Spanning Subgraphs of Graphs Partitioned into Two Isomorphic Pieces

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Abstract: A graph has the neighbor-closed-co-neighbor, or ncc property, if for each of its vertices x, the subgraph induced by the neighbor set of x is isomorphic to the subgraph induced by the closed non-neighbor set of x. As proved by Bonato and Nowakowski [5], graphs with the ncc property are characterized by the existence of perfect matchings satisfying certain local conditions. In the present article, we investigate the spanning subgraphs of ncc graphs, which we name sub-ncc. Several equivalent characterizations of finite sub-ncc graphs are given, along with a polynomial time algorithm for their recognition. The infinite sub-ncc graphs are characterized, and we demonstrate the existence of a countable universal sub-ncc graph satisfying a strong symmetry condition called pseudo-homogeneity.

Keywords: *spanning subgraph; ncc graph; matching; local property; polynomial time algorithm; pseudo-homogeneous graph*

Dedicated to the memory of Günter Bruns.
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1. INTRODUCTION

All the graphs in this article are undirected and simple. The *infinite random graph* R is the unique isomorphism type of countably infinite graph satisfying the *existentially closed* or *e.c.* adjacency property: for all finite sets of vertices X and Y, there is vertex not in $X \cup Y$ that is joined to each vertex of X and to no vertex of Y. The graph R derives its name from the fact that with probability 1, a countably infinite graph with edges chosen independently and with fixed probability is e.c. The graph R exhibits many remarkable graph-theoretic, algebraic, and topological properties as documented in the surveys of Cameron [8,9]. Let $N(x)$ and $N^c(x)$ denote the set of neighbors and non-neighbors (not including x) of a vertex x, respectively. A graph G has property $(\mathcal{N})$ if the subgraph induced in G by $N(x)$ is isomorphic to G; property $(\mathcal{N}^c)$ is defined similarly. It is not hard to show using the e.c. property that R has both properties $(\mathcal{N})$ and $(\mathcal{N}^c)$. However, it is an open problem to determine if there are countably infinite graphs non-isomorphic to R with both properties $(\mathcal{N})$ and $(\mathcal{N}^c)$; see Problem 411 in [10].

The properties $(\mathcal{N})$ and $(\mathcal{N}^c)$ are examples of *vertex partition* or *fractal-type properties*, which are usually studied in the context of infinite graphs. Other vertex partition properties that have been studied include the *pigeonhole property* [2,4], *inexhaustibility* [3,6], and *indivisibility* [13,14].

The following analog of $(\mathcal{N})$ and $(\mathcal{N}^c)$ for finite graphs was introduced by Nowakowski and the author in [5]. We use the notation $G \restriction S$ for the subgraph of G induced by a set of vertices S, and the notation $G \cong H$ for isomorphic graphs. The *closed non-neighbor set* of x, written $N^c[x]$, is the set of non-neighbors of x, including x itself. A graph G has the *neighbor-closed-co-neighbor* or *ncc* property, if for all $x \in V(G)$, $G \restriction N(x) \cong G \restriction N^c[x]$. There are many examples of such graphs, including the bipartite cliques $K_{n,n}$ and the Cartesian products of cliques with K_2, written $K_n \square K_2$. A similar but unrelated property of *neighborhood symmetry* was introduced by Froncek [16].

A *disjoint neighbor perfect* or *dnp matching* M is a perfect matching with the additional property that if $e = ab \in M$, then $N(a) \cap N(b) = \emptyset$; we say the edge ab is *dn*. For example, every perfect matching in a bipartite graph is dnp, and there is a unique dnp matching in $K_n \square K_2$. The following characterization of the ncc graphs was given in [5].

Theorem 1. *A finite graph G is ncc if and only if there is a positive integer n so that the order of G is $2n$, G is n-regular, and G has a dnp matching.*

As a corollary of Theorem 1, it was shown in [5] that the ncc graphs may, perhaps surprisingly, be recognized in polynomial time. Another characterization of ncc graphs will be useful. A graph G has a *locally C_4 perfect matching* if G has a perfect matching M with the property that every pair of distinct edges of M induce a 4-cycle.

Theorem 2. *A finite graph G is ncc if and only if it has a locally C_4 perfect matching.*

Proof. If G is ncc, then by Theorem 1, there is some positive integer n so that G is n-regular and has a dnp matching M. If $ab \in M$, then it follows that $N(b) = N^c[a]$ and $N(a) = N^c[b]$. Therefore, M is a locally C_4 perfect matching.

