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BOUNDS AND CONSTRUCTIONS FOR N-E.C. TOURNAMENTS

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Abstract. Few families of tournaments satisfying the n-e.c. adjacency property are known. We supply a new random construction for generating infinite families of vertex-transitive n-e.c. tournaments by considering circulant tournaments. Switching is used to generate new n-e.c. tournaments of certain orders. With aid of a computer search, we demonstrate that there is a unique minimum order 3-e.c. tournament of order 19, and there are no 3-e.c. tournaments of orders 20, 21, and 22. We show that there are no 4-e.c. tournaments of orders 47 and 48 improving the lower bound for the minimum order of such a tournament.

1. Introduction

Adjacency properties of graphs and digraphs were discovered by Erdős and Rényi [8] in their pioneering work on random graphs. We focus here on the n-e.c. adjacency property of tournaments. For a positive integer n, a tournament is *n-existentially closed* or *n-e.c.* if for all disjoint sets of vertices A and B with $|A \cup B| = n$ (one of A or B may be empty), there is a vertex z not in $A \cup B$ such that there is an arc from z to each vertex of A and there is an arc from each vertex of B to z. We say that z is *correctly joined* or *c.j.* to A and B. Hence, for all n-subsets S of vertices, there exist 2^n vertices joined to S in all possible ways. For example, the tournament in Figure 1 is the unique minimum order 2-e.c. tournament.

Although the n-e.c. property is straightforward to define, it is not obvious from the definition that tournaments with the property exist. Let $T(m,p)$ be a random tournament on the vertex set $[m] = \{1, 2, \ldots, m\}$, where for each ordered pair of vertices (i,j), $i < j$ a directed edge from i to j occurs independently with probability p. Note that $p = p(m)$ may tend to zero with m. The probability space $T(m,p)$ may be viewed as a result of $\binom{m}{2}$ independent coin flips, one for each pair of vertices, where the probability of success is equal to p. The probability that a random tournament $T(m,1/2)$

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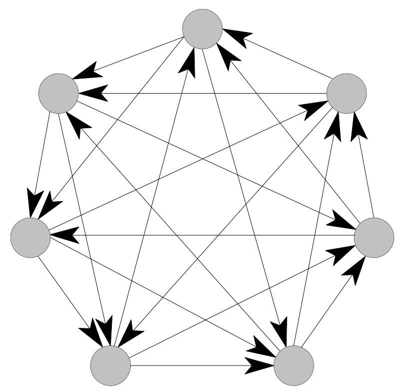


FIGURE 1. The smallest order 2-e.c. tournament.

is not n-e.c. is bounded from above by

$$\binom{m}{n} 2^n \left(1 - \frac{1}{2^n}\right)^{m-n},$$

which is smaller than one for m sufficiently large. Hence, n-e.c. tournaments exist for all n. Graham and Spencer [10] were the first to give explicit examples of tournaments satisfying adjacency properties. The *Paley tournament* of order q, for a prime power $q \equiv 3 \pmod 4$, written T_q, has vertices the elements of the finite field GF(q), where vertices x and y are joined if and only if $x - y$ is a non-zero quadratic residue in GF(q). In fact, T_7 is the unique isomorphism type of 2-e.c. tournament of order 7, and is the minimum order 2-e.c. tournament; see Figure 1 and [4]. By [1, 2, 10], if $q > n^2 2^{2n-2}$, then T_q is n-e.c. See [3] for more background on n-e.c. tournaments and graphs.

In this article, we give a new construction of n-e.c. tournaments satisfying certain properties. While our construction is randomized, it always generates regular tournaments; in fact, the tournaments are vertex-transitive; see Theorem 2.1. We demonstrate how switching in tournaments generates new non-isomorphic n-e.c. tournaments in Theorem 2.3. We investigate the function $t_{ec}(n)$, which is defined as the minimum order of an n-e.c. tournament. From [4], $t_{ec}(1) = 3$ and $t_{ec}(2) = 7$ (realized by the directed 3-cycle and T_7, respectively), but before this article no other values of this function were known. We show in Section 3 that $t_{ec}(3) = 19$, and using a computer search we found that there is a unique 3-e.c. tournament of order 19 and there is no other tournament of order less than 23. We show that there are no 4-e.c. tournaments of orders 47 and 48 improving the lower bound for the minimum order of such a tournament.

All tournaments we consider are finite unless otherwise stated. For a tournament T, if $x \in V(T)$, then let $\deg^+_T(x)$ and $\deg^-_T(x)$ be the out- and in-degrees of x, respectively. Let $N^+(x)$ and $N^-(x)$ be the out- and in-neighbourhoods of x, respectively. For a subset S of $V(T)$, define $T[S]$ to be the subtournament induced by S. We abbreviate isomorphism type by *isotype*. We denote the natural numbers (including 0) by $\mathbb{N}$, and the integers by $\mathbb{Z}$.

