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On Retracts of the Random Graph and Their Natural Order

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Abstract. We prove that the natural order on the idempotents of the endomorphism monoid of the countably infinite random graph is $\aleph_0$-universal; that is, it embeds every countable order. We therefore extend in a strong fashion a result of [2] which showed that the natural order embeds every countable linear order. We consider a refinement of the natural order which embeds every countable quasi-order.

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1. Introduction and Preliminaries

The random graph R is the unique (up to isomorphism) countable graph which is *n-existentially closed* or *n-e.c.* for every natural number n, that is, such that for every n-element subset S of vertices of R and every (possibly empty) subset T of S, there is a vertex not in S joined to each vertex of T and to no vertex of $S\backslash T$.

A graph G is *universal* if every countable graph is isomorphic to an induced subgraph of G, and is *homogeneous* if every isomorphism between finite induced subgraphs of G extends to an automorphism of G. The random graph is also the unique (up to isomorphism) countable graph which is universal and homogeneous. The random graph R has other intriguing properties, many of which are described in surveys by Cameron [3] and [4].

A *homomorphism* between two graphs is a vertex-mapping that is also an edge-mapping. The set of homomorphic self-maps of R, written *End*(R), is a monoid under composition, and is called the *monoid of* R. The automorphism group of R has been well-studied (see [1], [5], [8], [11], [12]), but the monoid of R was studied only recently, in [2]. An endomorphism $f \in \mathrm{End}(R)$ is a *retract* (or an *idempotent*) if $f^2 = f$. The set of all idempotents of R is denoted $E(\mathrm{End}(R))$. The relation $f \leqslant g$ defined by $fg = gf = f$ is an order on $E(\mathrm{End}(R))$, and is the well-known *natural order* (or *Rees order*; for background on the natural order and other aspects of semigroup theory, we direct the reader to [9]). In [2], the following theorems were proven. Let P be the natural order on $E(\mathrm{End}(R))$.

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Theorem 1 (Theorem 4.1 of [2]). *The order P has* $2^{\aleph_0}$*-many minimal elements and hence, contains a* $2^{\aleph_0}$*-antichain.*

Theorem 2 (Theorem 5.1 of [2]). *The order P embeds every countable linear order.*

Theorems 1 and 2 suggest that the natural order on $E(\mathrm{End}(R))$ has a rich structure which is of interest in its own right. Since R itself is universal, we might try to extend Theorem 2 and ask whether the natural order on $E(\mathrm{End}(R))$ embeds *any* countable order. (See Remarks after the proof of Theorem 4 on possible extensions of Theorem 1.)

In the present article, we answer this question in the affirmative. In particular, we embed the "colouring order" from graph homomorphism theory into the natural order; see Theorem 4. We then appeal to results of [7] and [10] which prove that the colouring order is universal. In Section 3, we consider a refinement of the natural order which embeds every countable quasi-order.

Unless otherwise stated, all graphs and orders considered are countable, and we consider only simple graphs and reflexive orders. If G is a graph, then $V(G)$ is the set of vertices of G, $E(G)$ is the set of edges of G. An edge between vertices x and y is written xy; we say that x and y are *joined*. If $B \subseteq V(G)$, then we write $G\restriction B$ for the subgraph induced by B; if H is an induced subgraph of G, then we write $H \hookrightarrow G$. We write $G \cong H$ if G and H are isomorphic. The set of natural numbers is written ω (considered as an ordinal), ω^* is the set of non-zero natural numbers, and $\aleph_0$ is the cardinality of ω. A *clique* of cardinality α, denoted K_α, is the graph on α vertices with each pair of distinct vertices joined. Given two graphs G, H, the *join* of G and H, written $G \vee H$, is the graph formed by adding all edges between vertices of G and those of H; the disjoint union of G and H is written $G \uplus H$. If G and H are graphs with $V(G) \cap V(H) \neq \emptyset$, then the *union of G and H over* $V(G) \cap V(H)$, denoted $G \cup H$, has vertices $V(G) \cup V(H)$ and edges $E(G) \cup E(H)$.

