Universal random semi-directed graphs

Anthony Bonato
Department of Mathematics
Wilfrid Laurier University, Canada
abonato@rogers.com

Dejan Delić, Changping Wang
Department of Mathematics
Ryerson University, Canada
ddelic@ryerson.ca, cpwang@ryerson.ca

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Abstract

Motivated by models for real-world networks such as the web graph, we consider digraphs formed by adding new vertices joined to a fixed constant m number of existing vertices of prescribed type. We consider a certain on-line random construction of a countably infinite graph with out-degree m, and show that with probability 1 the construction gives rise to a unique isomorphism type. We study algebraic properties of these so-called random semi-directed graphs; in particular, we prove that their automorphism groups embed all countable groups.

1. Introduction

In the last decade, there has been an enormous amount of research surrounding complex networks such as the web graph. The *web graph* has vertices representing web pages, and edges representing the links between pages. Many technological, social, biological networks have properties similar to those present in the web, such as *power law degree distributions* (the proportion of vertices of degree k is approximately $k^{-\beta}$, where $\beta > 1$ is a fixed real number) and the *small world property* (which implies that distances, measured either by diameter or average distance, are of smaller order than the order of the graph). For example, power laws and the small world property have been observed in protein-protein interaction networks, and networks formed by scientific collaborators. For more details on properties of the web graph and other complex networks, the reader is directed to the survey [3] and the books [4, 10].

A large number of stochastic models for complex networks have been proposed. For example, in the *preferential attachment model*, new vertices are born over time which have a greater probability to join to high degree vertices. It was proved in [1] with high probability that the in-degree distribution of graphs in the preferential attachment model follows a power law with exponent $\beta = 3$. The graphs in the preferential attachment models have the property that each vertex has exactly m out-neighbours. Constant out-degree is in fact, a common assumption in other models of complex networks; see, for example, [10, 16]. Hence, models of complex networks often generate directed graphs satisfying the following properties.

(1) *On-line*: digraphs are generated over a countably infinite set of discrete time-steps, with a countable (either finite or countably infinite) set of vertices born at each time-step. At time 0, a fixed *initial digraph* H is given.
(2) *Constant out-degree*: new vertices have edges directed only to existing vertices, and for $m > 0$ a fixed integer, there are exactly m such edges.

A digraph G satisfying these two properties is called *semi-directed with initial graph H and constant out-degree m*; we sometimes refer to G simply as *semi-directed*. The moniker "semi-directed" comes from [2] (see p. 17). It emphasizes that the orientation of edges in a semi-directed graph canonically arises according to time: new vertices may only point to vertices born at earlier time-steps. Note that semi-directed graphs have no infinite directed path emanating from any vertex.

In this paper, we consider the infinite semi-directed graphs that result when time tends to infinity. Analyzing models by considering the infinite limit is a common technique in the natural sciences. In particular, the existence of a unique limit indicates coherent behaviour of the model, while many distinct limits suggest a sensitivity to initial conditions that is an indicator of chaos. In [5, 6, 15], infinite limits of graphs generated by models of the web graph were investigated. Limits generated by on-line graph processes were, in fact, studied by Fraïssé [12] and others decades prior to birth of the internet.

One of the most studied example of an infinite limit graph arising from a stochastic model is the infinite random graph. The probability space $G(\mathbb{N}, p)$ consists of graphs with vertices $\mathbb{N}$, so that each distinct pair of integers is joined independently with a fixed probability $p \in (0, 1)$. Erdős and Rényi discovered that with probability 1, all $G \in G(\mathbb{N}, p)$ are isomorphic. The unique isomorphism type of countably infinite e.c. graph is named the *infinite random graph*, or the *Rado* graph, and is written R; see the survey [8].

Define a deterministic graph R^* as follows. Let R_0 be a K_1. Assume that for a nonnegative integer $t \geq 0$, the graph R_t is defined and finite. To form R_{t+1}, for each subset $S \subseteq V(R_t)$ (possibly empty) add a vertex z_S joined only to the vertices of S. The sets $\{V(R_t) : t \in \mathbb{N}\}$ and $\{E(R_t) : t \in \mathbb{N}\}$ are well-ordered sets or *chains*. We define

$$V(R^*) = \bigcup_{t\in\mathbb{N}} V(R_t), \quad E(R^*) = \bigcup_{t\in\mathbb{N}} E(R_t).$$

We write $\lim_{t\to\infty} R_t = R^*$, and say that R^* is the *limit of the chain* $(R_t : t \in \mathbb{N})$. The notion of limit extends to any chain $(G_t : t \in \mathbb{N})$ of graphs.

