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RESEARCH ARTICLE

The Monoid of the Random Graph

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Abstract

We investigate properties of the endomorphism monoid of the countable random graph R. We show that $End(R)$ is not regular and is not generated by its idempotents. The Rees order on the idempotents of $End(R)$ has $2^{\aleph_0}$ many minimal elements. We also prove that the order type of $\mathbb{Q}$ is embeddable in the Rees order of $End(R)$.

1. Introduction

The countable random graph R is the unique countable graph (up to isomorphism) with the property:

(♣) For every $n, m \geq 1$, if $x_1, \ldots, x_n$ and $y_1, \ldots, y_m$ are distinct vertices of R, then there is a vertex $x \in R - (\{x_1, \ldots, x_n\} \cup \{y_1, \ldots, y_m\})$ adjacent to the x_i and to none of the y_j.

The random graph has been widely studied (see [1] for a survey of results on R); in particular, the automorphism group of R has been thoroughly investigated; see [6], [9], and [10]. In this article, we study some of the semigroup-theoretic properties of the endomorphism monoid of R, $End(R)$. We show in Section 3. that, in contrast to $T(X)$ (the monoid of self-maps of a finite set X; see the results cited in [8]) that $End(R)$ is a non-regular monoid possessing a singular endomorphism that is not a product of idempotents. The Rees order on the idempotents of $End(R)$ has a rich structure: we show in Section 4. that the Rees order has $2^{\aleph_0}$ many minimal elements; in Section 5. we show that the Rees order embeds the ordering of the rationals, and thereby embeds every countable linear order.

2. Background and terminology

We remind the reader of basic terminology from graph and semigroup theory; the reader is directed to [2] for a graph theory reference and to [3] for a general reference on semigroups.

A *graph* G is a structure (G, E^G), where G is the set of vertices of G, and $E^G \subseteq G \times G$ is the edge set of G; we consider only *simple* graphs: E^G is irreflexive and symmetric. We will often write E for E^G if G is clear from context. For graphs G, H a *homomorphism* $f : G \to H$ is a mapping with the property that

$$xE^G y \text{ implies } f(x)E^H f(y). \qquad (1)$$

An *embedding* is an injective mapping with "implies" in (1) replaced by "if and only if".

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The *join* of G and H, $G \vee H$, is the graph with vertices $G \cup H$ and edges

$$E^G \cup E^H \cup \{(x, y), (y, x) : x \in G, y \in H\}.$$

If $S \subseteq G$, $G \restriction S$ is the *induced subgraph* of G on S. If G is an induced subgraph of H, we write $G \leq H$. If $G \restriction G \cap H = H \restriction G \cap H$ we can form the *union* of G and H, $G \cup H$ (over $G \cap H$), with vertices $G \cup H$ and edges $E^G \cup E^H$. If $G \cap H = \emptyset$ we write $G \uplus H$ for the union of G and H. The join or union of a set of graphs is defined similarly.

For $a \in G$, $N(a) = \{x : xEa\}$. The *degree* of a is $|N(a)|$. For $n \geq 2$, C_n is the chordless cycle of length n and K_n is the complete graph of order n. $K_{\aleph_0}$ is the complete graph on $\aleph_0$ vertices.

$End(G)$ is the monoid of homomorphic self-maps of the graph G. For $f \in End(G)$, $Im(f)$ is the image of G under f. For a set S, 1_S is the identity map on S.

$$E(End(G)) = \{f \in End(G) : f^2 = f\}$$

is the set of idempotents of $End(G)$; $f \in E(End(G))$ if and only if f is a *retraction* of G : $f \in End(G)$ and $f \restriction Im(f) = 1_{Im(f)}$. The *Rees (or natural) order* on $E(End(G))$ is the order defined by $e \leq f$ iff $ef = fe = e$. If $End(G)$ has no zero element, then $f \in E(End(G))$ is said to be *primitive* if it is minimal in the Rees order; that is, if $e \leq f$ then $e = f$. The set of primitive idempotents of $End(G)$ is denoted $PE(End(G))$.

ω is the set of natural numbers; $\omega^* = \omega - \{0\}$.

3. Negative results

A semigroup S is *regular* if for all $f \in S$, there is a $g \in S$ so that $fgf = f$ and $gfg = g$. For a graph G, $End(G)$ has *property* $(*)$ if for each singular $f \in End(G)$ (that is, f is not onto), there is an $n \geq 1$ so that f is the product of n idempotents of $End(G)$.

