, there is an increasing C-path $x_0, x_1, \ldots, x_{n-2}$ of length $n-2$ with $a = a_0$ and $b = a_{n-2}$.
2. There is an I-path $a = y_0, y_1, \ldots, y_n = b$ in D of length n.
3. As D satisfied (I), and since $d_D(a, x_j) \leq n$ for $1 \leq j \leq n-3$, a and x_j must be in same component. From the definition of $\mathbb{Q}_2$, for each $1 \leq j \leq n-3$, there is a length $J + 2$ I-path connecting a to x_j.

Note that any increasing sequence $x_0, x_1, x_2, \ldots, x_{n-2}$ with $a = x_0$ and $b = x_{n-2}$ satisfying both (2) and (3) above must necessarily satisfy $d_D(a, b) = n$. To see this, by (3), x_1 is not on the same chain as a, x_2 is not on the same chain as x_1, and so on. In general, x_i is not on the same chain as x_{i+1}, $0 \leq i \leq n-3$. So $x_0, x_1, x_2 \ldots, x_{n-2}$ is a C-path, so that

$$d_D(a, b) \geq n. \tag{2.2}$$

However, by (2),

$$\delta_D(a, b) \leq n. \tag{2.3}$$

Since a and b are in the same I-component of D, say D_i of D, $d_{D_i}(a, b) = d_D(a, b)$ and $\delta_{D_i}(a, b) = \delta_D(a, b)$. So by Lemma I.2.4 of [5],

$$d_D(a, b) \leq \delta_D(a, b). \tag{2.4}$$

The inequalities (2.2), (2.3), and (2.4) imply that $d_D(a, b) = n$.

Hence, the condition $C_D^+(a, n) \neq \emptyset$ may be expressed by an existential sentence with a parameter a from A. As A is e.c., there exists a strictly increasing set of points $a_1, \ldots, a_{n-2}$ with $\delta_A(a, a_{n-2}) \leq n$, so that for each $1 \leq j \leq n-3$, there with be an I-path in A of length $j+2$ connecting a to a_j. The last requirement implies that $a_0, a_1, \ldots, a_{n-2}$ with $a = a_0$ is a C-path in A. To see this note first that a and a_{n-2} must be in the same I-component A_i of A. Let $C_1 \cup C_2$ be the unique chain decomposition of A_i. Assume without loss of generality that $a \in C_1$. Note that the endpoint of an I-path beginning at a of even length is on C_1, and of odd length is on C_2. Hence, $a_1 \in C_2$, $a_2 \in C_1$, and so on. Hence, $a_0, a_1, \ldots, a_{n-2}$ is a C-path in A_i. Hence,

$$d_A(a, a_{n-2}) \geq n. \tag{2.5}$$

As $\delta_A(a, a_{n-2}) \leq n$, by (2.5) $d_A(a, a_{n-2}) = n$.

Before (H2) is shown to hold in A, we note a few facts. Let $a, b \in A$ be I-connected in a component A_i in A. Then

$$d_A(a, b) = d_{A_i}(a, b) \leq d_D(a, b) \tag{2.6}$$

and

$$\delta_A(a, b) = \delta_{A_i}(a, b) \geq \delta_D(a, b) \tag{2.7}$$

(the equality holding in both (2.6) and (2.7) because a and b are in the same I-connected component). However, using the technique described above to show that (H1) holds for A, any finite C- or I-path in D connecting a to b can also be realized inside of A, and hence, inside A_i (for the I-paths this is clear; for the C-paths, note that a and b are I-connected in A and thus, any C-path in A between a and b must be bounded above by $\delta_{A_i}(a, b)$ by Lemma I.2.4 of [5]). Hence,

$$d_A(a, b) = d_{A_i}(a, b) \geq d_D(a, b) \tag{2.8}$$

and

$$\delta_A(a, b) = \delta_{A_i}(a, b) \leq \delta_D(a, b). \tag{2.9}$$

(2.6), (2.7), (2.8), and (2.9) imply that the embedding of A into D is both a d and δ isometry. As each I-component of D is regular, it follows that each I-component of A is regular. From this it follows that if $A = \sum_{i \in \gamma} A_i$ is an I-component decomposition, then for any $a \in A_i$, for $i \in \gamma$, any hemisphere of a is contained in A_i. In particular, A_i is infinite for all $i \in \gamma$.