A graph G is *existentially closed* or *e.c.* if for all finite disjoint sets of vertices A and B (one of which may be empty), there is a vertex $z \notin A \cup B$ joined to all of A and to no vertex of B. By a back-and-forth argument, $R \cong R^*$ is the unique isomorphism type of countably infinite graphs that is e.c. Further, R is a *universal* graph: it contains as an induced subgraph an isomorphic copy of each countable graph.

In the present article, we consider structural and algebraic properties of certain infinite semi-directed graphs that arise naturally as limits of on-line random processes. Analogous to R, these so-called *random semi-directed* graphs have isomorphism types characterized via a set of adjacency properties (see Theorem 1). As an application of this characterization, random semi-directed graphs are universal (see Corollary 2). The automorphism group of R has been thoroughly investigated (see [8]). In Section 3 we show that all countable groups embed in the group of a random semi-directed graph.

All graphs we consider are simple, directed, and countable. If (x, y) is a directed edge, then y is an *out-neighbour* of x. We say that G *embeds in* H and write $G \leq H$ if G is isomorphic to an induced subgraph of H. If $S \subseteq V(G)$, then we write $\langle S\rangle_G$ for the subgraph induced by S (we omit the subscript G if it is clear from context). The *automorphism group* (or *group*) of G is written $\mathrm{Aut}(G)$. We write $\mathbb{N}$ for the natural numbers, $\mathbb{N}^+$ for the positive integers, and $\aleph_0$ for the cardinality of $\mathbb{N}$.

The vertices of a semi-directed graph may be ordered in the following way. A vertex x not in H has *height* k if there is a directed path of length k from x ending in a vertex of H; vertices in H have height 0. The height of a finite set S of vertices is the maximum height of a vertex in S.

2. Random semi-directed graphs

We consider the following general framework for limits of semi-directed graphs. A class $\mathcal{C}$ of digraphs closed under isomorphism is *good* if it contains infinitely many digraphs, and the class is *hereditary*: if $G \in \mathcal{C}$ and $H \leq G$, then $H \in \mathcal{C}$. For example, the class of all digraphs is good, as the class of linear orders (that is, transitive tournaments).

For the remainder of the article, fix $m > 0$ an integer, $\mathcal{C}$ a class of good digraphs, and H an m-vertex digraph in $\mathcal{C}$ (which exists as $\mathcal{C}$ is good). We define a countably infinite graph $R_{m,H}(\mathcal{C})$ as follows. Let R_0 be H. Assume that R_t is defined and countable so that $R_0 \leq R_t$. To form R_{t+1}, for each induced subgraph S of R_t that has m vertices and is in $\mathcal{C}$, add a vertex x_S that is joined to each vertex of S and no other vertices in R_t. Define $R_{m,H}(\mathcal{C}) = \lim_{t\to\infty} R_t$. The countably infinite digraph $R_{m,H}(\mathcal{C})$ is semi-directed by its construction. The idea behind the definition of

$R_{m,H}(\mathcal{C})$ is that all m-sets of vertices S that induce a graph in $\mathcal{C}$ are extended: the vertex x_S has its out-neighbours equalling S. Observe that vertices born at time t are exactly the vertices with height t.

One of our main results is that the isotype of $R_{m,H}(\mathcal{C})$ may be captured by a set of simple set of properties. We say that a digraph G is $(\mathcal{C}, m)$-*e.c.* if for each set A of m-vertices which induces a graph in $\mathcal{C}$ and each finite set B of vertices disjoint from A, there is a vertex $z \notin A \cup B$ so that $(z, a) \in E(G)$ for all a in A, but there are no directed edges between z and vertices of B. The $(\mathcal{C}, m)$-e.c. property is a directed analogue of the e.c. property, relativized by the parameter m and by the restriction that $\langle A \rangle \in \mathcal{C}$.