Conversely, if G has a locally C_4 perfect matching M, then M is clearly dnp. If $ab \in M$, then for all $x \in V(G)$, x is joined to exactly one of a or b. Hence, if M contains n edges, then G is n-regular. Hence, G is ncc by Theorem 1. ∎

The following theorem of [5] proves that the class of all induced subgraphs of ncc graphs is not interesting.

Theorem 3. *If G is a graph of order n, then G is isomorphic to an induced subgraph of an ncc graph of order at most $2n$.*

When studying a graph property, it is often useful to consider the class of *spanning* subgraphs of graphs with the property. An example of this is the rich theory of partial k-trees, which are spanning subgraphs of k-trees. A different situation emerges if we consider spanning subgraphs of ncc graphs, which we refer to as *sub-ncc*. For example, a *balanced bipartite* graph (that is, one where the two vertex classes have the same cardinality) is sub-ncc, while a clique with at least three vertices is not.

The goal of the present article is to investigate the sub-ncc graphs. Several equivalent characterizations of finite sub-ncc graphs are given in Theorem 5. With the aid of Theorem 5, we present a polynomial time algorithm for the recognition of sub-ncc graphs; see Corollary 10. We introduce a new graph parameter, the dn number, which measures in a sense how close a graph is to being sub-ncc. A polynomial time algorithm to compute the dn number of a graph is given in Theorem 12.

The infinite sub-ncc graphs are characterized in Theorem 13. This theorem is applied to prove the existence of a countable sub-ncc graph M containing all the countable sub-ncc graphs as induced subgraphs, and satisfying strong symmetry conditions. The graph M is the so-called universal pseudo-homogeneous sub-ncc graph; see Theorem 14.

2. THE STRUCTURE OF SUB-NCC GRAPHS

All graphs in this and the next section are finite. Theorem 5 of this section characterizes the finite sub-ncc graphs. Before this result can be stated, a few definitions are required.

A *pairing* P is a set of 2-element subsets of $V(G)$ called *pairs*, so that if p and p' are distinct pairs of G, then $p \cap p' = \emptyset$. In particular, a pairing is a matching if each pair forms an edge of the graph. Trivially, all even order graphs have a pairing containing all vertices. A *dnp pairing* is a pairing P with the following properties:

(1) for all $x \in V(G)$, there is a pair $p \in P$ so that $x \in p$;
(2) for all $p = \{a,b\} \in P$, $N(a) \cap N(b) = \emptyset$.

We call $\{a,b\} \in P$ a *dn pair*. An ncc graph, a triangle-free graph with a perfect matching, or a balanced bipartite graph has dnp pairings. An odd order graph or a graph of order $2n$, whose maximum degree is strictly greater than n, does not have a dnp pairing.

A graph G satisfies the *weak ncc property* if for all $x \in V(G)$ there are subsets of vertices $A(x), B(x)$ partitioning $V(G)$ so that

(A) the vertex x is isolated in $G \restriction A(x)$, and there is a bijection $f_x : A(x) \to B(x)$ with $f_x(x)$ is isolated in $G \restriction B(x)$;
(B) for all $x, y \in V(G)$,

$$f_x(x) = \begin{cases} f_y(x) & \text{if } x \in A(y); \\ f_y^{-1}(x) & \text{if } x \in B(y). \end{cases}$$

The name of this property derives in part from the following lemma.

Lemma 4. *If G is a finite ncc graph, then G has the weak ncc property.*

Proof. Let G be ncc with a dnp matching $\{a_i b_i : 1 \le i \le n\}$. Theorem 2 implies that the mapping

$$M : G \restriction \{a_i : 1 \le i \le n\} \to G \restriction \{b_i : 1 \le i \le n\}$$

defined by $M(a_i) = b_i$ is an isomorphism. For all $x \in V(G)$, let $A(x) = N^c[x]$ and $B(x) = N(x)$. By Theorem 2, for all $i \in \{1, \ldots n\}$ and $x \in V(G)$,

$$a_i \in N^c[x] \text{ if and only if } b_i \in N(x). \tag{2.1}$$

For all $x \in V(G)$, define $f_x : A(x) \to B(x)$ by

$$f_x(z) = \begin{cases} M(z) & \text{if } z \in A(x) \cap \{a_i : 1 \le i \le n\}; \\ M^{-1}(z) & \text{if } z \in A(x) \cap \{b_i : 1 \le i \le n\}. \end{cases}$$

For a fixed $x \in V(G)$, the function f_x is a well-defined bijection by (2.1), and is an isomorphism by Theorem 2. Hence, condition (A) holds in the definition of the weak ncc property. For condition (B), suppose that $x = a_i$ for some i (the case when $x = b_i$ is similar, and so is omitted). Fix $y \in V(G) \setminus \{x\}$. If $x \in A(y)$, then $f_x(x) = b_i = f_y(x)$. If $x \in B(y)$, then $f_x(x) = b_i = f_y{}^{-1} f_y(b_i) = f_y{}^{-1}(x)$. ∎

If G is a graph, then define $G^{-\triangle}$ to be spanning subgraph of G formed by deleting each edge of every K_3 in G. A *Tutte set* in G is a set T of vertices so that the number of odd order connected components of $G - T$ has cardinality strictly greater than $|T|$.