For graphs G, H a *homomorphism* (or *H-colouring*) $f : G \to H$ is a vertex-mapping such that if $xy \in E(G)$, then $f(x)f(y) \in E(H)$; we say that the edge xy is *preserved* by f. If there is a homomorphism from G to H we write $G \to H$. A map $f : G \to H$ is an *embedding* if f is an injective homomorphism so that $xy \notin E(G)$ implies that $f(x)f(y) \notin E(H)$; we say G *embeds* in H. If $f : G \to H$ is a vertex-mapping, then $\mathrm{Im}(f) = \{f(x) : x \in V(G)\}$ is the *image* of f. For a set or graph S, 1_S is the identity map on S.

We will abuse notation and identify an order with its vertex set. If $(A, \leqslant)$ is an order, and $x \in A$, then we define the *order ideal* (*or down set*) of x, written $\downarrow x$, to be the set $\{y : y \leqslant x\}$. If x and y are incomparable elements in an order, then we write $x \parallel y$. A map $f : A \to B$ is an *order-embedding* if it is an injective mapping satisfying $a \leqslant b$ if and only if $f(a) \leqslant f(b)$. An order is *α-universal* if it embeds every order of cardinality α.

A *quasi-order* is a reflexive transitive relation. If $(A, \leqslant)$ is a quasi-order, then define $x \sim y$ if $x \leqslant y$ and $y \leqslant x$. The set of $\sim$-classes is written $A/\sim$, and it is well-known that $(A/\sim, \leqslant)$ is an order. A *quasi-order-embedding* is defined in the same way as an order-embedding.

Define the *colouring quasi-order* on the class of finite graphs by $G \leqslant H$ if and only if $G \to H$. The $\sim$ relation here is $G \to H$ and $H \to G$, and is written $G \leftrightarrow H$; an $\sim$-class is called a $\leftrightarrow$-class. The corresponding quotient of the colouring quasi-order is the *colouring order* $\mathscr{C}$. In [10] $\mathscr{C}$ is referred to as the *colouring poset*. We identify a graph with its $\leftrightarrow$-class. Each countable graph admits a homomorphism to $K_{\aleph_0}$. Hence, $K_{\aleph_0}$ acts as a maximal element of $\mathscr{C}$, which we will call 1. The order that results by adjoining 1 to $\mathscr{C}$ is denoted by $\mathscr{C}^*$. The graph K_1 is the minimum element in $\mathscr{C}^*$, and K_1 is covered by K_2. The order $\mathscr{C}$ is dense above K_2 (see [13]). For further background on graph homomorphisms, the reader is directed to the survey of Hahn and Tardif [6].

2. The Natural Order is $\aleph_0$-Universal

The following theorem was proved first by Hedrlín [7] using category-theoretic techniques. An elegant combinatorial proof was recently given by Nešetřil [10].

Theorem 3. *The colouring order $\mathscr{C}$ is $\aleph_0$-universal.*

Our main result is the following theorem, which provides a strong extension of Theorem 2.

Theorem 4. *The natural order $(E(\mathrm{End}(R)), \leqslant)$ is $\aleph_0$-universal.*

Before we begin the proof of Theorem 4, we recall a result from [2]. A graph G is *algebraically closed* or *a.c.* if for every finite set of vertices S of G, there is an $x \in V(G)$ joined to every vertex of S. The following result is Proposition 4.2 of [2].

Theorem 5. *Let A be a graph. There is $f \in E(\mathrm{End}(R))$ with $R \restriction \mathrm{Im}(f) \cong A$ if and only if A is a countable a.c. graph.*

Proof of Theorem 4. We will assume, by taking isomorphic copies if necessary, that if G and H are distinct elements of $\mathscr{C}^*$, then $V(G) \neq V(H)$. Let $K'_{\aleph_0}$ be a copy of $K_{\aleph_0}$ with the property that $V(K'_{\aleph_0}) \cap V(H) = \emptyset$, for all $H \in \mathscr{C}^*$.