To show (H2) holds in A, fix distinct elements a, b of A.

Case 1. $a \in A_i, b \in A_j$ for $i \neq j$. Without loss of generality, assume that $A_i > A_j$ (that is, for all $x \in A_i$ and $y \in A_j$, $x > y$).

Let P and Q be two hemispheres of a and b respectively, that are on the same chain of A. Then $P \subset A_i$ and $Q \subset A_j$ by regularity. Hence, $P > Q$. As A satisfies

(H1), there is a point z (in A_i by the preceding discussion) strictly below P, on the same chain as P. Clearly, $x > Q$. z is the desired point.

Case 2. $a, b \in A_i$, for some $i \in \gamma$. Let P be a hemisphere of distance p to a, Q of distance q to b, so that P and Q are on the same chain, and $P \cap Q = \emptyset$. Let P' be the corresponding hemisphere of a in D of distance p, Q' of distance q in D to b. Since the embedding of A into D is a d-isometry,

$$P \subseteq P', \; Q \subseteq Q'.$$

If $P' \cap Q' \neq \emptyset$, then there is point z of distance p in D to a, and of distance q in D to b. By the same observations used above to show that A satisfies (H1), there would be a point z' in A of distance p in A to a, and of distance q in A to b. This contradicts that $P \cap Q = \emptyset$. As D satisfied (H2), there is a point y between P' and Q', say with

$$d_D(a, y) = p', \qquad d_D(b, y) = q', \tag{2.10}$$

with $p \neq p', q \neq q'$. Again, as A is e.c., by (2.10) there is a point z' in A of distance p' in A to a, and of distance q' in A to b. By choice of p' and q', z' must be between P and Q in A.

For (I), note that D satisfies (I). To see that A satisfies (I), fix $a \in A$. We show (I_n^+) holds in A, the argument for (I_n^-) being similar. In D, there exists an increasing sequence $a_0, a_1, a_2 \ldots, a_{n-2}$ of length $n-2$ with $a = a_0$ and $b = a_{n-2}$, and I-paths of length $i + 2$ connecting a to a_i. As A is e.c., there exists b' in A so that there is an increasing sequence $a_0, a'_1, a'_2 \ldots, a'_{n-2}$ with $b' = a'_{n-2}$ of length $n - 2$, and I-paths of length $i + 2$ connecting a to a'_i.

For the reverse direction of the proposition, we note the following facts from [5].

4. $\mathbb{Q}_2$ is pseudo-homogeneous (Theorem III.3.3 of [5]).
5. Every finite subset of $\mathbb{Q}_2$ is embeddable in a finite regular width-two sub-order of $\mathbb{Q}_2$. It follows that every finite width-two order embeds in a finite regular width-two order (Corollary III.2.5 of [5]).
6. Any two I-connected d-homogeneous width-two orders are back-and-forth equivalent. This can be inferred from the remark after Definition II.1.2 of [5]. From this, it follows that any two I-connected d-homogeneous width-two orders are $L_{\infty\omega}$-equivalent (see [3]). We note the following additional facts.
7. Let A satisfy (H1), (H2), and (I). Then each I-connected component A_i of A is d-homogeneous. To show this it is enough to show that A_i satisfies (H1) and (H2). Let $a \in A_i$ be fixed. We show $C^+_{A_i}(a, n)$ and $C^-_{A_i}(a, n)$ are nonempty for all $n \geq 0$. For instance, let $b > a$ be of distance n to a in A. Clearly, if $n = 0$ or 1, then $b \in A_i$. For $n > 1$, if b were not in A_i, we obtain a contradiction. As A satisfies (I), A_i is infinite and contains arbitrarily long C-paths with starting point a. Hence, if $b \in A_j$ with $A_j > A_i$, then b could not be of finite d-distance to a in A. By this contradiction, each hemisphere in A_i of a is nonempty.