Proposition 3.1. *1. $End(R)$ has no zero element.*

2. $End(R)$ is not regular.

3. $End(R)$ does not have property $(*)$.

Proof. (1) is left as an exercise to the reader.

(2) As R embeds each countable graph as an induced subgraph (see Proposition 4.1 of [1]), we may fix a copy of $K_{\aleph_0} \leq R$. Let $R = \{x_i : i \in \omega\}$, $K_{\aleph_0} = \{y_i : i \in \omega\}$.

Define $f : R \to R$ by $f(x_i) = y_i$, for $i \in \omega$. Then f is a bijective homomorphism.

Suppose there is a $g \in End(R)$ so that $fgf = f$.

For $x_i \in R$, $f(x_i) = fgf(x_i) = f(g(y_i))$. As f is injective, $g(y_i) = x_i$.

Now fix x_i not adjacent to x_j in R with $i \neq j$. As y_iEy_j and g is a homomorphism, we obtain a contradiction.

(3) We first prove the following Claim.

Claim 1 If $f \in E(End(R)) - \{1\}$ then $Im(f)$ is infinite-coinfinite in R.

To see that $Im(f)$ is infinite, observe that since $K_{\aleph_0} \leq R$,

$$K_{\aleph_0} \leq R \restriction Im(f).$$

If $Im(f)$ is cofinite, let $Im(f) = R - \{a_i : 1 \leq i \leq n\}$, for some $n \geq 1$. We claim that

$$N(a_1) - \{a_i : 2 \leq i \leq n\} \subseteq N(f(a_1)). \tag{2}$$

To verify (2), fix $x \in N(a_1) - \{a_i : 2 \leq i \leq n\}$, so that $x \in Im(f)$.

As xEa_1, $f(x)Ef(a_1)$, so that $xEf(a_1)$ and thus, $x \in N(f(a_1))$.

But by (♣), there is a z in R that is adjacent to $\{a_i : 1 \leq i \leq n\}$ and non-adjacent to $f(a_1)$. But then $z \in (N(a_1) - \{a_i : 2 \leq i \leq n\}) - N(f(a_1))$, contradicting (2).

Claim 2 If $f_1, \ldots, f_n \in E(End(R)) - \{1\}$, for $n \geq 1$, then $Im(f_n \cdots f_1)$ is infinite-co-infinite in R.

That $Im(f_n \ldots f_1)$ is infinite follows again as $K_{\aleph_0} \leq R \restriction Im(f_n \cdots f_1)$. As $Im(f_n \cdots f_1) \subseteq Im(f_n)$, by Claim 1, $Im(f_n \ldots f_1)$ is coinfinite.

Fix $x \in R$. As the reader can check, $R \restriction R - \{x\} \cong R$. Let $f \in End(R)$ be an embedding of R onto $R - \{x\}$; then f is singular. $Im(f)$ is cofinite, so by Claim 2, f is not equal to a product of idempotents of $End(R)$. ■

4. $2^{\aleph_0}$ many primitive idempotents

The main result of this section is the following theorem.

Theorem 4.1. $|PE(End(R))| = 2^{\aleph_0}$; *in particular, there is a* $2^{\aleph_0}$*-antichain in* $E(End(R))$.

To prove Theorem 4.1, we first prove the following proposition. Recall that a countable graph G is *algebraically closed* or *ac* if for each finite set S of vertices of G, there is an x in G adjacent to each vertex of S (hence, an ac graph G is infinite).

The following proposition characterizes the idempotents of the random graph.

Proposition 4.2. *Let A be a graph. There is $f \in E(End(R))$ with $Im(f) \cong A$ if and only if A is a countable ac graph.*

We first prove the following lemma.

Lemma 4.3. *Let A be a countable ac graph. Then there is a set $\{A_i : i \in \omega^*\}$ of finite induced subgraphs of A so that:*

1. *For $1 \leq i < j < \omega$, $A_i < A_j$;*
2. *$\bigcup_{i \in \omega^*} A_i = A$;*
3. *$A_1 = K_1$;*
4. *For each $i \geq 2$, A_i contains a vertex adjacent to every vertex in A_{i-1}.*

Proof. Let $A = \{a_i : i \in \omega^*\}$ be an enumeration of the vertices of A.

Let $A_1 = A \restriction \{a_1\}$. We proceed by induction on $i \geq 1$.