Theorem 1. *A countable digraph G is isomorphic to $R_{m,H}(\mathcal{C})$ if and only if G is semi-directed with initial graph H and constant out-degree m, each out-neighbour set induces a subgraph in $\mathcal{C}$, and G is $(\mathcal{C}, m)$-e.c.*

Proof. As the forward direction is immediate, we prove only the reverse direction. Let H' be the initial copy of H in G. The set $V(G)\backslash V(H')$ has a *special enumeration*: an enumeration $(x_t : t \in \mathbb{N}^+)$ of $V(G)\backslash(H')$ with the property that if (x_i, x_j) is a directed edge, then $i > j$. To see this, we may choose x_1 to be any vertex with height 1. Assuming that $\{x_1, \ldots, x_t\}$ were chosen, consider a vertex u of $V(G)\backslash(V(H') \cup \{x_1, \ldots, x_t\})$. If u has out-degree 0 in $\langle V(G)\backslash(V(H') \cup \{x_1, \ldots, x_t\})\rangle$, then let $x_t = u$. Otherwise, in $\langle V(G)\backslash(V(H') \cup \{x_1, \ldots, x_t\})\rangle$ there is a maximal directed finite path from u. The end point v of this path has out-degree 0, and we choose $x_t = v$.

As G is arbitrary with the given properties, it follows that $R_{m,H}(\mathcal{C})$ also has a special enumeration. Now, let $(x_t : t \in \mathbb{N}^+)$ and $(y_t : t \in \mathbb{N}^+)$ be special enumerations of $V(G)\backslash V(H')$ and $V(R_{m,H}(\mathcal{C}))\backslash V(R_0)$, respectively. We proceed by a back-and-forth argument, with f_0 isomorphically mapping H' in G to H at time 0 in $R_{m,H}(\mathcal{C})$. For a fixed t, suppose that f_t is a partial isomorphism with domain X_t containing $V(H') \cup \{x_1, \ldots, x_t\}$ and range Y_t containing $V(R_0) \cup \{y_1, \ldots, y_t\}$. We will assume as an additional inductive hypothesis that X_t and Y_t are *closed*: all out-neighbours of vertices in the set are in the set itself.

Suppose first that $t + 1 \geq 1$ is odd. In this case, we go forward. Let x be the lowest indexed vertex of $(x_t : t \in \mathbb{N})$ not in X_t. As the enumeration is special, the set of out-neighbours S_t of x are in X_t. By hypothesis, $|S_t| = m$ and $\langle S_t \rangle \in \mathcal{C}$. As $R_{m,H}(\mathcal{C})$ satisfies the $(\mathcal{C}, m)$-e.c. property, there is a vertex y whose out-neighbours are exactly $f_t(S_t)$. Extend f_t to f_{t+1} by mapping x to y, and let $X_{t+1} = X_t \cup \{x\}$ and $Y_{t+1} = Y_t \cup \{y\}$. It is straightforward to see that f_{t+1} is an isomorphism, and that the sets X_{t+1} and Y_{t+1} are closed.

The case $t + 1$ is even is similarly proven by going back, and so is omitted. We therefore have that the union of the chain of partial isomorphisms $(f_t : t \in \mathbb{N})$

$$F = \bigcup_{t \to \infty} f_t$$

is an isomorphism of G with $R_{m,H}(\mathcal{C})$. □

Analogous to the situation for R and all countable graphs, the graph $R_{m,H}(\mathcal{C})$ has the following universal property.

Corollary 2. *If G is a countable semi-directed graph with initial graph H and constant out-degree m, so that each out-neighbour set of a vertex of G induces a subgraph in $\mathcal{C}$, then $G \leq R_{m,H}(\mathcal{C})$.*

Proof. Let H' be the initial copy of H in G, and let f_0 isomorphically map H' in G to H at time 0 in $R_{m,H}(\mathcal{C})$. As in the proof of Theorem 1, let $(x_t : t \in \mathbb{N}^+)$ be a special enumeration of $V(G)\backslash V(H')$.

For a fixed t, suppose that f_t is an isomorphism with domain $X_t = V(H') \cup \{x_1, \ldots, x_t\}$ whose range is an induced subgraph Y_t of $R_{m,H}(\mathcal{C})$. Consider the vertex x_{t+1}. As the enumeration is special, the set of m out-neighbours S_t of x_{t+1} are in X_t. By hypothesis, S_t induces a subgraph in $\mathcal{C}$. It follows that in $R_{m,H}(\mathcal{C})$, there is a vertex y_{t+1} not in Y_t whose out-neighbours are exactly $f_t(S_t)$. We extend f_t to the isomorphism f_{t+1} which maps x_{t+1} to y_{t+1}.