2. New constructions of N-e.c. tournaments

We give a new construction of n-e.c. tournaments. The family we construct in Subsection 2.1 is not only regular, but vertex-transitive. In Subsection 2.2, we explain how switching provides a structural approach to generate an exponential number of non-isomorphic n-e.c. tournaments, using our vertex-transitive n-e.c. tournaments as building blocks.

2.1. **N-e.c. circulant tournaments.** Fix an integer $m \geq 1$. In the remainder of the subsection, all arithmetic is modulo $2m+1$. Fix

$$J \subseteq [2m] = \{1, 2, \ldots, 2m\} = \{-m, -m+1, \ldots, -2, -1, 1, 2, \ldots, m-1, m\}$$

with the property that $|J \cap \{j, -j\}| = 1$ for all $j \in [m]$. Hence, $|J| = m$, and $j \in J$ if and only if $-j \notin J$. A *circulant tournament* $G(J)$ has vertices $\mathbb{Z}_{2m+1}$ (for simplicity, we identify the elements (or residues) of the ring $\mathbb{Z}_{2m+1}$ with $0, 1, \ldots, 2m$) and directed edges (i, j) if $i - j \in J$. We call J the *connection set* of $G(J)$. The tournament $G(J)$ is vertex-transitive and so is regular.

For a fixed $0 < p \leq 1/2$, the *random circulant tournament* $CT(m, p)$ consists of $G(J)$ where the connection set J has each element of the set $[m]$ chosen with probability p. More precisely, for each $1 \leq i \leq m$, independently with probability p add i to J; otherwise add the element $-i$ to J. Without loss of generality, we can assume that $p \leq 1/2$, since in order to construct $CT(m, p)$ with $p > 1/2$ one can take the dual of $CT(m, 1-p)$, which is n-e.c. if and only if $CT(m, 1-p)$ is. Note that the two events "$i \in J$" and "$-i \in J$" are dependent.

We now state the main result of this section, which generates a family of vertex-transitive n-e.c. tournaments. We say that an event A holds *asymptotically almost surely* (a.a.s.) in $CT(m, p)$ if A holds with probability tending to 1 as $m \to \infty$. The probability of A is denoted by $\mathbb{P}(A)$.

Theorem 2.1. *Let $p \in (0, 1/2]$. A.a.s. $CT(m, p)$ is n-e.c. with*

$$n = \log_{1/p} m - 4 \log_{1/p} \log m - O(1).$$

Proof of Theorem 2.1. Let $n = \log_{1/p} m - 4 \log_{1/p} \log m - C$, where C will be determined later. Fix $X = \{x_1, x_2, \ldots, x_n\}$ an n-set in $G = CT(m, p)$, and fix $z \notin X$. Define the *projection* $\pi_X(z)$ to be the set of elements of $[m]$ of the form $z - i$ or $i - z$, where $i \in X$. Observe that $|\pi_X(z)| \leq n$ (it may happen that $|\pi_X(z)| < n$; for example, in the case $z = (i_1 + i_2)/2$ for some $i_1, i_2 \in X$). We would like to form a *template set* U disjoint from X such that for all distinct $z, z' \in U$,

$$\pi_X(z) \cap \pi_X(z') = \varnothing \qquad \text{and} \qquad |\pi_X(z)| = n.$$

These properties ensure that edges between X and U are generated independently. Further, we would like to choose $|U| = r = \lfloor m/n^2 \rfloor$. We construct

U as a union of a chain of sets U_k of vertices, where for all $k \geq 1$ exactly one vertex is added to U_k to form U_{k+1}. In particular, the set $U = U_r$. The sets U_k are constructed by induction on $k \geq 1$, with the induction stopping at $k = r$.

In the base step of the induction, we require that $U_1 \subseteq \{0, 1, \ldots, 2m\} \setminus X$. Remove all vertices z with $|\pi_X(z)| < n$ from $\{0, 1, \ldots, 2m\} \setminus X$. Since each pair of vertices from X can eliminate at most one vertex, there are at least

$$2m + 1 - n - \binom{n}{2}.$$

vertices remaining, which is positive if m is sufficiently large and by the choice of n. Choose an arbitrary remaining vertex z_1 to form U_1.

Suppose that for a fixed $k \geq 1$ with $r > k$, the set U_k has been constructed with $|U_k| = k$. To form U_{k+1}, some vertices from $\{0, 1, \ldots, 2m\} \setminus (X \cup U_k)$ must be removed. As in the base step, by considering all the pairs of vertices from X, $\binom{n}{2}$ vertices are eliminated. Each vertex z in U_k satisfies $|\pi_X(z)| = n$. To ensure that $\pi_X(z) \cap \pi_X(z') = \varnothing$ for $z \in U_k$ and $z' \in U_{k+1}$, we must eliminate another $2kn$ vertices. Hence, there are at least

$$2m + 1 - n - \binom{n}{2} - 2kn$$

remaining vertices, which is positive for large m and by the choice of n and r. Add an arbitrary remaining vertex z_{k+1} to U_k to form U_{k+1}.