Assume A_i has been defined, so that

i) $A_i \geq A_1$;

ii) $A_i = A \restriction \{a_1, \ldots, a_r\}$ for some $r \geq i$.

iii) If $i \geq 2$, A_i satisfies (4) in the statement of the lemma.

As A is ac, there is a minimal $s > r$ so that a_s is adjacent to each vertex in A_i.

Let $A_{i+1} = A \restriction \{a_1, \ldots, a_s\}$. Then A_{i+1} satisfies items i), ii), and iii) of the inductive hypothesis.

Items (1)-(4) of the lemma hold for the set $\{A_i : i \in \omega^*\}$ as the reader can check. ∎

4.1. Proof of Proposition 4.2

For necessity, observe that the property of a graph being ac is expressible by a set of positive sentences Φ (see p. 49 of [5] for the definition of a positive sentence). As R satisfies Φ, A satisfies Φ by Lyndon's preservation theorem (see [7]). Hence, A is ac.

We now prove sufficiency. By Lemma 4.3, there is a set of induced subgraphs of A $\{A_i : i \in \omega^*\}$ with properties (1)-(4) as in the statement of the lemma.

Let $R_1(A) = A_1$, with $f_1 = 1_{A_1}$.

For $n \geq 1$, assume $R_n(A)$ has been defined so that:

1. $R_n(A)$ is finite;
2. $A_n \leq R_n(A)$;
3. there is $f_n \in E(End(R_n(A)))$ with $Im(f_n) = A_n$.

By taking isomorphic copies, we may assume that $R_n(A) \cap A_{n+1} = A_n$. Define $R_{n+1}(A)' = R_n(A) \cup A_{n+1}$ over A_n. Let $n+1$ be a vertex in A_{n+1} adjacent to each vertex of A_n.

Let $S_1, \ldots, S_r$ be an enumeration of the n-element subsets of $R_n(A)$. Fix $1 \leq j \leq r$. Let $T_1, \ldots, T_s$ be a list of the non-isomorphic graphs of order $n+1$ with S_j as an induced subgraph (there are only finitely many).

By taking isomorphic copies, we can assume that for $1 \leq i < k \leq s$, $T_i \cap T_k = S_j$, and for $1 \leq i \leq s$, $T_i \cap R_{n+1}(A)' = S_j$. Let

$$R_{n+1}(A)'_j = R_{n+1}(A)' \cup \bigcup_{1 \leq i \leq s} T_i$$

over S_j. For each $1 \leq i < j \leq r$, assume that $R_{n+1}(A)'_i \cap R_{n+1}(A)'_j = R_{n+1}(A)'$. Let $R_{n+1}(A) = \bigcup\{R_{n+1}(A)'_j : 1 \leq j \leq r\}$ over $R_{n+1}(A)'$. Then $R_{n+1}(A)$ satisfies items (1) and (2) in the inductive hypothesis.

For item (3), define $f_{n+1}(x) = \begin{cases} f_n(x) & \text{if } x \in R_n(A); \\ x & \text{if } x \in A_{n+1} - A_n; \\ n+1 & \text{else.} \end{cases}$

Then f_{n+1} will satisfy item (3) once we prove the following claim.

Claim 1 $f_{n+1} \in E(End(R_{n+1}(A)))$.

We show that f_{n+1} is a homomorphism.

Let $x, y \in R_{n+1}(A)$, so that xEy. The only important case to check is when $x \in R_n(A)$ and $y \notin R_{n+1}(A)'$. Then y is a vertex adjacent to n vertices of $R_n(A)$ (including x).

Then $f_{n+1}(x) = f_n(x) \in A_n$, and $f_{n+1}(y) = n+1$. As $n+1Ez$ for each $z \in A_n$, the claim follows.

Observe that for every $n \geq 1$, $R_n(A) < R_{n+1}(A)$, and $f_{n+1} \restriction R_n(A) = f_n$. Now, let $R^* = \bigcup_{n \in \omega^*} R_n(A)$.

The proposition now follows from the following facts which are easily verified.

i) $R^* \cong R$. To see this, we show that R^* satisfies (♣). Let S and T be fixed finite subsets of R^* of size m and n respectively. Then there is some $j \geq 1$ so that $S \cup T \subseteq R_j(A)$. Let $k = \max(j, m+n)$. Then $S \cup T \subseteq R_k(A)$. By construction, we can find a vertex $x \in R_{k+1}(A) - (S \cup T)$ adjacent to the vertices in S and nonadjacent to the vertices in T.

ii) $A < R^*$.

iii) $f = \bigcup_{n \in \omega^*} f_n \in E(End(R^*))$, and $Im(f) = A$.