Define

$$F = \bigcup_{t \to \infty} f_t.$$

Then F witnesses that $G \leq R_{m,H}(\mathcal{C})$. □

We next introduce a random graph process which we name the *Age Dependent Process* (*ADP*). The parameters of the process are m, $\mathcal{C}$, and H. Start with $G_0 \cong H$ with vertices labelled $v_1, \ldots v_m$. For $t \geq 1$ fixed, assume that a digraph G_{t-1} has been defined and there are finitely many vertices in G_{t-1}. At time t, add a new vertex v_{m+t}, and choose a set S of m distinct vertices from $V(G_{t-1})$ so that S induces a subgraph of $\mathcal{C}$, where the probability that a vertex v_i is included in the set is exponentially proportional to its height. More precisely, denote

$$\begin{aligned} L_{t-1} &= \{(j_1, \ldots, j_m) \in \mathbb{N}^m : \langle v_{j_1}, \ldots, v_{j_m} \rangle \in \mathcal{C}, \\ &\quad v_{j_1}, \ldots, v_{j_m} \in V(G_{t-1}) \text{ are distinct}\}. \end{aligned}$$

For each $S = \{v_{i_1}, \ldots, v_{i_m}\}$ where $(i_1, \ldots, i_m) \in L_{t-1}$, define

$$\mu(S) = 2^{-(i_1 + \cdots + i_m)}$$

and

$$N_t = \sum_{(j_1, \ldots, j_m) \in L_{t-1}} 2^{-(j_1 + j_2 + \cdots + j_m)}.$$

In particular, N_t is the sum of all the $\mu(S)$, where S is a subset of cardinality m from $V(G_{t-1})$ such that $\langle S \rangle \in \mathcal{C}$. The probability that S is chosen from $V(G_{t-1})$ equals $\mu(S)/N_t$; this clearly defines a probability measure on m-subsets S with $\langle S \rangle \in \mathcal{C}$ in G_t. If S is so chosen, then add directed edges from v_{m+t} to each vertex of S.

Theorem 3. *Let $G = \lim_{t \to \infty} G_t$, where G_t is generated by* ADP *with parameters m, H, and $\mathcal{C}$. Then with probability 1, G is $(\mathcal{C}, m)$-e.c.*

Proof. Fix disjoint finite subsets A and B of $V(G)$ so that $|A| = m$ and $\langle A \rangle \in \mathcal{C}$. Let $A = \{v_{i_1}, \ldots, v_{i_m}\}$, where the vertex v_{i_j} was born before $v_{i_{j+1}}$ for all j. Let t_0 be an integer greater than the height of $A \cup B$. For each $t \geq t_0$, let V_t be the event that v_t is pointing to exactly all vertices in A. Note that v_t has out-degree m when it is born, so that if V_t occurs, then there are no edges between v_t and any vertex of B. Then the probability that V_t occurs, written $\mathbb{P}(V_t)$, equals $2^{-(i_1 + \cdots + i_m)}/N_t$, where N_t is the normalizing factor defined above.

Note that

$$\begin{aligned} N_t &\leq \sum_{1 \leq j_1 < j_2 < \cdots < j_m \leq t+m-1} 2^{-(j_1 + j_2 + \cdots + j_m)} \\ &\leq \left(\sum_{j=1}^{t+m-1} 2^{-j} \right)^m \\ &\leq 1, \end{aligned}$$

for all t. Therefore, for all $t \geq t_0$,

$$\mathbb{P}(V_t) \geq 2^{-(i_1 + \cdots + i_m)} \geq 2^{-mt_0}.$$

Hence, the probability that there exists no vertex in G that is joined to all vertices in A and none of B is at most

$$\begin{aligned} \mathbb{P}\left(\bigcap_{t=t_0}^{\infty} \overline{V_t} \right) &= \prod_{t=t_0}^{\infty} (1 - \mathbb{P}(V_t)) \\ &\leq \lim_{t \geq t_0} (1 - 2^{-mt_0}) = 0. \end{aligned}$$

As there are only countably many finite subsets A and B and a countable union of measure 0 events is a measure 0 event, the proof follows. □

The following corollary follows immediately from Theorems 1 and 3. It supplies an analogue of the Erdős and Rényi isomorphism result for R.