Now, suppose we use a template set $U = U_r$ with $|U| = r$. For a fixed z in U and x_i in X,

$$\mathbb{P}(z \text{ is c.j. to } x_i) \geq p.$$

The events "$(z, x_i) \in E$" and "$(z, x_j) \in E$" are independent (since $|\pi_X(z)| = n$). Since U is a template set, for z, z' distinct elements of U, the events "$(z, x_i) \in E$" and "$(z', x_i) \in E$" are independent as well. Hence,

$$\mathbb{P}(z \text{ is not c.j. to } X) \leq 1 - p^n,$$

and

$$\mathbb{P}(\text{No } z \text{ in } V(T) \text{ is c.j. to } X) \leq (1 - p^n)^r.$$

We therefore have that the probability P of the event that G is not n-e.c. satisfies

$$\begin{aligned}
P &\leq m^n 2^n (1 - p^n)^r \\
&= \exp\left(n(\log m + \log 2) - (1 + o(1))\frac{p^n m}{n^2}\right) \\
&= \exp\left(O(\log^2_{1/p} m) - (1 + o(1))\frac{p^{-4\log_{1/p}\log n - C}}{n^2}\right) \\
&= \exp\left(O(\log^2_{1/p} m) - p^{-C}\Omega(\log^2_{1/p} m)\right) \\
&= \exp\left(-\Omega(\log^2_2 m)\right) = o(1),
\end{aligned}$$

for C sufficiently large. □

Another adjacency property related to n-e.c. property was introduced by Schütte in [7]. Given a positive integer n, a tournament T satisfies *property* P_n if for any set S of n vertices of a tournament T, there is a vertex z which dominates all elements of S. Define $t_P(n)$ to be the minimum order of a tournament with property P_n. Note that $t_P(n) \leq t_{ec}(n)$ for any $n \geq 1$. In [17] Szekeres and Szekeres, via a clever argument (which we think deserves to be better known) showed that

$$t_P(n) \geq (n+2)2^{n-1} - 1, \tag{2.1}$$

so the same lower bound holds for $t_{ec}(n)$ (see Section 3 for more details). On the other hand, from Theorem 2.1 with $p = 1/2$, it follows that $t_{ec}(n) = O(n^4 2^n)$. Using random tournaments we have that $t_{ec}(n) = O(n^2 2^n)$. It follows that

$$\lim_{n\to\infty} t_{ec}(n)^{1/n} = 2.$$

However, the asymptotic order of $t_{ec}(n)$ is not known. An open problem is to determine whether the limits

$$\lim_{n\to\infty} \frac{t_{ec}(n)}{n2^n}, \qquad \lim_{n\to\infty} \frac{t_{ec}(n)}{t_P(n)}$$

exist, and if so to find their values.

Theorem 2.1 naturally extends to the infinite case. The connection set J is chosen with defining properties similar to the finite case, but $J \subseteq \mathbb{Z}\backslash\{0\}$, and we work in ordinary, non-modular arithmetic. With probability 1, a random choice of J gives rise to a countably infinite circulant tournament $G(J)$ which is n-e.c. for all $n \geq 1$ (that is, it is *e.c.*). There is a unique isotype, written T_∞, of countable tournament that is e.c. (see [5]). An analogous construction was given in [6] for circulant graphs, and was used in [18] to examine the cycle structure of the automorphisms of the infinite random graph. For an explicit construction of an e.c. $G(J)$, consider all finite binary sequences under the lexicographic order (according to increasing length), and form an infinite binary sequence Z by concatenating all of these sequences. Hence,

$$Z = 0100011011000001010\dots .$$

Let $Y = \{y_i\}_{i=1}^{\infty}$ be an enumeration of the terms of Z. Hence, $y_1 = 0$, $y_2 = 1$, $y_3 = 0$, $y_4 = 0$, and so on. Now, include $i \in J$ if and only if $y_i = 1$ (otherwise, $-i \in J$). It is not hard to see that $G(J)$ is e.c.

2.2. **Switching and new examples.** If T is a tournament and $A \subseteq V(T)$, then the tournament T_A is formed by reversing the arcs between A and $V(T) \setminus A$, and leaving all other arcs unaltered. We say that T_A is the tournament formed from T *by switching on* A. If H is an induced subtournament of T, then we will abuse notation and write T_H for $T_{V(H)}$. Using switching, we develop a method for explicitly constructing many n-e.c. tournaments from our circulant examples.