4.2. Proof of Theorem 4.1

A graph G is *retract rigid* if $E(End(G)) = \{1\}$.

Claim 1 $PE(End(R)) = \{f \in E(End(R)) : R \restriction Im(f) \text{ is retract rigid}\}$.

Let $f \in PE(End(R))$ with $R \restriction Im(f) = S$. Let $h \in E(End(S)) - \{1\}$ with $S \restriction Im(h) = T < S$.

Define $g = hf$. Then $g \in E(End(R))$ and $Im(g) = T$. Observe that $gf = hff = hf = g$.

Fix $x \in R$. Then $fg(x) = g(x)$ as $g(x) \in T < S$; thus, $fg = g$.

Hence, $g \leq f$. But then by hypothesis, $g = f$; which is a contradiction (as $T = Im(g) < Im(f) = S$).

Conversely, let $f \in E(End(R))$ with $R \restriction Im(f)$ retract rigid.

Assume that for $e \in E(End(R))$,

$$fe = ef = e, \tag{3}$$

so that $Im(e) = Im(fe) \subseteq Im(f)$.

Define $e' = e \restriction Im(f)$.

Then $e' \in E(End(R \restriction Im(f)))$. By hypothesis, $Im(e') = Im(f)$.

For $x \in R$, $ef(x) = e'f(x) = f(x)$, the first equality follows as $f(x) \in Im(f)$, the second equality as e' is the identity on $Im(e') = Im(f)$.

Hence, by (3), $e = ef = f$.

Claim 2 Let $\{G_i : i \in I\}$ be a set of graphs, and let $f \in E(End(\bigvee_{i\in I} G_i))$.

1. For each $i \in I$, $f(G_i) \subseteq G_i$.
2. For each $i \in I$, $f \restriction G_i \in E(End(G_i))$.

For (1), fix $i \in I$ and $x \in G_i$. If $f(x) \in G_j$ with $i \neq j$, then $xEf(x)$. Then $f(x)Ef(x)$, which is a contradiction.

(2) follows from (1).

Claim 3 Let $\{G_i : i \in I\}$ be a set of retract rigid graphs. Then $\bigvee_{i\in I} G_i$ is retract rigid.

Let $f \in E(End(\bigvee_{i\in I} G_i))$. Then for each $i \in I$, $f \restriction G_i \in E(End(G_i))$ by Claim 2 (2). But then for each $i \in I$, $f \restriction G_i = 1_{G_i}$, so f is also the identity.

Let $X = \{n : n \text{ odd}, n \geq 5\}$.

For $i \geq 5$, recall that we denote the i-cycle by C_i.

Claim 4 If $I, J \subseteq X$ and $I \neq J$, then $\bigvee_{i\in I} C_i \not\cong \bigvee_{j\in J} C_j$.

To see this, it is enough to show that for $i \in I - J$, $C_i \not\leq \bigvee_{j\in J} C_j$.

We use the fact that the degree of each vertex in C_i is 2.

Case i) C_i embeds in a single C_j. This is impossible as $C_i \not\leq C_j$ if $i \neq j$.

Case ii) C_i embeds in $C_j \vee C_k$ in $\bigvee_{j\in J} C_j$ so that $C_i \cap C_z \neq \emptyset$ for $z \in \{j, k\}$. As $|C_i| \geq 5$, at least one of C_j or C_k contain three vertices of C_i; without loss of generality, suppose C_j contains three vertices of C_i. But then there is a vertex in $C_i \cap C_k$ of degree ≥ 3, which is a contradiction.

Case iii) C_i embeds in $C_j \vee C_k \vee C_l$ so that $C_i \cap C_z \neq \emptyset$ for $z \in \{j, k, l\}$. Then C_z contains 2 vertices of C_i, for some $z \in \{j, k, l\}$; say $z = j$. But then $C_i \cap C_k$ contains a vertex of degree ≥ 3.

Case iv) C_i embeds in $\bigvee_{k=1}^{n} C_{j_k}$, for $n \geq 4$. But then each $C_i \cap C_{j_k}$ contains a vertex of degree ≥ 3.

Recall that a finite graph G is *core* if each $f \in End(G)$ is onto.