Theorem 5. *The following are equivalent for a graph G of order n.*

(1) *The graph G is sub-ncc.*
(2) *The graph G has the weak ncc property.*
(3) *At most n edges may be added to G to obtain a dnp matching.*
(4) *At most n edges may be added to $G^{-\triangle}$ to obtain a perfect matching.*
(5) *At most n edges may be added to $G^{-\triangle}$ so the resulting graph has no Tutte set.*
(6) *The graph G has a dnp pairing.*

Proof. $(1 \Rightarrow 2)$ Let G be a spanning subgraph of some ncc graph H. For $x \in G$, let $A(x) = N^c[x]$ in H, and let $B(x) = N(x)$ in H. Define $f_x : N^c[x] \to N(x)$ in H as in the proof of Lemma 4. Then $f_x : A(x) \to B(x)$ in G is a bijection. For (A) in G, note that x is isolated in $A(x)$, and $f_x(x)$ is isolated in $B(x)$, as an isolated vertex of H is also isolated in G. The proof of (B) follows as in the proof of Lemma 4.

$(2 \Rightarrow 3)$ Fix $x \in V(G)$, and let $x' = f_x(x)$. By item (A) for G, $\{x, x'\}$ is a dnp pair. Delete the vertices x and x' and all edges incident with either x or x' to form the induced subgraph G'. We claim that G' has the weak ncc property. To see this, note that by item (B) for G, for all vertices y, exactly one of x and x' is in each of $A(y)$ and $B(y)$. Suppose that $x \in A(y)$ and $x' \in B(y)$ (the case when $x \in B(y)$ and $x' \in A(y)$ is similar and so is omitted). In G', define $A'(y)$ to be $A(y)\backslash\{x\}$ and $B'(y) = B(y)\backslash\{x'\}$.

For all vertices y, define $f'_y : A'(y) \to B'(y)$ to be the restriction of f_y to $A'(y)$. This map is well defined since by (B), $f_y(x) = x'$. Since f_y is a bijection, the map f'_y is a bijection. Further, y is isolated in $A'(y)$ and $f_y(y)$ is isolated in $B'(y)$. Hence, (A) holds for G'. For (B), fix $z \in V(G')\backslash\{y\}$. Suppose first that $z \in A(y)$. As $f_z(z) \neq x, x'$, we have that

$$f'_z(z) = f_z(z) = f_y(z) = f'_y(z),$$

where the second equality follows by (B) for G. If $z \in B(y)$, then

$$f'_z(z) = f_z(z) = f_y^{-1}(z) = f'^{-1}_y(z),$$

where the second equality holds by (B) for G.

By induction, we obtain a dnp pairing P of G. We may therefore add at most n edges to P to obtain a dnp matching, so item (3) of the statement of the theorem follows.

The proof that $(3 \Leftrightarrow 4)$ follows since a dnp matching in G is equivalent to perfect matching in $G^{-\triangle}$. The proof of $(4 \Leftrightarrow 5)$ follows by a theorem of Tutte [19] which states that a graph has a perfect matching if and only if it has no Tutte set. The proof of $(3 \Rightarrow 6)$ is immediate.

$(6 \Rightarrow 1)$ Let $P = \{\{a_i, b_i\} : 1 \leq i \leq n\}$ be a dnp pairing of G. Let G' be the graph formed by adding the edges $a_i b_i$ if necessary, so P becomes a dnp matching M in G'. Let H be an edge maximal spanning subgraph of K_{2n} containing G' so that no edge of M in H is in a K_3. The proof will follow once we show that H is ncc. To see this, since M is a dnp matching in H, by Theorem 2, it is enough to verify that M has the property that for all i if $a_i b_i \in M$, then $N^c(a_i) \cap N^c(b_i) = \emptyset$. Suppose otherwise, and without loss of generality, there is an i and j so that $b_j \in N^c(a_i) \cap N^c(b_i)$. If $a_i a_j \in E(H)$, then by maximality of H and the fact that M is dnp, we must have that $b_i b_j \in M$, which is a contradiction. If $a_i a_j \notin E(H)$, then add $a_i b_j$ to H to form H'. In H', the matching M is dnp, which is a contradiction of the maximality of H. ■

Theorem 5 provides some insight into the class of sub-ncc graphs. For instance, "most" finite graphs are not sub-ncc. For background on random graphs, the reader is directed to [1]. *Almost no* $G \in G(n,p)$ has a graph property $\mathcal{P}$ if the probability that $\mathcal{P}$ holds for G tends to 0 as n tends to ∞. The term "almost all" is defined in a similar fashion.