An n-e.c. *problem* is a pair (B, σ), where B is an ordered n-subset of vertices and σ is binary n-sequence; if $B = (x_1, \ldots, x_n)$ and $\sigma = (i_1 \ldots i_n)$, then a *solution* to (B, σ) is a vertex z not in B so that there is a directed edge from z to x_j if and only if $i_j = 1$. If (B, σ) is an n-e.c. problem, then the (B, σ)-*solution set* is the set of all solutions to the n-e.c. problem (B, σ). If B and σ are clear from context, then we will just say *solution set*. For simplicity, if B is clear from context, we will identify σ with the (B, σ)-solution set. For example, if B consists of 3 vertices x, y, and z, then (101) consists of $N^-(x) \cap N^+(y) \cap N^-(z)$

For a positive integer n, we say that T is *n-good* if:

(i) T is odd order and regular.
(ii) For all n-e.c. problems (B, σ), the solution set determined by B and σ has cardinality at least $n + 1$.

We note that an n-good tournament is n-e.c. A.a.s. random circulant tournaments are n-good. In particular, if $m = \Omega(n^4 2^n)$, then with positive probability $CT(m, 1/2)$ is n-good. By [1, 2, 10], a Paley tournament with sufficiently many vertices is n-good.

Theorem 2.2. *Let $n \geq 2$ be an integer, and let T be an n-good tournament. Then for all n-vertex induced subtournaments H of T, we have that T_H is n-e.c.*

Proof. Fix an n-subset A of $V(T)$. Let $0' = 1$ and $1' = 0$. Consider the n-e.c. problem (A, σ). Write $A = B \cup C$, where $B = A \cap V(H)$ and $C = A \backslash B$ (note that A or B may be empty). Let $\sigma_B = (i_1 \ldots i_k)$ be the subsequence of σ that corresponds to the elements of B, and let σ_C be the subsequence of σ that corresponds to the elements of C. Define $\sigma'_B = (i'_1 \ldots i'_k)$. Consider the n-e.c. problem $(A = B \cup C, \sigma'_B \sigma_C)$ with a solution z in T chosen outside H (which is permissible since G is n-good; see property (ii)). Then z solves (A, σ) in T_H. □

If T is a tournament with n vertices and out-degrees $d_1 \leq \cdots \leq d_n$, then the n-tuple $(d_1, \ldots, d_n)$ is called the *out-degree sequence* of T. Note that two distinct length n out-degree sequences must correspond to non-isomorphic tournaments (but the converse may fail). Let $\mathrm{ods}(n)$ be the number of distinct out-degree sequences of order n. We now apply Theorem 2.2 to give many non-isomorphic examples of n-e.c. tournaments.

Theorem 2.3. *Let $n \geq 2$ be any integer and let T be an n-good tournament. Then there are at least* $\mathrm{ods}(n)$*-many non-isomorphic n-e.c. tournaments of order $|V(T)|$.*

Proof. Say that T has constant in-degree r (and so has constant out-degree r). Fix H an n-vertex tournament. Since an n-e.c. tournament is $(n+1)$-universal (that is, includes an isomorphic copy of each tournament of order at most $n + 1$ as an induced subtournament), there is an isomorphic copy of H that is an induced subtournament of T. Let $J = T_H$; by Theorem 2.2, J is n-e.c. and has order $|V(T)|$.

Fix a vertex x in H, and suppose that $\deg^+_H(x) = k_x \geq 0$ (so $\deg^-_H(x) = n - 1 - k_x$). Then x is joined to $(r - n + k_x + 1)$-many vertices outside of H in J. Therefore,

$$\deg^+_J(x) = r - n + 2k_x + 1. \tag{2.2}$$

Now consider all the 2^n distinct solution sets $(i_1 \dots i_n)$, where $i \in \{0, 1\}$, in T determined by the n vertices of H (each of which is nonempty since T is n-e.c.). These solution sets partition $V(T)\backslash V(H)$ into 2^n sets. The out-degree of a vertex in $(1 \dots 1)$ in J is $r - n$; the out-degree of a vertex in $(01 \dots 1)$ in J is $r - n + 2$; the out-degree of a vertex in $(0 \dots 0)$ is $r + n = r - n + 2n$. In general, the out-degrees of vertices y in $V(T)\backslash V(H)$ in J are always one of the integers

$$r - n + 2j, \tag{2.3}$$

where $0 \leq j \leq n$.

For all $x \in V(H)$, $y \in V(T)\backslash V(H)$, we have that

$$\deg^+_J(x) \neq \deg^+_J(y).$$

This follows since by (2.2) $\deg^+_J(x)$ is of the form $r - n + m_1$, where m_1 is odd, and by (2.3) $\deg^+_J(y)$ is of the form $r - n + m_2$, where m_2 is even.