Claim 5 Let $n \geq 5$ be odd.

(1) C_n is retract rigid.

(2) $K_{\aleph_0}$ is retract rigid.

For (1), use the fact that C_n is a core graph (and the fact that every finite core graph is retract rigid; see [4]). For (2), if $f \in E(End(K_{\aleph_0}))$ and $f(x) \neq x$ in $K_{\aleph_0}$, then $xEf(x)$. But then $f(x)Ef(x)$, which is a contradiction. Hence, $f = 1_{K_{\aleph_0}}$.

The proof of the following Claim is an exercise.

Claim 6 $G \vee K_{\aleph_0}$ is ac for each countable graph G.

The proof of the theorem is completed as follows.

By Proposition 4.2 and Claim 6, for each $I \subseteq X$, there is $f_I \in E(End(R))$ with $Im(f_I) = \bigvee_{i \in I} C_i \vee K_{\aleph_0}$.

By Claims $1, 3, 4$, and 5, each $f_I \in PE(End(R))$, and if $I \neq J$, $f_I \neq f_J$ (as $Im(f_I)$ is distinct from $Im(f_J)$).

Hence, $|PE(End(R))| \geq 2^{\aleph_0}$, and since $|PE(End(R))| \leq |End(R)| \leq 2^{\aleph_0}$, the proof of Theorem 4.1 follows.

5. Embedding $\mathbb{Q}$ into the Rees order

The present section is devoted to the proof of the following theorem.

Theorem 5.1. *$\mathbb{Q}$ embeds in $E(End(R))$; in particular, each countable linear order embeds in $E(End(R))$.*

Proof. For finite graphs A, B, write $A \to B$ if there is a homomorphism from A to B but no homomorphism from B to A.

Fix $\mathcal{C}$ a $\to$-chain of finite graphs of order type $\eta = OT(\mathbb{Q})$ (we appeal to Theorem 1 of [4]). We therefore write: $\mathcal{C} = \{G_p : p \in \mathbb{Q}\}$, so that for distinct $p, q \in \mathbb{Q}$, $G_p \to G_q$ iff $p < q$.

Let $S = K_{\aleph_0} \vee \biguplus \mathcal{C}$; that is, S is the join of $K_{\aleph_0}$ with the disjoint union of each member of $\mathcal{C}$. By Proposition 4.2, as S is ac, there is an $\alpha \in E(End(R))$ with $Im(\alpha) = S$.

Fix $G_p \in \mathcal{C}$. Define

$$[p, \infty) = \{G_i : i \geq p\};$$
$$(-\infty, p) = \{G_i : i < p\}.$$

Define $S_p = K_{\aleph_0} \vee \biguplus[p, \infty)$. Define $S_p \sqsubseteq S_q$ if and only $[p, \infty) \subseteq [q, \infty)$; we write $S_p \sqsubset S_q$ if $S_p \sqsubseteq S_q$ and $S_p \neq S_q$.

Then $(S_p, \sqsubseteq)$ is an order and $S_p \sqsubseteq S_q$ implies $S_p \subseteq S_q$.

Claim 1 If $p < q$, $S_q \sqsubset S_p$.

To see this, fix $G_i \in S_q$. Then $q \leq i$. But $p < q$ so that $p < i$, so $G_i \in S_p$.

Fix $p \in \mathbb{Q}$.

Claim 2 There is $\beta_p \in E(End(R))$ so that $Im(\beta_p) = S_p$.

For each $G_i \in (-\infty, p)$ fix $f_i : G_i \to G_p$ a homomorphism.

For $x \in S$ define $h_p(x) = \begin{cases} f_i(x) & \text{if } x \in G_i \in (-\infty, p); \\ x & \text{else.} \end{cases}$

Then h_p is a homomorphism. For this, fix $x, y \in S$ so that xEy.

Case 1) At least one of x, y is in $K_{\aleph_0}$; say $x \in K_{\aleph_0}$. Then $h_p(x) = x$ and $h_p(y)$ is either y or $f_i(y)$ for some $i < p$; in either case, $h_p(x)Eh_p(y)$.

Case 2) Neither x nor y are in $K_{\aleph_0}$. Then xEy only if x, y are in the same G_i. But h_p preserves the edges in each G_i.

Define $\beta_p = h_p\alpha$. Then $\beta_p \in End(R)$ and β_p is the identity on S_p, which finishes the proof of Claim 2.