Corollary 6. (1) *If G has diameter* 2 *and has the property that every edge is in a K_3, then G is not sub-ncc.*
(2) *For a fixed $p \in (0,1)$, almost no $G \in G(n,p)$ is sub-ncc.*

Proof. Item (1) follows since G has no dn pair. Item (2) follows from item (1), since it is well known that almost all $G \in G(n,p)$ are diameter 2 and have the property that each edge is in a triangle. ■

If G and H are graphs, then we write the *Cartesian product* of G and H as $G \,\square\, H$, the *categorical product* of G and H as $G \times H$, and the *strong product* of G and H as $G \boxtimes H$. Information on these and other products may be found in Imrich and Klavzar [17]. The disjoint union of graphs G and H is written $G + H$.

The example with $G = H \cong K_2$ demonstrates that $G \boxtimes H$ need not satisfy the sub-ncc property even if both G and H do. However, the Cartesian and categorical products do preserve the sub-ncc property.

Corollary 7. *Let G and H be finite graphs.*

(1) If *G and H are sub-ncc, then $G + H$ is sub-ncc.*
(2) *If G is sub-ncc and H is any graph, then both $G \,\square\, H$ and $G \times H$ are sub-ncc.*
(3) *If G is a graph of order n, then $G + \overline{K_n}$ is sub-ncc. In particular, every graph G is the induced subgraph of a sub-ncc graph with the same clique and chromatic number as G.*

Proof. Item (1) is immediate from Theorem 5. If G has a dnp pairing $\{\{a_i, b_i\} : 1 \leq i \leq n\}$, then it is not hard to see that the set

$$\{\{(a_i, x), (b_i, x)\} : 1 \leq i \leq n, x \in V(H)\}$$

is a dnp pairing of both $G \,\square\, H$ and $G \times H$. Hence, item (2) follows from Theorem 5. For item (3), a dnp pairing is formed by pairing each vertex of G with a unique vertex of $\overline{K_n}$. ■

Let G be a connected graph. Define $e(x)$ to be the set of vertices of G which are of even distance to x (including x, which we take as distance 0 from itself). Define $o(x)$ to be the set of vertices of G which are of odd distance to x. The graph G has the *ncc-parity property* if for all $x \in V(G)$, the subgraph induced by $e(x)$ is isomorphic to the subgraph induced by $o(x)$. For example, ncc graphs are ncc-parity, since they are diameter 2. Any connected balanced bipartite graph is ncc-parity.

A problem posed in [5] was to give a characterization of the ncc-parity graphs. (In [5], the ncc-parity graphs were referred to as ncc(e) graphs.) This problem remains open, but the following holds.

Theorem 8. *Every finite ncc-parity graph is sub-ncc.*

Proof. Let G be an ncc graph of order $2n$, where n is a fixed positive integer, and let x be a vertex of G. The vertex x is isolated in $G \restriction e(x)$. Since $G \restriction e(x) \cong G \restriction o(x)$, there are the same number, say k_x many, of isolated vertices in $G \restriction e(x)$ and $G \restriction o(x)$. Let $\{x_i : 1 \leq i \leq k_x\}$ and $\{y_i : 1 \leq i \leq k_x\}$ be the set of isolated vertices in $G \restriction e(x)$ and $G \restriction o(x)$, respectively. Note that $\{x_i, y_i\}$ is a dnp pair for all $1 \leq i \leq k_x$.

Define

$$P_x = \{\{x_i, y_i\} : 1 \leq i \leq k_x\},$$

and define $P_{x_i} = P_{x_j} = P_{y_i} = P_{y_j}$, for all $1 \leq i, j \leq k_x$. Proceed inductively to define

$$P = \bigcup_{z \in V(G)} P_z.$$

The set P is a dnp pairing. To see that P is a pairing, suppose to the contrary that there are two pairs $\{x, y\}$ and $\{x, y'\}$ in P. But then y and y' are in the set $\{y_i : 1 \leq i \leq k_x\}$. But x is paired with a unique element of $\{y_i : 1 \leq i \leq k_x\}$. The pairing P is dnp by construction. The proof now follows by Theorem 5. ■

3. RECOGNIZING SUB-NCC GRAPHS

Theorem 5 may be used to give a polynomial time algorithm for recognizing sub-ncc graphs.