Suppose that H has out-degree sequence $\alpha = (d_1, \dots, d_n)$, with $d_1 \leq \dots \leq d_n$. If s is a sequence of positive integers, let $\langle s \rangle$ be the sequence with the same elements but sorted in non-decreasing order. By the above discussion, T_H has out-degree sequence $\langle(\widehat{\alpha}, \sigma)\rangle$, where $\widehat{\alpha} = (r - n + 2d_1 + 1, \dots, r - n + 2d_n + 1)$ is a subsequence consisting of out-degrees from vertices $V(H)$ in T_H, and σ is a subsequence containing the out-degrees $r - n, r - n + 2, \dots, r + n$ from the solution sets

$$(1 \dots 1),\ (01 \dots 1),\ \dots,\ (0 \dots 0),$$

respectively. Note that the elements of σ depend only on n and r, and not on the degrees in H. Furthermore, for any tournament H, by previous discussion, none of the terms of $\widehat{\alpha}$ can equal a term in σ. Suppose that H and H' have distinct out-degree sequences α and β, respectively. Therefore, by the above discussion, T_H and $T_{H'}$ have distinct out-degree sequences $\langle(\widehat{\alpha}, \sigma)\rangle$ and $\langle(\widehat{\beta}, \sigma)\rangle$, respectively. Hence, $T_H \ncong T_{H'}$ and the result follows. □

A straightforward inductive argument establishes that $2^{n-1} \leq \text{ods}(n)$. Hence, we obtain the following corollary, which gives new n-e.c. tournaments. We emphasize that the corollary gives an explicit, non-randomized method for constructing new n-e.c. tournaments (for example, when applied to Paley tournaments).

Corollary 2.4. *If there is an n-good tournament of order r, then there are at least 2^{n-1} non-isomorphic n-e.c. tournaments of order r.*

3. The unique minimum order 3-e.c. tournament

By the results of [17] it follows that $t_P(3) = 19$, and so $t_{ec}(3) \geq 19$. We verified that T_{19} is 3-e.c., and so $t_{ec}(3) = 19$. In this section, we describe our computer search which demonstrated that this is the only isotype of 3-e.c. tournament with 19 vertices. It seems that one should be able to do this by hand, although we have not found a simple way to do it. See Section 5 for more details.

Moreover, we checked that there is no other 3-e.c. isotype of order less than 23 vertices. We summarize our search results on small order 2- and 3-e.c. tournaments, respectively, in the following tables.

2-e.c. tournaments	
Order	Isotypes
7	1
8	0
9	14
10	1083

3-e.c. tournaments	
Order	Isotypes
19	1
20	0
21	0
22	0

The theoretical tools and methodology used in our computer search have some similarities to those used in [9], but some substantial adjustments were required.

Before we describe our computer search, we state some results on 3-e.c. tournaments which are of interest in their own right. We start with the following necessary condition for a tournament to be n-e.c. The proof of (3.1) uses an argument similar to the one of Szekeres and Szekeres [17], so let us start with restating their lemmas in terms of the n-e.c. property.

Lemma 3.1. *Let $n > 0$. If a tournament T is n-e.c., then for any set X of vertices with $|X| = n - 1$,*

$$\left|\bigcap_{x \in X} N^-(x)\right| \geq n+1 \quad \textit{and} \quad \left|\bigcap_{x \in X} N^+(x)\right| \geq n+1.$$

Proof. Let $X \subseteq V(T)$, $|X| = n - 1$. We prove the first inequality only, as the second one can be shown analogously. For a contradiction, suppose that the set $Z = \bigcap_{x \in X} N^-(x)$ has at most n vertices. As T is n-e.c., then there exists a vertex $v \in \bigcap_{z \in Z} N^-(z)$. Furthermore, there exists a vertex

$$u \in \bigcap_{x \in X \cup \{v\}} N^-(x) \subseteq Z.$$

But $u \in Z$ implies that $u \in N^+(v)$, which gives us a contradiction. □

Lemma 3.2. *Let $n > m > 0$. If a tournament T is n-e.c., then for any set X of vertices with $|X| = m$,*

$$\left|\bigcap_{x\in X} N^-(x)\right| \geq 2^{n-m-1}(n+2)-1 \quad \text{and} \quad \left|\bigcap_{x\in X} N^+(x)\right| \geq 2^{n-m-1}(n+2)-1.$$

Proof. Let $X \subseteq V(T)$, $|X| = m$. Again, we prove the first inequality only. We prove it by (downward) induction on m with $1 \leq m \leq n-1$. The base case ($m = n-1$) is obtained directly from Lemma 3.1.

For the inductive step, suppose that for a given $m+1 < n$ and for any set $Y \subseteq V(T)$ with $|Y| = m+1$ an inequality $\left|\bigcap_{x\in Y} N^-(x)\right| \geq 2^{n-m-2}(n+2)-1$ holds. Now, let $X \subseteq V(T)$ be any set of m vertices, $Z = \bigcap_{x\in X} N^-(x)$, and $k = |Z|$. Since T is n-e.c. (so, in particular, m-e.c.), $|Z|$ is not empty. Let us consider a subtournament $T[Z]$. Clearly, the number of edges of $T[Z]$ is $\binom{k}{2}$. It follows from the inductive hypothesis that the in-degree of any vertex in $T[Z]$ is at least $2^{n-m-2}(n+2)-1$. Hence, we obtain that

$$\binom{k}{2} \geq k\left(2^{n-m-2}(n+2)-1\right),$$

which gives $k \geq 2^{n-m-1}(n+2)-1$. □

Now, we are ready to present a tool that is needed to achieve our goal. We conjecture that the stronger statement holds, that is, (3.1) can be replaced by $|Z| \geq 2^{n-m-1}(n+2)-1$, as in Lemma 3.2. This is shown to be the case when X or Y is empty but the general case still remains an open problem.