Claim 3 Let f_p, f_q be retractions of S onto S_p and S_q, respectively. Then

1. $\alpha_p = f_p\alpha \in E(End(R))$ so that $Im(\alpha_p) = S_p$.
2. $f_p < f_q$ iff $\alpha_p < \alpha_q$.

(1) follows as in the proof of Claim 2.

For (2) suppose $f_p < f_q$. Fix $x \in R$.

Then $\alpha_p\alpha_q(x) = f_p\alpha f_q\alpha(x) = f_p f_q\alpha(x) = f_p\alpha(x) = \alpha_p(x)$, the second equality holding as α is the identity on $S \supseteq S_q$, the third equality by hypothesis.

Similarly, $\alpha_q\alpha_p(x) = \alpha_p(x)$.

Now suppose that $\alpha_p < \alpha_q$. Fix $x \in S$.

Then $f_p f_q(x) = f_p\alpha f_q\alpha(x) = \alpha_p\alpha_q(x) = \alpha_p(x) = f_p\alpha(x) = f_p(x)$, the first and fifth equality as α is the identity on S, the third equality by hypothesis.

The equality $f_q f_p = f_p$ is proved in a similar fashion.

Claim 4 Fix $p, q \in \mathbb{Q}$ and let f_p, f_q be retractions of S onto S_p and S_q, respectively. Then $f_p < f_q$ implies $q < p$.

If $q = p$, then for each $x \in S$, $f_p(x) = f_p f_q(x) = f_q(x)$, the first equality by hypothesis, the second as f_p is the identity on $S_p = S_q$. But then $f_p = f_q$; which is a contradiction.

If $p < q$ then $S_q \sqsubset S_p$ by Claim 1.

But then $f_p(x) = f_p f_q(x) = f_q(x)$.

Contradiction.

Define $\Pi = \{f_p : p \in \mathbb{Q},\ f_p$ a retraction of S onto S_p so that $f(\bigcup(-\infty, p)) \subseteq \bigcup[p, \infty)$, and for each $G_i \in (-\infty, p)$, there is $j \geq p$ so that $f_p(G_i) \subseteq G_j\}$.

By the proof of Claim 2, $\Pi \neq \emptyset$.

Claim 5 With the induced ordering of $E(End(S))$, Π is dense without endpoints.

Before we prove Claim 5, we show that Claim 5 finishes the proof of the theorem.

By Claim 5, Π contains a chain of order type η; enumerate this chain as $\{f_p : p \in \mathbb{Q}\}$.

But by Claim 3 (2), $\{\alpha_p : p \in \mathbb{Q}\} \subseteq E(End(R))$ is order-isomorphic to $\{f_p : p \in \mathbb{Q}\}$. In this way, we can embed $\mathbb{Q}$ in $E(End(R))$.

We now complete the proof of Claim 5.

We first show density.

Fix $f_p < f_q$ in Π. By Claim 4, $q < p$. Choose $r \in \mathbb{Q}$ so that $q < r < p$.

$$\text{Define } f_r(x) = \begin{cases} f_p(x) & \text{if } x \in \bigcup[q, r) \text{ or} \\ & \text{if } x \in \bigcup(-\infty, q) \text{ and } f_q(x) \in \bigcup[q, r); \\ f_q(x) & \text{if } x \in \bigcup(-\infty, q) \text{ and } f_q(x) \in \bigcup[r, \infty); \\ x & \text{else.} \end{cases}$$

Claim 6 $f_r \in \Pi$.

1) f_r is a retraction from S to S_r. For this it is enough to check that f_r is a homomorphism.

Let $x, y \in S$, xEy. As in Claim 2, we may assume that x, y are not in $K_{\aleph_0}$. The only important case is if x, y are in $G_i \in (-\infty, q)$. But then as $f_q \in \Pi$, either $\{f_q(x), f_q(y)\}$ is in $\bigcup[q, r)$ or in $\bigcup[r, \infty)$; in either case, f_r preserves the edge $\{x, y\}$.

2) It is immediate that $f_r(\bigcup(-\infty, r)) \subseteq \bigcup[r, \infty)$.

Now, fix $G_i \in (-\infty, r)$.

Case i) $q \leq i < r$. Then $f_r(G_i) = f_p(G_i)$, and we may use the fact that $f_p \in \Pi$.

Case ii) $i < q$.

As $f_q \in \Pi$, $f_q(G_i)$ is either in single member of $[q, r)$ or $[r, \infty)$; in each case, $f_r(G_i)$ is in a single G_j using again the fact that $f_p, f_q \in \Pi$.

