A NOTE ON UNIQUELY H-COLOURABLE GRAPHS

ANTHONY BONATO

Abstract. For a graph H, we compare two notions of uniquely H-colourable graphs, where one is defined via automorphisms, the second by vertex partitions. We prove that the two notions of uniquely H-colourable are not identical for all H, and we give a condition for when they are identical. The condition is related to the first homomorphism theorem from algebra.

1. Introduction

A *homomorphism* from G to H is an edge-preserving vertex-mapping. If there is a homomorphism from G to H, then we say that G is *H-colourable*. For background and notation on graph homomorphisms, the reader is directed to [6]. Uniquely H-colourable graphs, where H is a fixed finite graph, have been studied by many authors; see, for example, [1, 2, 4, 9, 10]. The usual definition of uniquely H-colourable uses automorphisms of H, and as such, makes no explicit mention of vertex partitions. To be more explicit, recall that a graph H is a *core* if every endomorphism of H is an automorphism. For a core H, G is *uniquely H-colourable* if G is H-colourable, every homomorphism from G to H is surjective, and for all homomorphisms f, h from G to H, there is an automorphism g of H so that $f = gh$. We denote the class of uniquely H-colourable graphs by $C_u(H)$.

Given a homomorphism f from G to H, define $\ker(f) = \{(x, y) \in V(G) \times V(G) : f(x) = f(y)\}$. Then $\ker(f)$ is an equivalence relation whose equivalence classes, called *colour blocks*, are independent sets partitioning $V(G)$. A graph G is *weakly uniquely H-colourable* if the last condition in the definition of uniquely G-colourable is replaced by: for all homomorphisms f, h from G to H, $\ker(f) = \ker(h)$. The class of weakly uniquely H-colourable graphs is written $C_{wu}(H)$. It is straightforward to see that $C_u(H) \subseteq C_{wu}(H)$. The classical notion of

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uniquely n-colourable graph introduced in [6] corresponds to weakly uniquely K_n-colourable graphs. Further, $C_u(K_n) = C_{wu}(K_n)$.

However, perhaps surprisingly, there are cores H such that $C_u(H) \subsetneq C_{wu}(H)$. We demonstrate that there are infinitely many graphs H with this property; see Theorem 2. If $C_u(H) = C_{wu}(H)$, then we say that the core H is *good*; otherwise, we say that H is *bad*. Our goal in this short note is to present a condition for a core to be good that applies to a large number of cases; see Theorem 3 (2). We give an equivalent algebraic formulation of this condition in Theorem 3 (1). For related work on H-colourings and properties of cores, the reader is directed to [7, 8].

2. Good, great, and bad cores

All graphs we consider are finite, undirected, and simple. Define $Hom(G,H)$ be the set of homomorphisms from G into H. The monoid $Hom(H,H)$ of endomorphisms of H under composition is denoted $End(H)$. We write $Aut(H)$ for the group of automorphisms of H under composition. If X is a set, then we write $Sym(X)$ for the set of bijections from X to itself.

Before we can give examples of bad cores, we need the following straightforward lemma.

Lemma 1. *Let H be a core graph. Then $G \in C_{wu}(H)$ if and only if G is H-colourable, every element of $Hom(G,H)$ is surjective, and for all $f, h \in Hom(G,H)$, there is a $g \in Sym(V(H))$ so that $f = gh$.*

Proof. Assume $|V(H)| = n$, for some $n \geq 1$. For the forward direction, fix $f, h \in Hom(G,H)$. Label the colour blocks of $\ker(f)$ as $B_1, \ldots, B_n$. For each $1 \leq i \leq n$, choose $b_i \in B_i$. Define $g : V(H) \to V(H)$ by $g(h(b_i)) = f(b_i)$. Then g is well-defined as $\ker(f) = \ker(h)$ and as h is surjective. As f is surjective, it follows that g is surjective, and hence, $g \in Sym(V(H))$. As $f = gh$, the forward direction follows.