Theorem 3.3. *Let $n > m > 0$. If a tournament T is n-e.c., then for all disjoint sets of vertices X and Y with $|X \cup Y| = m$ (one of X or Y can be empty), the set $Z = Z(X,Y)$ of vertices defined as*

$$Z = \Big(\bigcap_{x\in X} N^-(x)\Big) \cap \Big(\bigcap_{y\in Y} N^+(y)\Big)$$

satisfies

$$|Z| \geq 2^{n-m-1}(n+1)-1, \tag{3.1}$$

and a tournament induced by Z is $(n-m)$-e.c.

Proof. Let $X, Y \subseteq V(T)$, $X \cap Y = \varnothing$, and $|X \cup Y| = m$. We will show that $T[Z]$ is $(n-m)$-e.c. and that (3.1) holds. First note that $|Z| \geq 2^{n-m}$. Indeed, since T is n-e.c., for any $(n-m)$-subset $S \subseteq V(T) \setminus (X \cup Y)$ of vertices, there exist 2^{n-m} vertices dominating each vertex of X, no vertex of Y, and connected to S in all possible ways.

Let now $A, B \subseteq Z$, $A \cap B = \varnothing$, and $|A \cup B| = n-m$. In order to prove that $T[Z]$ is $(n-m)$-e.c. it is enough to show that there is a vertex $z \in Z \setminus (A \cup B)$ that is correctly joined to A and B. Since T is n-e.c., $X \cap Z = \varnothing$, $Y \cap Z = \varnothing$, there is a vertex $z' \in V(T)$ correctly joined to $A \cup X$ and $B \cup Y$, and $z' \in Z \setminus (A \cup B)$.

We prove (3.1) by (downward) induction on m with $1 \le m \le n-1$. For the base case ($m = n-1$), we would like to show that $|Z| \ge n$. For a contradiction, suppose that $|Z| \le n-1$. But then, according to Lemma 3.1, Z is dominated by at least $n+1$ vertices. So there exists a vertex $v \in V(T)\backslash Y$ which dominates all the vertices in Z. In addition, since T is n-e.c., there exists a vertex

$$u \in N^-(v) \cap \Big(\bigcap_{x \in X} N^-(x) \Big) \cap \Big(\bigcap_{y \in Y} N^+(y) \Big) \subseteq Z.$$

But $u \in Z$, implies that $u \in N^+(v)$, which gives us a contradiction.

For the inductive step, suppose that (3.1) holds for a given m, $2 \le m \le n-1$. In other words, for any two sets $A, B \subseteq V(T)$, $A \cap B = \varnothing$, and $|A \cup B| = m$,

$$|Z(A,B)| \ge 2^{n-m-1}(n+1) - 1.$$

Let $X, Y \subseteq V(T)$, $X \cap Y = \varnothing$, and $|X \cup Y| = m-1$. We would like to show that (3.1) holds for $Z(X,Y)$; that is,

$$|Z(X,Y)| \ge 2^{n-m}(n+1) - 1.$$

Let $v \in Z(X,Y)$. By inductive hypothesis, both $Z_1 = Z(X \cup \{v\}, Y)$ and $Z_2 = Z(X, Y \cup \{v\})$ have at least $2^{n-m-1}(n+1) - 1$ elements. Then the set $Z = Z(X,Y) = Z_1 \cup Z_2 \cup \{v\}$ must have at least

$$2\left(2^{n-m-1}(n+1) - 1\right) + 1 = 2^{n-m}(n+1) - 1$$

elements, which finishes the proof. □

According to the necessary condition we can construct a 3-e.c. tournament using 2-e.c. ones as building blocks. By a computer search, there are 14 and 1083 isotypes of 2-e.c. tournaments on 9 and 10 vertices, respectively. However, only 2 and 295 of them have both out- and in-degree at least 4, which is necessary according to Lemma 3.2. We now give a high-level description of the computational approach that we used to determine that T_{19} is the only 3-e.c. tournament on 19 vertices. Suppose that G has 19 vertices and is 3-e.c.; each vertex has out- and in-degree 9 by Lemma 3.2. Fix a vertex v_0 and insert a 2-e.c. tournament on 9 vertices on vertex set $X = N^+(v_0)$. It remains to check that we get a tournament isomorphic to T_{19} when edges between X and $Y = N^-(v_0)$ and those within Y are distributed to satisfy the necessary condition stated in Theorem 3.3. In order to do this, we can take any vertex $v_1 \in X$ and assign to this vertex in-neighbours from Y so that both out- and in-neighbourhoods induce a 2-e.c. tournament. This assignment may be done in many different ways. Next, we can take any other vertex v_2 and try to assign in-neighbours to keep the required property. We take v_2 from the set of vertices that are not *processed* (in this case, not equalling v_0 nor v_1) for which the number of determined incident arcs is maximized; this helps to minimize the number of cases. We repeat this process to discover that there is no chance to create a 3-e.c. tournament different than T_{19}.