   Now, let a, b be two distinct points in A_i, with hemispheres P and Q on the same chain C_1 of A_i so that $P \cap Q$ is nonempty. As A satisfies (H2) there is a point z between P and Q on C_1. Say $P > z > Q$ (the case $P < z < Q$ is similar). Let $c \in P, d \in Q$. As $\delta_{A_i}(c, d)$ is finite, and since $c > z > d$, z must be of finite δ_{A_i}-distance to both c and d. Hence, $z \in A_i$.

8. Condition (5) above is expressible in $L_{\infty\omega}$. For each $r \geq 1$, let Γ_r be the $L_{\omega_1\omega}$-sentence:

$$\forall x_1 \ldots x_r \bigvee \exists \bar{y} \left(\bigvee_{|A|=m+r} \mathrm{diag}_A(x_1 \ldots x_r, \bar{y}) \right),$$

where the infinitary disjunction is over all $m \geq 0$, and the finitary disjunction is over all isomorphism types of finite regular I-connected width-two orders A of cardinality $m + r$ (as there are only finitely many isomorphism types of width-two orders for each finite cardinality, this is well-defined). Here $\mathrm{diag}_A(\bar{z})$, for a finite structure A of cardinality $|\bar{z}|$ is the conjunction of all literals satisfied by all elements of A (namely, the *diagram* of A). The combined meaning of $\Gamma_r, r > 0$, are: for every finite subset S, there exists a finite superset T of S, so that T is isomorphic to a finite regular width-two order.

It follows that each d-homogeneous width-two order satisfies $\Gamma_r, r > 0$ by (6) above.

Combining (4), (6), (7), and (8), it follows that if A satisfies (H1), (H2), and (I), then A is a linear sum of d-homogeneous orders, all of which satisfy $\Gamma_r, r > 0$. In particular, it follows that each such A is a linear sum of (not necessarily countable) pseudo-homogeneous width-two orders.

Using this fact, it can be shown that A is existentially closed. If A is I-connected, then A itself is pseudo-homogeneous, so the desired conclusion follows by [2].

Assume now that $A = \sum_{i \in \gamma} A_i$, where the A_i are the I-connected components of A.

Let $B \in W_2$ so that

$$A \leq B \models \exists \bar{x} \theta(\bar{x}, \bar{a}),$$

where $\theta(\bar{x}, \bar{a})$ is quantifier-free, and $\bar{a}$ is a finite tuple of elements from A. Let $\bar{a} = \bigcup_{i=1}^{n} \overline{a_i}$ where $\overline{a_i} \subseteq A_{ji}$, where $j_i \in \gamma$, and $A_{j_1} > \cdots > A_{j_n}$. Let $\bar{b}$ be a tuple from B so that $B \models \theta(\bar{b}, \bar{a})$. Then

$$\bar{b} = \bigcup_{i=1}^{n} \overline{b_i} \cup \bigcup_{j=1}^{n+1} \overline{c_j},$$

where each $\overline{b_i}$ is of finite d-distance in B to $\overline{a_i}$, and each $\overline{c_j}$ is of strong infinite d-distance in B to $\bar{a}$, and

$$\overline{c_1} > A_{j_1} > \overline{c_2} > \cdots > \overline{c_n} > A_{j_n} > \overline{c_{n+1}}.$$

(This definition makes sense, since via (I), each $\overline{a_i}$ and $\overline{a_j}$ with $i \neq j$ are of strong infinite d-distance in A, and hence, strong infinite d-distance in B. Hence, a tuple

$\overline{b_k}$ cannot be of finite B-distance to both $\overline{a_i}$ and $\overline{a_j}$.) Using the pseudo-homogeneity of the A_i and Lemma 1, amalgamate each $\overline{b_i}$ into A_{j_i} witnessed by $\overline{b'_i}$. Note that the $\overline{c_j}$ are of strong infinite d-distance in B to each of the $\overline{b_i}$ and in B:

$$\overline{c_1} > \overline{b_1} > \overline{c_2} > \cdots > \overline{c_n} > \overline{b_n} > \overline{c_{n+1}}.$$