For the converse, let $f, h \in Hom(G,H)$. By hypothesis, there is a $g \in Sym(V(H))$ so that $f = gh$. Then for $x, y \in V(G)$, $f(x) = f(y)$ if and only if $g^{-1}f(x) = g^{-1}f(y)$. But the latter is equivalent to $h(x) = h(y)$. Hence, $(x,y) \in \ker(f)$ if and only if $(x,y) \in \ker(h)$, so $\ker(f) = \ker(h)$. □

For an integer $n \geq 2$, let $G = K_{2n-2}$ with $1, 2$ fixed distinct vertices of G. Define a graph H_n as follows. Let G_1 and G_2 be two disjoint copies of G, so that the vertices of G_2 are $\{x' : x \in V(G)\}$. The vertices of H_n are the vertices of G_1 and G_2, along with three new vertices a, b, and c. The edges of H_n include the edges of G_1 and G_2; the vertex

a is joined to every vertex in $V(G_1) \cup V(G_2)$ and no other vertex; the vertex b is joined to vertices in $\{c\} \cup V(G_1)$ and no other vertex; the vertex c is joined to vertices in $\{b\} \cup V(H_2)$ and no other vertex; the only edge "between" $V(G_1)$ and $V(G_2)$ is $12'$. See Figure 1 for H_2.

If x is a vertex of G, then $G - x$ is the graph that results when x is deleted. A nontrivial graph is *critical* if $\chi(G - x) < \chi(G)$ for all $x \in V(G)$.

Theorem 2. *Each graph H_n is a bad core.*

Proof. The graph $H = H_n$ is critical with $\chi(H) = 2n$. Hence, H is a core. Define J by deleting the edge $12'$, so that 1 and 2 have the same neighbors in J. Then J is critical with $\chi(J) = 2n$.

We next show that $J \in C_{wu}(H) \setminus C_u(H)$, which will witness that H is bad. To see this, note first that $J \in C_{wu}(H)$: clearly J is H-colourable; since J and H are critical with chromatic number $2n$, every $f \in Hom(J, H)$ is surjective; and as $|V(J)| = |V(H)|$, $\ker(f)$ has only singleton colour blocks, so any two elements of $Hom(J, H)$ have the same kernel. Let $f = id_J \in Hom(J, H)$ and define h so that it interchanges 1 and 2 and fixes all other vertices. Then $h \in Hom(J, H)$ as 1 and 2 have the same neighbors in J. If there is a $g \in Aut(H)$ so that $f = gh$, then g interchanges 1 and 2 leaving all other vertices of H fixed. But 1 and 2 have different neighbours in H, so that $g \notin End(H)$, which is a contradiction. □

By Theorem 2, there are infinitely many bad cores. By a direct check, each core of order at most 6 is good. Hence, the minimum order of an bad core is 7, with an explicit example given in Figure 1.

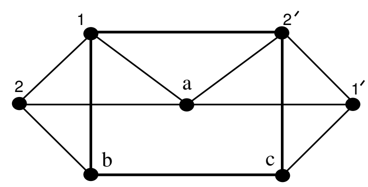

FIGURE 1. An bad core of minimum order.

Let G and H be graphs and let $f \in Hom(G, H)$ be surjective. The *quotient graph* $G/\ker(f)$ has vertices the colour blocks of $\ker(f)$, and two colour blocks B and C are joined if and only if there is some vertex in B joined to some vertex in C. Note that the colour blocks are just the preimages under f of vertices of H. The *natural map*

$\eta_f : V(G/\ker(f)) \to V(H)$ defined by $\eta_f(f^{-1}(x)) = x$ is a well-defined homomorphism. Observe that if $G \in C_{wu}(H)$, then η_f is a bijection.

The next definition is inspired by the homomorphism theorems that hold in varieties of algebraic systems. Let H be a core graph. The class $C_{wu}(H)$ satisfies the *first homomorphism theorem* if for all $G \in C_{wu}(H)$ and all $f : Hom(G, H)$, the homomorphism $\eta_f : V(G/\ker(f)) \to V(H)$ is an isomorphism (that is, f is a *complete* homomorphism). By preceding remarks we need only show that η_f is an *embedding* (an injective homomorphism which preserves non-edges). The classes $C_{wu}(H)$ satisfying the first homomorphism theorem can be characterized by an intrinsic condition of H. If e is an edge of H, then $H - e$ is the graph formed by deleting e. We say that a graph H is *great* if for all $e \in E(H)$, there is some $f \in Hom(H-e, H)$ so that f is not surjective. For example, each clique and cycle is great. We now state and prove our main result.