Algorithm for recognizing sub-ncc graphs:
Input: A graph G.
Output: YES if G is sub-ncc, NO otherwise.

(1) If $|V(G)|$ is odd, then output NO. Otherwise, say $|V(G)| = 2n$ for some positive integer n and proceed to (2).
(2) For each edge $e = ab$ of G, if $N(a) \cap N(b) \neq \emptyset$, then give e weight 2. Otherwise, give e weight 1. For each non-edge consisting of vertices a, b of G, if $N(a) \cap N(b) \neq \emptyset$ in G, then add the edge $e = ab$ to G and give e weight 2. Otherwise, add the edge $e = ab$ and give it weight 1. Let K^*_{2n} be the resulting edge-weighted clique.
(3) Determine if K^*_{2n} has a perfect matching of weight n. If there is such a matching, then return YES. If K^*_{2n} has no such matching, then return NO.

The proof of the validity of this algorithm relies on the following Theorem.

Theorem 9. *A graph G of order $2n$ is sub-ncc if and only if K^*_{2n} has a perfect matching of weight n.*

Proof. Suppose first that G is sub-ncc. By Theorem 5, G has a dnp pairing P. But then in K^*_{2n}, P is a perfect matching of weight n. Conversely, suppose that K^*_{2n} has a perfect matching M of weight n. Since some edges of M may be deleted in passage from K^*_{2n} back to G, M is a pairing P in G. Since each edge of M is weight 1, P is a dnp pairing in G. ■

Corollary 10. *Recognizing whether a finite graph is sub-ncc may be done in polynomial time.*

Proof. The construction of K^*_{2n} from G may be done in polynomial time. A minimum weight perfect matching in graph with positive integer edge weights may be found in polynomial time by the results of Edmonds [12]. The proof now follows by Theorem 5 since a minimum weight perfect matching in K^*_{2n} must have weight n. ■

We introduce a strengthening of the above algorithm, which, in a certain sense, computes how far a graph is from having an dnp pairing. A *dn pairing* is a pairing satisfying item (2) in the definition of dnp pairing. Define the *dn pairing number* of G, written $dn(G)$, to be the cardinality of a dn pairing in G with the maximum number of pairs. For example, a graph G with $2n$ vertices is sub-ncc if and only if $dn(G) = n$. A graph G has $dn(G) = 0$ if and only if G is diameter 2 and every edge of G is contained in a K_3.

There are examples to demonstrate that $dn(G)$ may obtain any value in $\{0, 1, \ldots, \lfloor \frac{n}{2} \rfloor\}$. Fix $n \geq 2$, and let $1 \leq j \leq \lfloor \frac{n}{2} \rfloor$ be fixed. Define a graph $K(n,j)$ by joining j endvertices to some fixed set of j distinct vertices in K_{n-j}. Then $dn(K(n,j)) = j$, since a dn pair must have at least one element not in K_{n-j}.

Algorithm for computing $dn(G)$:
Input: A graph G with n vertices.
Output: $dn(G)$.

(1) Add edges to distinct non-joined vertices of G. Assign the weight $\epsilon = 1/(n+1)$ to all edges xy so that $N(x) \cap N(y) \neq \emptyset$ in G. Assign the weight 1 to all edges xy so that $N(x) \cap N(y) = \emptyset$ in G. Let K_n^* be the resulting edge-weighted clique.
(2) Find a maximum weight matching M in K_n^*. The sum of weights of edges in M is written $wt(M)$. If $\lfloor wt(M) \rfloor = j$, then output $dn(G) = j$.

The following theorem proves the validity of this algorithm.

Theorem 11. *The finite graph G has $dn(G) = j$ if and only if all maximum weight matchings of K_n^* have weight of the form $j + k\epsilon$, where j and k are integers satisfying $0 \leq j, k \leq \lfloor \frac{n}{2} \rfloor$, and $j + k = \lfloor \frac{n}{2} \rfloor$.*

Proof. Note that the any maximum weight matching M of K_n^* is perfect, or *near-perfect:* all vertices but one are matched (the latter case occurs only if n is odd). Otherwise, add edges to M to obtain a matching with larger weight. Hence, if $wt(M) = j + k\epsilon$, where $0 \leq j, k \leq \lfloor \frac{n}{2} \rfloor$, then $j + k = \lfloor \frac{n}{2} \rfloor$.