Two improvements are crucial. We improve the running time of the algorithm dramatically by checking (at each step) the necessary condition stated in Theorem 3.3. After vertex v_1 is processed, we check the condition with $m = 2$ for the two vertices that are processed at this point, that is, vertices v_0, v_1. All configurations that fail this test are removed. In the next steps, after satisfying a new vertex v_i, the additional test is checked for $m = 2$ and $m = 3$, and for all sets of processed vertices containing vertex v_i we deal with at the current round.

In order to remove unnecessary configurations we use McKay's `nauty` software package [13] for computing automorphism groups of graphs and digraphs. We cannot use, however, the package directly since it does not support removing isomorphisms in digraphs. Furthermore, we need to keep the information of which vertices are processed (note that this cannot be determined; having all out- and in-arcs determined is only a necessary condition for a vertex to be processed). To overcome this problem we introduce a bijection from our configuration to an undirected graph H on $3|V(G)|+4$ vertices. Let

$$V(G) = \{v_1, v_2, \ldots, v_n\}$$

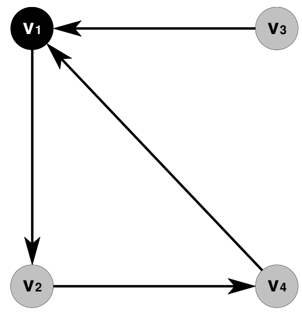
and let

$$V(H) = \{x_1, x_2, y, z\} \cup \{u_1, u_2, \ldots, u_n\} \cup \{s_1, s_2, \ldots, s_n\} \cup \{t_1, t_2, \ldots, t_n\}.$$

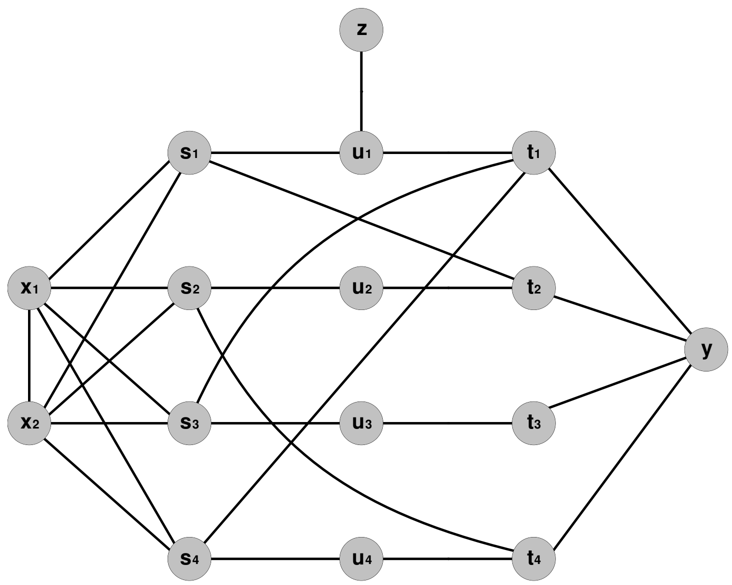
Now, we construct H as follows: $s_i t_j \in E(H)$ if and only if $(v_i, v_j) \in E(G)$ (this corresponds to the arcs of G), $s_i u_i \in E(H)$ and $u_i t_i \in E(H)$ for $i \in [n]$ (to match s_i's with t_i's), $x_1 x_2 \in E(H)$, $x_i s_j \in E(H)$ and $y t_j \in E(H)$ for $i = 1, 2$, $j \in [n]$ (to distinguish the input from the output). Note that x_1, x_2 are the only vertices of degree $n+1$ in H, while y has degree n. All other vertices have degree less than n. Finally, $u_i z \in E(H)$ if v_i is processed. An example of this transformation is depicted in Figure 2; vertex v_1 is processed. It is clear that we can reconstruct the graph G, together with the information of which vertices are processed, from H.

The operation of removing isomorphisms, together with checking the additional condition, can decrease the number of configurations by up to 90% in each round. The first operation works well during the first few rounds, whereas the second one works better later on.

The same approach can be used to show that there is no isotype of 3-e.c. tournament on 20, 21, or 22 vertices. In order to eliminate 21 and 22 we start by inserting a tournament on at most 10 vertices on $N^+(v_0)$ (one out of $297 = 2 + 295$ ones). It is worth noting that the Paley tournament T_{23} of order 23 is also 3-e.c. Moreover we have checked that it is the unique 3-e.c. doubly-regular (Hadamard) tournament of such order (see Section 4 for the definition and more details). We were not able to check all possibilities of construction of 3-e.c. tournament order 23 in the way shown, because the number of building blocks on 11 vertices (that is, the number of isotypes of 2-e.c. tournaments of out- and in-degree at least 4) is high: there are $131,396$ such tournaments.