Theorem 3. *Let H be a core graph.*

(1) *The class $C_{wu}(H)$ satisfies the first homomorphism theorem if and only if H is great.*
(2) *If H is great, then H is good.*

Proof. For the forward direction of item (1), to obtain a contradiction we assume that for all $e \in E(H)$, every $f \in Hom(H-e, H)$ is surjective, and therefore, a bijection. Fix $f, h \in Hom(H - e, H)$. Hence, $\ker(f)$ and $\ker(h)$ have all singleton blocks. In particular,

$$(H - e) \,/\, \ker(f) \cong H - e. \tag{1.1}$$

The mapping $g : V(H) \to V(H)$ defined by $g(f(x)) = h(x)$ is well-defined and bijective as $\ker(f) = \ker(h)$ and f is surjective. Hence, $gf = h$ for some $g \in Sym(V(H))$. As f and h were arbitrary, $H - e \in C_{wu}(H)$ by Lemma 1. Since $C_{wu}(H)$ satisfies the first homomorphism theorem, $\eta_f : V((H - e) \,/\, \ker(f)) \to V(H)$ is an isomorphism, so that $H - e \cong H$ by (1.1), which is a contradiction.

For the reverse direction of (1), fix H a great core. Fix $G \in C_{wu}(H)$ and $h \in Hom(G, H)$. If the natural map $\eta_h : V(G/\ker(h)) \to V(H)$ is not an isomorphism, then it is not an embedding. Hence, there are $x, y \in V(H)$ such that $xy \in E(H)$ but the colour blocks $h^{-1}(x)$ and $h^{-1}(y)$ are not joined in $G/\ker(h)$. Let $e = xy \in E(H)$. Note that $h : V(G) \to V(H - e)$ is a homomorphism. As H is great, there is some $f \in Hom(H-e, H)$ that is not surjective. But then $fh : V(G) \to V(H)$ is a homomorphism that is not surjective, which contradicts that $G \in C_{wu}(H)$.

For item (2), as H is great, $C_{wu}(H)$ satisfies the first homomorphism theorem by item (1). Fix $G \in C_{wu}(H)$ and fix $f, h \in Hom(G, H)$. Then by Lemma 1 there is a $g \in Sym(V(H))$ so that $f = gh$. We show that $g \in Aut(H)$. As H is a core, it is enough to show that $g \in End(H)$. To see this, fix $xy \in E(H)$. By hypothesis, $h^{-1}(x)h^{-1}(y) \in E(G/ker(h))$, so that there is some $a \in h^{-1}(x)$ and $b \in h^{-1}(y)$ with $ab \in E(G)$. Now $f(a) = g(h(a)) = g(x)$; similarly, $f(b) = g(y)$. As f is a homomorphism, we have that $f(a)f(b) \in E(H)$, and hence, $g(x)g(y) \in E(H)$. □

Cliques and odd cycles are great cores, and hence, are good by Theorem 3 (2). If G and H are graphs, recall that their *join*, written $G + H$, is the graph formed by adding edges between each vertex of G and H. If G is a great core, then an analysis of cases demonstrates that $G + K_n$ is a great core for each $n \geq 1$. In particular, the odd wheels W_{2n+1} for $n \geq 2$ are great cores. For a large class of great cores, we consider a recent construction of [3]. If G is a graph, define $C(G)$ to be G with edge replaced by a path with 3 edges. As proven in [3], if G is connected with at least three vertices, then fG is a core if and only if $C(G)$ is. It is not hard to see that for all graphs G, $C(G)$ is great.

We note that not all good cores are great. Direct checking (which is tedious, and so omitted) demonstrates that the Petersen graph is a good but not great core. Hence, the converse of Theorem 3 (2) is false.

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Department of Mathematics, Wilfrid Laurier University, Waterloo, ON, Canada, N2L 3C5
E-mail address: abonato@rogers.com