To see this, fix $i \in \{1, \dots, n\}$. As $\overline{a_i}$ is of strong infinite d-distance below $\overline{c_i}$ in A, and hence in B, for each a in $\overline{a_i}$, and each $c \in \overline{c_i}$, there are arbitrarily long increasing (or if $c \in \overline{c_{n+1}}$, decreasing) C-paths $P_k, k \geq 1$ in B between a and c. Let $b \in \overline{b_i}$. Then b is of finite d-distance in B from a by hypothesis. Assume that $c > a$ and so the P_k are increasing. If $b \geq a$, clearly $d_B(b, c) = \infty$. If $b \geq a$, then there exists $k \geq 1$ so that $b < x_m$, where x_m is an element of some P_k (not equal to a). Then again, $d_B(b, c) = \infty$. The argument for $c < a$ is similar.

Now using Lemma 1 amalgamate the $\overline{c_k}$ above $\overline{a_k} \cup \overline{b'_k}$ in A_{j_k}, witnessed by $\overline{c'_k}$ for $1 \leq k \leq n$, and below $\overline{a_n} \cup \overline{b'_n}$ in A_{j_n}, witnessed by $\overline{c'_{n+1}}$. Define

$$\bar{b}' = \bigcup_{i=1}^{n} \overline{b'_i} \cup \bigcup_{j=1}^{n+1} \overline{c'_j}.$$

Then $A \models \theta(\bar{b}', \bar{a})$, so that $A \models \exists \bar{x} \theta(\bar{x}, \bar{a})$. □

COROLLARY 1. *Let $A \in W_2$. Then $A \in W_2^{\mathrm{ec}}$ if and only if A is a linear sum of d-homogeneous width-two orders.*

Proof. If $A \in W_2^{\mathrm{ec}}$ then $A \models$ (H1), (H2), (I) by Proposition 1. By the proof of Proposition 1, A is a linear sum of d-homogeneous width-two orders.

Conversely, assume that A is a linear sum of d-homogeneous width-two orders. Then from the proof of Proposition 1, A is a linear sum of pseudo-homogeneous width-two orders, and hence is existentially closed. □

COROLLARY 2. *W_2^{mc} exists, is complete, decidable, non-finitely axiomatizable, and has $2^{\aleph_0}$ countable models.*

Proof. By Proposition 1, W_2^{ec} is first-order axiomatizable. Hence, W_2^{ec} exists and is axiomatized by (H1) $\cup$ (H2) $\cup$ (I). We claim that $\mathrm{Th}(W_2^{\mathrm{mc}}) = \mathrm{Th}(\mathbb{Q}_2)$. $\mathbb{Q}_2 \models$ (H1) $\cup$ (H2) $\cup$ (I), so consider $B \models$ (H1) $\cup$ (H2) $\cup$ (I). Let $A \preceq B$ be countable. Let $A = \sum_{i \in \gamma}$ be a decomposition of A into I-connected components, where γ is a countable linear order. As $A \models$ (H1) $\cup$ (H2) $\cup$ (I), A_i is d-homogeneous for all i, so that $A_i \cong \mathbb{Q}_2$, by Lemma II.1.3a of [5]. Hence, $A \cong \mathbb{Q}_2 \cdot \gamma \equiv \mathbb{Q}_2$ (see Remark II.3.6 of [5]).

As (H1) $\cup$ (H2) $\cup$ (I) is recursive, decidability of $\mathrm{Th}(W_2^{\mathrm{mc}})$ follows. By Corollary 1, the countable e.c. models are of the form: $\mathbb{Q}_2 \cdot \gamma$, where γ is a countable linear ordering. It is not difficult to see that if γ and ε are non-isomorphic countable linear orderings, then $\mathbb{Q}_2 \cdot \gamma \ncong \mathbb{Q}_2 \cdot \varepsilon$. Hence, there are continuum many countable models of W_2^{mc} (as there are continuum many non-isomorphic countable linear orderings; see, for example, [6]). Non-finite axiomatizability follows since $\mathbb{Q}_2$ is not-finitely axiomatizable, as noted in [8]. □

As noted in [7], the theory of W_2 is undecidable. However, as $\mathrm{Th}(W_2)$ and $\mathrm{Th}(W_2^{\mathrm{mc}})$ have the same universal consequences, by Corollary 2, the following corollary is immediate.