For each $G \in \mathscr{C}^* \setminus \{1\}$, fix a homomorphism $f_G : G \to 1$. Define the countable graph

$$\mathscr{C}' = K'_{\aleph_0} \vee \left(\left(\biguplus_{G \in \mathscr{C}^* \setminus \{1\}} G \right) \uplus 1 \right).$$

Then $\mathscr{C}'$ is a.c. To see this, note that for any finite set $S \subseteq V(\mathscr{C}')$, a vertex of $K'_{\aleph_0}$ not in S is joined to every vertex of S. By Theorem 5, there is a retract $\alpha \in E(\mathrm{End}(R))$ with $R \restriction \mathrm{Im}(\alpha) \cong \mathscr{C}'$. For $H \in \mathscr{C}^* \setminus \{1\}$ define

$$H' = K'_{\aleph_0} \vee \left(\left(\biguplus_{G \in \downarrow H} G \right) \uplus 1 \right).$$

Define $f^H : \mathscr{C}' \to \mathscr{C}'$ as follows:

$$f^H(x) = \begin{cases} x & \text{if } x \in (\biguplus_{G \in \downarrow H} V(G)) \cup V(1) \cup V(K'_{\aleph_0}); \\ f_{G_0}(x) & \text{if } x \in V(G_0), \text{ where } G_0 \in (\mathscr{C}' \setminus \{1\}) \setminus (\downarrow H). \end{cases}$$

See Figure 1.

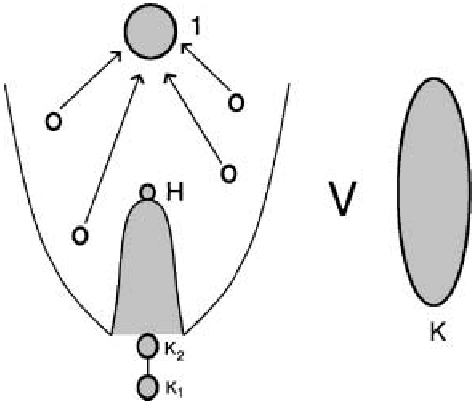

Figure 1. A depiction of the map f^H. The map f^H is the identity on the shaded subgraphs, and $K = K'_{\aleph_0}$

Claim 1: The map f^H is in $E(\mathrm{End}(\mathscr{C}'))$ with $\mathrm{Im}(f^H) = V(H')$.

To see this, it is enough to show that f^H is a homomorphism. Fix distinct x, y in $V(\mathscr{C}')$ with $xy \in E(\mathscr{C}')$.

Case 1. The vertex x is in $V(K'_{\aleph_0})$. In this case, $f^H(x) = x$. If $y \in V(K'_{\aleph_0})$, then $f^H(y) = y$ and $f^H(x)f^H(y) = xy \in E(\mathscr{C}')$. In all other cases, y is mapped to a vertex outside $K'_{\aleph_0}$, and hence, the edge xy is preserved by f^H.

Case 2. The vertex x is not in $V(K'_{\aleph_0})$. We may assume that $y \notin V(K'_{\aleph_0})$. The vertices x and y must be in the same connected component of $\mathscr{C}'$, in which case f^H acts either as the identity map or as f_G, for some G. In each case, the edge xy is preserved by f^H.

The endomorphism $\alpha^H = f^H\alpha$ is a retract of R whose image is $V(H')$.

Claim 2: We have that $f^G \leqslant f^H$ if and only if $\alpha^G \leqslant \alpha^H$.