Let $dn(G) = j$, and suppose M is a maximum weight matching of K_n^* with weight $j' + k'\epsilon$, with $0 \leq j', k' \leq \lfloor \frac{n}{2} \rfloor$, and $j' + k' = \lfloor \frac{n}{2} \rfloor$. If P is a dn pairing in G with j pairs, then in K_n^* the pairing P corresponds to a matching with weight j that may be enlarged to a maximum weight perfect matching with weight at least $j + k\epsilon$, where $0 \leq j, k \leq \lfloor \frac{n}{2} \rfloor$, and $j + k = \lfloor n/2 \rfloor$. Hence, $wt(M) \geq j + k\epsilon$.

Suppose that $j' < j$. Hence, $k' > k$. Then by the maximality of M', $j + k\epsilon \leq j' + k'\epsilon$, which implies that $j - j' \leq (k' - k)\epsilon$. Thus, $(k' - k)\epsilon \geq 1$, which contradicts the choice of ϵ. Therefore, $j' \geq j$. Observe that for all possible values of k', $k'\epsilon < 1$. Hence, M has exactly j' many edges of weight 1. Now if $j' > j$, then the j' many weight 1 edges of M correspond to j' many dn pairs in G, which contradict that $dn(G) = j$. Hence, $j = j'$, and so $k = k'$. As M was arbitrary, the forward direction follows.

For the reverse direction, assume that a maximum weight matching M of K_n^* has weight $j + k\epsilon$, where $0 \leq j, k \leq \lfloor n/2 \rfloor$, and $j + k = \lfloor n/2 \rfloor$. From the proof of the forward direction, we obtain that $dn(G) \geq j$. Suppose for a contradiction that $dn(G) = j' > j$. As in the proof of the forward direction, we may then obtain a maximum weight perfect matching with weight $j' + k'\epsilon$, with $0 \leq j', k' \leq \lfloor n/2 \rfloor$, and $j' + k' = \lfloor n/2 \rfloor$. As $j' + k'\epsilon > j + k\epsilon$ for all possible values of k and k', this is a contradiction. Hence, $dn(G) = j$. ■

Corollary 12. *The dn number of a finite graph may be found in polynomial time.*

Proof. The construction of K_n^* from G may be done in polynomial time. A maximum weight matching in K_n^* may be found in polynomial time, since this may be reduced to a minimum weight perfect matching problem; see Section 5.3 of Cook et al. [11]. The proof now follows by Theorem 11. ■

4. THE INFINITE CASE

We next characterize infinite sub-ncc graphs via dnp pairings. Note first that the definitions of ncc and sub-ncc graphs and dnp pairings apply equally to graphs of finite or infinite order.

Theorem 13. *Let α be an infinite cardinal. A graph G of order α is sub-ncc if and only if it has a dnp pairing.*

Proof. Let G be a spanning subgraph of some ncc graph H with $|V(H)| = \alpha$. We prove that H has a dnp pairing. The forward direction will then follow since the property of having a dnp pairing is preserved by taking spanning subgraphs. Let x be a vertex of G. As $G\restriction N^c[x] \cong G\restriction N(x)$, the set of isolated vertices in $G\restriction N^c[x]$ has the same cardinality as the set of isolated vertices in $G\restriction N(x)$. Let $\{x_i : 1 \leq i \leq \beta_x\}$ and $\{y_i : 1 \leq i \leq \beta_x\}$ be the set of isolated vertices in $G\restriction N^c[x]$ and $G\restriction N(x)$, respectively, where $\beta_x \leq \alpha$ is a cardinal.

Define

$$P_x = \{\{x_i, y_i\} : 1 \leq i \leq \beta_x\},$$

and define $P_{x_i} = P_{x_j} = P_{y_i} = P_{y_j}$, for all $1 \leq i, j \leq \beta_x$. Define

$$P = \bigcup_{z \in V(H)} P_z.$$

The proof that P is a dnp pairing is similar to the last paragraph of the proof of Theorem 8, and so is omitted.

For the converse, let G have a dnp pairing $P = \{\{x_i, y_i\} : 1 \leq i \leq \alpha\}$. Add edges, if necessary, to form the graph G', where $x_iy_i \in E(G')$ for all $1 \leq i \leq \alpha$. Define H to be a maximal subgraph of K_α subject to the conditions that no edge x_iy_i is in a subgraph isomorphic to K_3 (the existence of H follows by Zorn's lemma). The matching $M = \{x_iy_i : 1 \leq i \leq \alpha\}$ is a locally C_4 perfect matching in H. It is straightforward to verify that any graph with a locally C_4 perfect matching is ncc. ■

As a side remark, it is an open problem of [5] to characterize infinite ncc graphs. As noted in the proof of the reverse direction of Theorem 13, if a graph (of any cardinality) has a locally C_4 perfect matching, then it is ncc.