COROLLARY 3. *W_2 has a decidable universal theory.*

Remark 5. As stated above in remark (5) in the proof of Proposition 1, Pouzet proved that every finite $A \in W_2$ embeds in a finite regular B. Since $\mathbb{Q}_2$ is not $\aleph_0$-categorical, this cannot be done uniformly. That is, there does not exist a function $f\colon \mathbb{N} \to \mathbb{N}$ so that $|B| = f(|A|)$ (see [1] or [4]). This fact is not immediate a priori.

Remark 6. In response to a question posed in Remark III.3.4 of [5], we show that $\mathbb{Q}_2$ is in fact δ-homogeneous. It then follows, as noted in [5] that d-homogeneity is equivalent to δ-homogeneity.

Let A be a width-two order, $a \in A$, and $n \geq 0$. As δ is a metric, we define a δ-sphere in A with center a to be

$$I(a, n) = \{y : \delta_A(a, y) = n\}.$$

The corresponding δ-hemispheres are defined in the same manner as hemispheres with respect to the d-metric of [5]. Let $a \neq b$ in $\mathbb{Q}_2$ with $\delta_A(a, b) = r$. Let $P = I^{\varepsilon}(a, p)$ and $Q = I^{\varepsilon'}(b, q)$ be hemispheres with $\varepsilon, \varepsilon' \in \{-, +\}$ with $P \cap Q \neq \emptyset$. Then for any pair a', b' in $\mathbb{Q}_2$ with $\delta_A(a', b') = r$, if $P' = I^{\varepsilon}(a', p)$ and $Q' = I^{\varepsilon'}(b', q)$, then $P' \cap Q' \neq \emptyset$. To see this, consider isomorphism type of $\mathbb{Q}_2$ with $\alpha = \pi$ and the chains $\mathbb{Q}$. For all points x of $\mathbb{Q}_2$, the δ-hemispheres of x are intervals in $\mathbb{R}$ with irrational endpoints. If $x \neq y$ in $\mathbb{Q}_2$, then none of the hemispheres of x and y are equal or share common endpoints in $\mathbb{R}$. Further, the hemispheres of y partition each hemisphere of x into two distinct open intervals in $\mathbb{Q}$. Now, if $P \cap Q \neq \emptyset$, then q must be one of $r - p + 2$ or $r - p$. By examining all cases, if a' and b' are as above, then $I^{\varepsilon}(a', p)$ intersects $I^{\varepsilon'}(b', r - p + 2)$ and $I^{\varepsilon'}(b', r - p)$.

Remark 7. We describe some uncountable existentially closed models. Such models exist by the Löwenheim–Skolem theorem, but the following construction produces explicit examples in each uncountable cardinal κ. Fix L_κ a dense uncountable linear ordering of power κ. Fix an irrational $\alpha > 1$. Define A_κ to have universe $(L_\kappa \cdot \mathbb{Q}) \times \{0, 1\}$ with ordering defined as follows. Enumerate the components of $L_\kappa \cdot \mathbb{Q}$ as L_q for $q \in \mathbb{Q}$. $(a, i) \leq (b, i)$, for $i \in \{0, 1\}$ iff $a \leq b$ in $L_\kappa \cdot \mathbb{Q}$. Consider (a, i), (b, j) with $i \neq j$, so that $a \in L_p$ and $b \in L_q$, for $p, q \in \mathbb{Q}$. Then let $(a, i) \leq (b, j)$ iff $p + \alpha \leq q$. Clearly, $A_\kappa \in W_2$, and $|A_\kappa| = \kappa$. It is not hard to see that A_κ is I-connected and satisfies (H1), (H2), and (I). That A_κ is e.c. now follows from Proposition 1.