Corollary 4. *With probability 1, a limit graph generated by* ADP *with parameters m, H, and $\mathcal{C}$ is isomorphic to $R_{m,H}(\mathcal{C})$.*

3. The group of $R_{m,H}(\mathcal{C})$

The graph R is *homogeneous*: isomorphisms between finite induced subgraphs extend to automorphisms. The homogeneous graphs were characterized in [17], while the homogeneous digraphs were characterized in [9]. The graph $R_{m,H}(\mathcal{C})$ is not homogeneous; it is not even vertex-transitive: two vertices with different heights are in different orbits of $\mathrm{Aut}(R_{m,H}(\mathcal{C}))$. Hence, the symmetries exhibited by R and $R_{m,H}(\mathcal{C})$ are quite different.

Henson [13] proved that $\mathrm{Aut}(R)$ embeds (that is, contains subgroups isomorphic to) all countable groups. We now prove that the group $\mathrm{Aut}(R_{m,H}(\mathcal{C}))$ shares this property with R. Given a set X, we use the notation $\mathrm{Sym}(X)$ for the group of permutations of X. For a set S of vertices and automorphism f, $f(S)$ is the image of S under f.

Theorem 5. *The group* $\mathrm{Sym}(X)$ *embeds in* $\mathrm{Aut}(R_{m,H}(\mathcal{C}))$*, where* X *is countably infinite. In particular, each countable group embeds in* $\mathrm{Aut}(R_m(\mathcal{C}))$*.*

Before we prove Theorem 5 we need the following lemma. The graph $R_{m,H}(\mathcal{C})'$ is defined analogously to $R_{m,H}(\mathcal{C})$, but at each time-step R_{t+1}, *infinitely many* vertices x_S are joined to each induced subgraph of order m from $\mathcal{C}$ in R_t.

Lemma 6. *The graph* $R_{m,H}(\mathcal{C})'$ *is isomorphic to* $R_{m,H}(\mathcal{C})$*.*

Proof. It is sufficient to prove that $R_{m,H}(\mathcal{C})'$ satisfies the hypotheses of Theorem 1. By its construction, the graph $R_{m,H}(\mathcal{C})'$ is semi-directed with initial graph H and constant out-degree m. Further, each vertex has its out-neighbour set inducing an m-vertex subgraph in $\mathcal{C}$. To see that $R_{m,H}(\mathcal{C})'$ satisfies the $(\mathcal{C}, m)$-e.c. property, suppose we are given A a set of m-vertices in $R_{m,H}(\mathcal{C})'$ which induces a graph in $\mathcal{C}$, and a finite set B of vertices in $R_{m,H}(\mathcal{C})'$ disjoint from A. Let t_0 be the height of $A \cup B$. A vertex joined to A and not to B may be found in R_{t_0+1}. □

Proof of Theorem 5. Without loss of generality, by Lemma 6 we will work with $R_{m,H}(\mathcal{C})'$ for the remainder of the proof. By Cayley's theorem, it is sufficient to prove that $\mathrm{Sym}(X)$ embeds in $\mathrm{Aut}(R_{m,H}(\mathcal{C})')$.

We first observe that $\mathrm{Sym}(X)$ embeds in $\mathrm{Aut}(R_1)$. To see this, label the vertices of $V(R_1)\backslash V(R_0)$ as $X = \{x_i : i \in \mathbb{N}\}$. Fix a bijective mapping $f : X \to X$. Define $F : R_1 \to R_1$ which acts as the identity on H, and otherwise acts as f on X. As the x_i have the same out-neighbours in R_1, it follows that F is an automorphism of R_1. Define $\beta : T(X) \to \mathrm{Aut}(R_1)$ by $\beta(f) = F$. It is straightforward to check that β is an injective group homomorphism.

We next prove that there exists an injective group homomorphism $\alpha : \mathrm{Aut}(R_1) \to \mathrm{Aut}(R_{m,H}(\mathcal{C})')$. Once this is established, then $\alpha\beta : T(X) \to \mathrm{Aut}(R_{m,H}(\mathcal{C})')$ supplies an embedding of $\mathrm{Sym}(X)$ into $\mathrm{Aut}(R_{m,H}(\mathcal{C})')$, and the assertion will follow.