The digraph G.



The graph $H = H(G, \{v_1\})$.

FIGURE 2. A transformation from G to H.

4. 4-E.C. TOURNAMENTS AND SKEW HADAMARD MATRICES

Before we move to the 4-e.c. property, we need to introduce a few definitions first. A *doubly regular tournament* of order n is a tournament T such that there exist integers m_1, m_2, satisfying $|N^+(v)| = m_1$ for every vertex $v \in V(T)$, and $|N^+(v) \cap N^+(u)| = m_2$ for every pair of distinct nodes $v, u \in V(T)$. It is not difficult to see that $m_1 = 2m_2 + 1$ and $n = 4m_2 + 3$ so that an order of any doubly regular tournament is congruent to 3 (mod 4). A *Hadamard matrix* H of order $4m$ is a $4m \times 4m$ matrix of ± 1's such that $HH^T = 4mI$, where I is the identity matrix. A Hadamard matrix is *skew* if $H + H^T = 2I$. In [15], it has been shown that the existence of a doubly regular tournament of order n is equivalent to the existence of a skew

Hadamard matrix of order $n+1$. For more on skew Hadamard matrices, see the survey [16].

Now, we are ready to come back to our problem. We know that $t_{ec}(4) \geq 47$. From Lemma 3.2 we derive that in any 4-e.c. tournament

$$\begin{gathered} |N^+(x)| \geq 23, \quad |N^+(x) \cap N^+(y)| \geq 11, \\ |N^+(x) \cap N^+(y) \cap N^+(z)| \geq 5, \\ |N^-(x)| \geq 23, \quad |N^-(x) \cap N^-(y)| \geq 11, \text{ and} \\ |N^-(x) \cap N^-(y) \cap N^-(z)| \geq 5, \end{gathered} \tag{4.1}$$

for all distinct triples of vertices x, y, z.

Hence, if there exists a 4-e.c. tournament on 47 vertices, then it must be doubly regular (in fact, even *triply regular*). Therefore, the connection with Hadamard tournaments and skew Hadamard matrices we discussed earlier may be of use. We know that there exist doubly regular tournaments on 47 vertices. The incomplete list was computed by Brendan McKay [12] using skew Hadamard matrices from Christos Koukouvinos's catalogue [11]. We have examined three tournaments on 47 vertices and $36,350$ tournaments on 51 vertices, but no such tournament was 4-e.c. As the list is incomplete no definitive conclusion can be obtained, but we do think that the connection may play an important role in determining the minimum order of 4-e.c. tournaments.

However, we emphasize that the Paley tournament T_{23} is the unique 3-e.c. doubly regular (Hadamard) tournament of order 23, so if there is a 4-e.c. tournament of order 47, then for every vertex v both $N^+(v)$ and $N^-(v)$ induce T_{23}. Similarly, from (4.1), a 4-e.c. tournament of order 48 must contain T_{23} as an induced subgraph by considering either the out- or in-neighbourhood of each vertex. With computer support, we verified that when the vertex set of T_{23} is partitioned into two subsets A, B, containing $|A| = 11$ and $|B| = 12$ vertices, respectively, then at least one of them does not induce a 2-e.c. tournament. (We have verified that there are 35 isotypes of 2-e.c. tournaments that can be obtained by appropriately choosing the set A.) This implies that there cannot be a 4-e.c. tournament of order either 47 or 48. (It may be possible to show this without computer support using the above property of T_{23}.)

The bound (2.1) gives that $t_P(5) \geq 111$. We verified that T_{67} is the smallest order Paley tournament that has property P_4, and it is also 4-e.c. Hence,

$$49 \leq t_{ec}(4) \leq 67.$$

We checked that T_{359} is the first Paley tournament that is 5-e.c. which implies that

$$111 \leq t_{ec}(5) \leq 359.$$

We note that T_{331} is the first Paley tournament that has property P_5.

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We would like to thank the anonymous referee for many valuable comments and suggestions. We verified that the Paley tournament is the unique 3-e.c. tournament of order 19 using a computer. As pointed out by the referee, proving that T_{19} is the unique 3-e.c. tournament of order 19 can be done by hand, and one might possibly learn something important during this process. It follows from Lemma 3.2 that a 3-e.c. tournament must have all degrees at least 9, and that corresponding 2-e.c. tournaments of order 9 must be regular. Using these facts, one may show by hand that there are only two doubly regular tournaments of order 19 (see also [12, 11] where this fact can be confirmed), and complete the analysis by hand.

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- (ii) the Atlantic Computational Excellence Network ACEnet, Memorial University of Newfoundland, St. John's, NL, Canada (www.acenet.ca): $3,324$ CPUs.

The programs used to obtain the results of Section 3 may be downloaded from [14].

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