We now consider countable ncc graphs. Let $\mathcal{K}$ be a class of finite graphs closed under isomorphisms. The class $\mathcal{K}$ has the *amalgamation property*, written (AP), if it satisfies the following. For graphs A, B, and C in $\mathcal{K}$, for any isomorphism f from

A onto an induced subgraph of B, and any isomorphism g from A onto an induced subgraph of C, there is a graph D in $\mathcal{K}$, an isomorphism f' from B onto an induced subgraph of D, and an isomorphism g' from C onto an induced subgraph of D, so that for all vertices x of A, $(f'f)(x) = (g'g)(x)$. The graph D is referred to as an *amalgam of B and C over A*. Informally, D results by gluing B and C over A, possibly by making some identifications.

The class of all finite graphs has (AP): choose D to be the graph $B \cup C$ that has vertices $V(B) \cup V(C)$ and edges $E(B) \cup E(C)$. Many otherwise favorable classes of finite graphs, such as the class of bipartite graphs, do not have (AP), however. The class $\mathcal{K}$ has the *joint embedding property* or *(JEP)* if for every pair B and C in $\mathcal{K}$, there is a $D \in \mathcal{K}$ so that B and C are induced subgraphs of D. (If graphs with no vertices are allowed, then (JEP) is a special case of (AP).)

Let $\mathcal{C}$ be a class of countable graphs closed under isomorphisms and closed under induced subgraphs, and let $\mathcal{C}' \subseteq \mathcal{C}$. A graph $M \in \mathcal{C}'$ is *universal pseudo-homogeneous for* $\mathcal{C}'$ if each of the following conditions hold.

- (PH1) Each finite graph in $\mathcal{C}$ is isomorphic to an induced subgraph of M.
- (PH2) Let A be a finite induced subgraph of M. Then there is an induced subgraph B of M such that $B \in \mathcal{C}'$ and $V(A) \subseteq V(B)$.
- (PH3) Let $A \in \mathcal{C}'$ be a finite induced subgraph of M, and let $B \in \mathcal{C}'$ contain A as an induced subgraph. Then there is an induced subgraph B' of M containing A as an induced subgraph, and an isomorphism $\beta : B \to B'$ which is the identity on A.

It can be shown that M is the unique (up to isomorphism) countable graph in $\mathcal{C}'$ with properties (PH1), (PH2), and (PH3). Pseudo-homogeneous graphs and relational structures were first investigated by Calais [7], and Fraïssé discussed them in Chapter 11 of [15]. As proved in Section 6.6 of [15], a universal pseudo-homogeneous graph exists in $\mathcal{C}$ if and only if $\mathcal{C}'$ satisfies (AP) and (JEP), and $\mathcal{C}$ and $\mathcal{C}'$ satisfy the following *cofinality condition*:

- (C) For each finite $A \in \mathcal{C}$, there is a finite $B \in \mathcal{C}'$ so that A is isomorphic to an induced subgraph of B.

The class $\mathcal{C}$ is called a *pseudo-amalgamation class* relative to $\mathcal{C}'$. The infinite random graph R mentioned in the Introduction is universal pseudo-homogeneous with $\mathcal{C} = \mathcal{C}'$ being the class of all countable graphs. The graph R is, in fact, *homogeneous*: every isomorphism between induced subgraphs extends to an automorphism. All countable homogeneous graphs have been classified by Lachlan and Woodrow [18]. None of these, however, are universal pseudo-homogeneous graphs for the class of sub-ncc graphs. For example, the graph R is not sub-ncc, since it is diameter 2 and each edge of R is in a K_3.