Fix j an automorphism of R_1. Let $J_1 = j$. For $t \geq 1$, assume that J_t is an automorphism of R_t, and the restriction of J_t to R_1 equals J_1. Let $N^+(z)$ be the set of out-neighbours of a vertex z. Define J_{t+1} by

$$J_{t+1}(z) = \begin{cases} J_t(z) & \text{if } z \in V(R_t); \\ x_{J_t(S)} & \text{if } z = x_S \text{ and } S = N^+(z). \end{cases}$$

From the definition of R_{t+1} and the fact that $J_t \in \mathrm{Aut}(R_t)$, it follows that J_{t+1} is an automorphism of R_{t+1}. Note that J_{t+1} restricted to R_t equals J_t.

The map $J = \bigcup_{t\in\mathbb{N}} J_t$ is an automorphism of $\mathrm{Aut}(R_{m,H}(\mathcal{C})')$. Hence, the function $\alpha : \mathrm{Aut}(R_1) \to \mathrm{Aut}(R_{m,H}(\mathcal{C})')$ defined by $\alpha(j) = J$ is well-defined. It is straightforward to see that α is injective, and that α preserves the identity automorphism.

Now fix $f, g \in \mathrm{Aut}(R_1)$ and $z \in V(R_H)$. We prove by induction on the height t of z that

$$\alpha(fg)(z) = \alpha(f)\alpha(g)(z). \tag{1}$$

Equation (1) will establish that α is an embedding of groups, and is immediate if $t = 0$. Fix $t \geq 1$. Suppose that z has height $t+1$ and so z is of the form x_S, where $S = N^+(z) \subseteq V(R_t)$. Then

$$\begin{aligned} \alpha(fg)(z) &= x_{\alpha(fg)(S)} \\ &= x_{\alpha(f)\alpha(g)(S)} \\ &= \alpha(f)\alpha(g)(z). \end{aligned}$$

The second equality follows since the height of S is strictly less than $t+1$, and by induction hypothesis. □

The property of extending automorphisms of R_1 to automorphisms of all of $\mathrm{Aut}(R_{m,H}(\mathcal{C}))$ in the proof of Theorem 5 clearly generalizes to any R_t with $t \geq 0$. In particular, $j \in \mathrm{Aut}(R_t)$ extends to $J \in \mathrm{Aut}(R_{m,H}(\mathcal{C}))$, and the map $\alpha_t : \mathrm{Aut}(R_t) \to \mathrm{Aut}(R_{m,H}(\mathcal{C})')$ defined by $\alpha_t(j) = J$ is an injective group embedding. Although $\mathrm{Aut}(R_{m,H}(\mathcal{C}))$ is not homogeneous, we may refer to the above property as *temporal homogeneity*: symmetries of the graphs R_t at time t lift to symmetries of the entire limit graph.

We consider some computational consequences of Theorem 5. We refer the reader to Hodges [14] for any terms not explicitly defined.

Corollary 7. *The group* $\mathrm{Aut}(R_{m,H}(\mathcal{C}))$ *does not satisfy any non-trivial group identity. In particular,* $\mathrm{Aut}(R_{m,H}(\mathcal{C}))$ *generates the variety of all groups.*

Proof. Since every countable group embeds into $\mathrm{Aut}(R_{m,H}(\mathcal{C}))$ by Theorem 5, so does the free group on a countable set of generators, written $F(X)$. If there were an equation $s = t$ in the language of groups that is not a consequence of the groups axioms, and satisfied by $\mathrm{Aut}(R_{m,H}(\mathcal{C}))$, then $s = t$ would be satisfied by $F(X)$, which is a contradiction. □

Corollary 8. *The universal theory of* $\mathrm{Aut}(R_{m,H}(\mathcal{C}))$ *is undecidable.*

Proof. We first note that the universal theory of $\mathrm{Aut}(R_{m,H}(\mathcal{C}))$ equals the universal theory of all groups. This follows since every countable group embeds into $\mathrm{Aut}(R_{m,H}(\mathcal{C}))$ by Theorem 5, every universal sentence true in $\mathrm{Aut}(R_{m,H}(\mathcal{C}))$ will be true in all countable groups and, by the Löwenheim-Skolem Theorem (see [14]), in all groups.

It is well-known that the universal theory of groups is undecidable. This fact follows this by the existence of a group with an undecidable word problem; see [7, 18]. Hence, the universal theory of $\mathrm{Aut}(R_{m,H}(\mathcal{C}))$ is undecidable. □

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