For the forward direction, for $x \in V(R)$,

$$\alpha^G\alpha^H(x) = f^G\alpha f^H\alpha(x) = f^G f^H\alpha(x) = f^G\alpha(x) = \alpha^G(x),$$

the second equality holding since α is the identity map on $\mathscr{C}'$, the third equality holding since $f^G \leqslant f^H$. Hence, $\alpha^G\alpha^H = \alpha^G$. The proof that $\alpha^H\alpha^G = \alpha^G$ is similar.

Suppose that $\alpha^G \leqslant \alpha^H$. Fix $x \in V(\mathscr{C}')$. Then

$$f^G f^H(x) = f^G\alpha f^H\alpha(x) = \alpha^G\alpha^H(x) = \alpha^G(x) = f^G\alpha(x) = f^G(x),$$

where we use the fact in the first and fifth equality that α is the identity map on $\mathscr{C}'$.

Define a map

$$F : (\mathscr{C}^*\backslash\{1\}, \leqslant) \rightarrow (E(\mathrm{End}(R)), \leqslant)$$

by $F(G) = \alpha^G$. We now show that F is an order-embedding.

To see that F is injective, assume that $G \neq H$, and suppose first that $G < H$, so that $V(H)\backslash V(G) \neq \emptyset$. (Recall that we are assuming that $V(G) \neq V(H)$.) Then for

$x \in V(H)\backslash V(G)$, $\alpha^H(x) = x$, but $\alpha^G(x) \in V(1)$, so $\alpha^G \neq \alpha^H$. The case when $G > H$ is similar. The last case is when $G \parallel H$. In this case, the sets $V(H)\backslash V(G)$ and $V(G)\backslash V(H)$ are nonempty. Then for $x \in V(G)\backslash V(H)$, $\alpha^G(x) = x$ but $\alpha^H(x) \in V(1)$, so $\alpha^G \neq \alpha^H$.

By Claim 2, it is sufficient to prove that

$$G \leqslant H \text{ if and only if } f^G \leqslant f^H. \tag{2.1}$$

Suppose first that $G \leqslant H$. Hence, $\downarrow G \subseteq \downarrow H$. Fix $x \in V(\mathscr{C}')$. Then

$$f^H f^G(x) = f^G(x),$$

since f^H is the identity map on $V(H') \supseteq V(G')$.

We must show that $f^G f^H(x) = f^G(x)$.

Case 1. The vertex x is in $(\biguplus_{J \in \downarrow G} V(J)) \cup V(1) \cup V(K'_{\aleph_0})$. Then $f^H(x) = x$ since every $J \in \downarrow G$ is also in $\downarrow H$ and since f^H is the identity map on 1 and $K'_{\aleph_0}$. Thus, $f^G f^H(x) = f^G(x)$.

Case 2. The vertex x is not in $(\biguplus_{J \in \downarrow G} V(J)) \cup V(1) \cup V(K'_{\aleph_0})$. Suppose first that $x \in (\biguplus_{J \in \downarrow H} V(J)) \backslash (\biguplus_{J \in \downarrow G} V(J))$. In this case, $f^H(x) = x$, so $f^G f^H(x) = f^G(x)$.

Now, let $x \notin (\biguplus_{J \in \downarrow H} V(J)) \cup V(1) \cup V(K'_{\aleph_0})$. Say $x \in V(J)$ with $J \in \mathscr{C}^* \backslash \{1\}$ so that $J \notin \downarrow H$. Then

$$f^G f^H(x) = f^G f_J(x) = f_J(x) = f^G(x),$$

the first equality by definition of f^H, the second equality since $f_J(x) \in V(1)$ and f^G is the identity on 1, and the third equality since $J \notin \downarrow G$ and by the definition of f^G.

We have therefore established the forward direction of (2.1). For the reverse direction, assume that $G \nleqslant H$. To obtain a contradiction, suppose that $f^G \leqslant f^H$. If $H \leqslant G$, then by the forward direction of (2.1) we would have $f^H \leqslant f^G$. Antisymmetry would then imply that $f^G = f^H$, contradicting the injectivity of F.