Theorem 14. *Let $\mathcal{C}$ be the class of all finite graphs, and let $\mathcal{C}'$ be the class of sub-ncc graphs. The class $\mathcal{C}$ is a pseudo-amalgamation class relative to the class $\mathcal{C}'$. Hence, there is a unique isomorphism type of universal pseudo-homogeneous countable sub-ncc graph M.*

Proof. We prove that $\mathcal{C}'$ has (JEP), (AP), and (C). The class $\mathcal{C}'$ has (JEP) by Corollary 7 (1), since if B and C are sub-ncc, then so is $D = B + C$. For (AP), let A, B, and C be sub-ncc graphs, and fix an isomorphism f from A onto an induced subgraph of B, and any isomorphism g from A onto an induced subgraph of C. For simplicity, we will discard f and g and identify A with its images in B and C. Hence, without loss of generality, assume that $V(B) \cap V(C) = V(A)$. Let $P = \{\{a_i, b_i\} : 1 \leq i \leq m\}$ be a dnp pairing of A. Let $|(V(B) \cup V(C)) \backslash V(A)| = n$. Let

$$D = (B \cup C) + \overline{K_n}.$$

In other words, form the union of B and C (over A), and add a suitably large independent set of vertices. Enlarge P to a dnp pairing of all of D by pairing each vertex of $(V(B) \cup V(C)) \backslash V(A)$ with a vertex of $\overline{K_n}$. More precisely, let $(V(B) \cup V(C)) \backslash V(A) = \{y_1, \ldots, y_n\}$ and let $V(\overline{K_n}) = \{z_1, \ldots, z_n\}$. Then

$$P' = \{\{a_i, b_i\} : 1 \leq i \leq m\} \cup \{\{y_i, z_i\} : 1 \leq i \leq n\}$$

is a dnp pairing of D. Hence, D is a sub-ncc amalgam of B and C over A.

For property (C), let A be a fixed finite graph. Form the graph $B = A + A$, and pair each vertex of A in one copy to the same vertex in the other copy. This gives a dnp pairing of B. ■

Corollary 15. *There is a unique isomorphism type of universal pseudo-homogeneous countable sub-ncc graph M.*

The graph M is highly symmetric: from the theory in [15], an isomorphism between finite induced sub-ncc subgraphs of M extends to an automorphism. It follows that M is vertex- and edge-transitive. We now give an explicit recursive construction of the graph M as a limit of finite sub-ncc graphs.

Let M_0 be a copy of K_2, and let P_0 be the pair formed by the vertices of M_0. Assume that M_n is a finite sub-ncc graph with a dnp pairing P_n, and assume that M_n contains M_0 as an induced subgraph. Enumerate all the induced subgraphs of M_n that are sub-ncc as G_i, $1 \leq i \leq j_n$. Fix i in $\{1, \ldots, j_n\}$. Enumerate all the sub-ncc graphs of order at most $n + 1$ that contain G_i as a subgraph as H_t, $1 \leq t \leq k_i$. Without loss of generality, assume that $V(M_n) \cap V(H_t) = V(G_i)$ for all t, and if $s \neq t$, then $V(H_s) \cap V(H_t) = V(G_i)$. Let

$$r = \sum_{1 \leq t \leq k_i} |V(H_t) \backslash V(G_i)|.$$

Define

$$M'_{n+1,i} = \left(M_n \cup \bigcup_{1 \leq t \leq k_i} H_t\right) + \overline{K_r}.$$

Define

$$M'_{n+1} = \bigcup_{1 \leq i \leq j_n} M'_{n+1,i}.$$

By a similar argument as given in the proof of (AP) in Theorem 14, the graph M'_{n+1} has a dnp pairing P'_{n+1} extending the dnp pairing P_n. Form the graph M''_{n+1} by adding disjoint copies of each graph of order at most $n+1$ to M'_{n+1}. Let $r' = |V(M''_{n+1}) \backslash V(M'_{n+1})|$. Let M_{n+1} be the graph $M''_{n+1} + \overline{K_{r'}}$. The graph M_{n+1} has a dnp pairing P_{n+1} extending P'_{n+1}, and hence, P_n.
Let

$$M' = \bigcup_{i \in N} M_i \text{ and } P' = \bigcup_{i \in N} P_i.$$

The graph M' is sub-ncc by Theorem 13, since P' is a dnp pairing of M'.

Theorem 16. *The graph M is isomorphic to the graph M'.*

Proof. We must verify properties (PH1), (PH2), and (PH3). Property (PH1) follows the definition of M''_{n+1} in the construction of M. For property (PH2), let A be a finite induced subgraph of M. Then A is an induced subgraph of some finite M_n which is sub-ncc. For property (PH3), fix A an induced subgraph of M and B a sub-ncc graph containing A as an induced subgraph. Suppose that A is an induced subgraph of M_n. From the definition of M'_{n+1}, there is an isomorphic copy B' of B in M containing A as an induced subgraph. ■

Apart from the symmetry exhibited by the graph M, little is known about the properties of this graph. For instance, like R, does M have spanning one- and two-way paths? What can be said of the endomorphism monoid and automorphism group of M? We plan on addressing these problems in future work.

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