Remark 8. As $\mathrm{Th}(\mathbb{Q}_2)$ is not $\aleph_0$-categorical, by the Ryll–Nardzewski theorem, for some n, there are infinitely many n-orbits. As was noted in [8], there is only one 1-orbit. We demonstrate that for each $n > 1$, there are infinitely many n orbits. As $\mathbb{Q}_2$ is regular, every automorphism is both a d and δ isometry. Choose the isomorphism type of $\mathbb{Q}_2$ with $\alpha = \pi$ and chains both equalling $\mathbb{Q}$. We define a set $\{x_i : i \geq 0\}$ in $\mathbb{Q}_2$ so that $d_{\mathbb{Q}_2}(x_0, x_i) = i$, for each $i \geq 0$. Define $x_0 = (0, 0)$, $x_1 = (1, 1)$, and $x_2 = (2, 0)$. For $i \geq 3$, choose a rational $a_i \in ((i-2)\pi, i\pi)$. For $i \geq 3$, define

$$x_i = \begin{cases} (a_i, 0) & i \text{ even}, \\ (a_i, 1) & i \text{ odd}. \end{cases}$$

Define $y_i = (x_0, x_i, x_0, \ldots, x_0) \in \mathbb{Q}_2^n$, $n \geq 2$. Then the y_i are all in distinct n-orbits.

3. The Width-n Case, for $n > 2$

The classes W_n, for $n > 2$, appear to be less well-behaved than the $n = 2$ case. As noted by [8], even in the case $n = 3$, there is no countable universal model. We do not know if there is a model companion in the general case. Throughout this section we assume $n > 2$.

3.1. UNIQUE CHAIN DECOMPOSITIONS

Recall that to every order P, there is an associated incomparability graph, $I(P)$: the vertices of $I(P)$ are just the elements of P, and the graph relation R is defined $aR^{I(P)}b$ iff $a \|^P b$.

Note that by Dilworth's theorem on chain decompositions of width-n orders, an order P is width-n iff $I(P)$ is n-colourable (a graph G is *n-colourable* if there is a partition of the vertices of G into n independent sets). It follows that P has a unique chain decomposition (UCD) iff $I(P)$ is uniquely n-colourable (an n-colourable graph G is *uniquely n-colourable* if every n-colouring of G induces the same partition of the vertex set of G). As noted in Lemma I.1.5 of [5], in the W_2 case, having a unique chain decomposition is equivalent to being I-connected.

DEFINITION 2. Let $A \in W_n$.

1. Define an *I-path* in A to be a path in $I(A)$. A is *I-connected* if $I(P)$ is a connected graph.
2. Let A be I-connected. If $a, b \in A$, define $\delta_A(a, b)$ to the length of a minimal I-path.

As in the width-two case, δ_A is a metric, and I-connectedness is equivalent to indecomposability. Further, since every uniquely n-colourable graph (for $n > 1$)

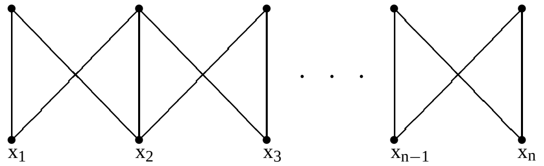

Figure 2. I-connected, not UCD.

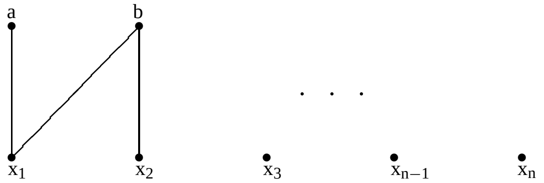

Figure 3. UCD, not d-metrizable.

is connected, UCD implies I-connectedness. The converse fails, however. That is, there exists a width-n order A, for each $n > 2$, so that A is I-connected, but does not have UCD. For example, the width-n order in Figure 2 has this property.

3.2. D-METRIZABILITY

DEFINITION 3. Let $A \in W_n$ with UCD. A C-*path* of length n in A is a strictly increasing or decreasing sequence $x_0, \ldots, x_n$ so that x_i is not on the same chain in the decomposition as x_{i+1}, for $1 \leq i \leq n - 1$.