The next two Claims will finish the proof of Claim 6.

Claim 7 $f_p < f_r$.

1) $f_r f_p = f_p$ as $S_p \sqsubset S_r$.

2) Fix $x \in S$.

$$\text{Then } f_p f_r(x) = \begin{cases} f_p f_p(x) = f_p(x) & \text{if } x \in \bigcup[q, r) \text{ or} \\ & \text{if } x \in \bigcup(-\infty, q) \text{ and } f_q(x) \in \bigcup[q, r); \\ f_p f_q(x) = f_p(x) & \text{if } x \in \bigcup(-\infty, q) \text{ and } f_q(x) \in \bigcup[r, \infty); \\ f_p(x) & \text{else,} \end{cases}$$

so that $f_p f_r = f_p$.

Claim 8 $f_r < f_q$.

1) $f_q f_r = f_r$ as $S_r \sqsubset S_q$.

2) Fix $x \in S$.

Case i) $x \in \bigcup[q, \infty)$.

Then $f_r f_q(x) = f_r(x)$, as f_q is the identity on x.

Case ii) $x \in \bigcup(-\infty, q)$.

Subcase a) $f_q(x) \in \bigcup[q, r)$.

Then $f_r f_q(x) = f_p f_q(x) = f_p(x) = f_r(x)$, the second equality follows since $f_p < f_q$, the first and third equality follow by definition of f_r.

Subcase b) $f_q(x) \in \bigcup[r, \infty)$.

Then $f_r f_q(x) = f_q(x) = f_r(x)$, the first equality holds as f_r is the identity on $\bigcup[r, \infty)$, the second equality holds by the definition of f_r.

We now show that Π has no least point.

Fix $f_p \in \Pi$. Let $q > p$, so $S_q \sqsubset S_p$ by Claim 1.

For each $G_i \in [p, q)$ fix $f_i : G_i \to G_q$ a homomorphism.

$$\text{For } x \in S_p, \text{ define } g_q(x) = \begin{cases} f_i(x) & \text{if } x \in G_i \in [p, q); \\ x & \text{else.} \end{cases}$$

A similar argument to the above establishes that $g_q \in E(End(S_p))$.

Define $f_q = g_q f_p$. Then $f_q \in E(End(S))$ with $Im(f_q) = S_q$.

To show that $f_q \in \Pi$, first note that $f_q(\bigcup(-\infty, q)) \subseteq \bigcup[q, \infty)$.

Fix $G_i \in (-\infty, q)$.

Case i) $G_i \in [p, q)$. Then $f_q(G_i) = g_q f_p(G_i) = g_q(G_i) \subseteq G_q$, the second equality follows as f_p is the identity on $\bigcup[p, q)$.

Case ii) $G_i \in (-\infty, p)$. As $f_p \in \Pi$, there is a j so that

$$f_p(G_i) \subseteq G_j \in [p, \infty).$$

Subcase a) $G_j \in [p, q)$. Then $f_q(G_i) = g_q f_p(G_i) \subseteq G_q$, by the definition of g_q.

Subcase b) $G_j \in [q, \infty)$. Then $f_q(G_i) = g_q f_p(G_i) = f_p(G_i) \subseteq G_j$, the second equality follows since g_q is the identity on $\bigcup[q, \infty)$.

Claim 9 $f_q < f_p$.

1) $f_p f_q = f_q$ as $S_q \sqsubset S_p$.
2) $f_q f_p = g_q f_p f_p = g_q f_p = f_q$.

Finally, we show that Π has no greatest point.

Fix $f_p \in \Pi$. Choose $q < p$.

For $x \in S$, define $f_q(x) = \begin{cases} f_p(x) & \text{if } x \in G_i \in (-\infty, q); \\ x & \text{else.} \end{cases}$

The verification that f_q is a retraction of S to S_q is similar to previous arguments. Observe that $f_q \in \Pi$ as $f_p \in \Pi$.

Claim 10 $f_p < f_q$.

1) $f_q f_p = f_p$ as $S_p \sqsubset S_q$.
2) For $x \in S$, $f_p f_q(x) = \begin{cases} f_p f_p(x) = f_p(x) & \text{if } x \in G_i \in (-\infty, q); \\ f_p(x) & \text{else,} \end{cases}$
so that $f_p f_q = f_p$. ∎

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