DEFINITION 4. Let $A \in W_n$ with UCD. Define the partial function $d_A\colon A \times A \to \mathbb{N}$, where $d_A(a, b)$ is defined as in Lemma I.2.5 of [5] for $(a, b) \in S = \{(x, y) \in A \times A$: there are bounded C-paths in A from x to $y\}$. (Note that if A is finite or more generally has bounded C-paths between its elements, $S = A \times A$, and d_A is a total function.)

Even when d is a total function, d may fail to be a metric. For example, the order A in Figure 3 is width-n, has UCD, but $d_A(x_1, b) = 3$ and $d_A(x_1, x_3) = d_A(x_3, b) = 1$. We would like to capture the subclass of W_n where d is a metric. Such orders we call d-*metrizable*.

DEFINITION 5. Let $A \in W_n$ with UCD. Then A satisfies (dM) if for all C-paths a, b, if $x \| a$ then $x < b$ if $a < b$, and $x > b$ if $a > b$.

The proof of the following lemma is similar to the proof of Lemma I.2.4 in [5].

LEMMA 2. *Let* $A \in W_n$ *with UCD and satisfying* (dM). *Let* $a, b \in A$. *Then the length of any* C-*path from* a *to* b *is* $\leq \delta_A(a, b) - 2$.

LEMMA 3. *Let $A \in W_n$ with UCD.*

1. *Then d_A is a metric on A iff (dM) holds.*
2. *Assume A satisfies (dM). If $z \not\leq x$ and $y \not\leq z$ then*
$$d_A(x, y) \leq d_A(x, z) + d_A(y, z) \leq d_A(x, y) + 2.$$

Proof. The necessity of (1) is obvious. For the sufficiency of (1), first note that by Lemma 2, d_A is a total function. The verification of the triangle inequality is similar to the proof in Lemma I.2.5 in [5], so is omitted. For (2), the proof is the same as in Lemma I.2.5 in [5]. □

3.3. D-HOMOGENEITY

For a d-metrizable width-n order with UCD the notions of d-isometry and d-homogeneity generalize in the obvious way from the $n = 2$ case.

For a d-metrizable width-n order with UCD, A, the notions of hemispheres $C^{\pm}(x, m)$, for $x \in A, m \geq 0$, may be defined as in Section II.2 of [5]. Unlike the case $n = 2$, in general, hemispheres may not be on a unique chain.

DEFINITION 6. Let A be a d-metrizable width-n order with UCD. If $A = C_1 \cup \cdots \cup C_n$ is a chain decomposition, define $C_i^{\pm}(x, m) = C^{\pm}(x, m) \cap C_i$ for $m \geq 0$, and $1 \leq i \leq n$. These are the *induced hemispheres*. Define an *allowable induced hemisphere* to be an induced hemisphere that is not always empty.

Remark 9. Note that if $m \geq 3$ each induced hemisphere is allowable (if $n > 2$, a C-path starting on an arbitrary x may end on any chain). If $m = 0$, only the chain x is on is allowable. If $m = 1$, all chains except the chain x is on are allowable. If $m = 2$, only the chain x is on is allowable.

Using the notion of allowable induced hemisphere, and the inequality in 2 of Lemma 3, one may prove the following lemma, in a manner analogous to the proof of Lemma II.2.6 of [5].

LEMMA 4. *Let A be a d-metrizable width-n order with UCD. Then a sufficient condition for A to be d-homogeneous is that A satisfy the following conditions.*

1. *All allowable induced hemispheres are nonempty.*
2. *If two allowable induced hemispheres P and Q with distinct centers a and b, respectively, are disjoint, then there exists a point z on the same chain as P and Q so that z is strictly between P and Q.*

We do not have an explicit example of a d-metrizable width-n order with UCD that is d-homogeneous, for $n > 2$. Further, we do not know the answer to the following problem.

PROBLEM. If $n > 2$, does W_n have a model companion?

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