We may therefore assume that $G \parallel H$. Since $f^G \leqslant f^H$, we have $f^H f^G = f^G$. But then

$$\mathrm{Im}(f^G) = \mathrm{Im}(f^H f^G) \subseteq \mathrm{Im}(f^H). \tag{2.2}$$

Now, $V(G) \subseteq \mathrm{Im}(f^G)$, and so by (2.2) we have $V(G) \subseteq \mathrm{Im}(f^H)$. Therefore, $f^H(x) = x$ for every $x \in V(G)$. But since $G \parallel H$, f^H maps vertices of G to vertices in 1. This is a contradiction, and the reverse direction of (2.1) follows.

By Theorem 3, $\mathscr{C} = \mathscr{C}^* \backslash \{1\}$ is $\aleph_0$-universal. Now apply F to embed $(\mathscr{C}^* \backslash \{1\}, \leqslant)$ into $(E(\mathrm{End}(R)), \leqslant)$. □

Theorem 1 demonstrates that large antichains embed in the natural order P on $E(\mathrm{End}(R))$. We do not know which of the uncountable orders embed in P. Another open problem (whose solution may in fact rely on set-theoretic assumptions) is whether P is $2^{\aleph_0}$-universal or $\aleph_1$-universal.

3. An $\aleph_0$-Universal Quasi-Order Refining the Natural Order

A quasi-order A is *α-universal* if it embeds each quasi-order of cardinality α. With Theorem 4 at hand, we may ask if there is a "natural" refinement of

$(E(\mathrm{End}(R)), \leqslant)$ to an $\aleph_0$-universal quasi-order. Define the relation $\preceq$ on $E(\mathrm{End}(R))$ by $f \preceq g$ if $R\restriction(\mathrm{Im}(f))$ embeds into $R\restriction(\mathrm{Im}(g))$. Then $(E(\mathrm{End}(R)), \preceq)$ is a quasi-order. If $f \leqslant g$, then $f = gf$, so $\mathrm{Im}(f) = \mathrm{Im}(gf) \subseteq \mathrm{Im}(g)$. Thus, $f \leqslant g$ implies $f \preceq g$. In $(E(\mathrm{End}(R)), \preceq)$, 1_R is the unique maximal element, and the $\sim$-class of endomorphisms whose images each induce $K_{\aleph_0}$ are the minimal elements. (The latter statement follows since each countable a.c. graph contains $K_{\aleph_0}$ as an induced subgraph.)

We prove the following theorem.

Theorem 6. *The quasi-order* $(E(\mathrm{End}(R)), \preceq)$ *is* $\aleph_0$*-universal.*

We first recall a lemma of [2].

Lemma 1. *Let A be a countable a.c. graph. Then there is a set*

$$\{A_j : j \in \omega\}$$

of finite induced subgraphs of A so that

1. *For* $0 \leqslant i < j < \omega$, $A_i \hookrightarrow A_j$;
2. *the graph A equals* $\bigcup_{j \in \omega} A_j$;
3. *the subgraph* A_0 *equals* K_1;
4. *for every* $j \geqslant 1$, *there is a vertex of* A_j *joined to each vertex of* A_{j-1}.

We now prove the following.

Lemma 2. *For every* $f \in E(\mathrm{End}(R))\backslash\{1\}$ *there is a set* $X = \{f_i \in E(\mathrm{End}(R))\backslash \{1\} : i \in \omega\}$ *with the following properties*:

1. *If* $i, j \in \omega$ *with* $i \neq j$, *then* $f_i \neq f_j$;
2. *for all* $i \in \omega$, $R\restriction(\mathrm{Im}(f)) \cong R\restriction(\mathrm{Im}(f_i))$.

Lemma 2 implies that each $\sim$-class in $(E(\mathrm{End}(R))\backslash\{1\}, \preceq)$ is infinite. Lemma 2 does not apply to the identity map 1_R as it is the unique retract of R with image $V(R)$.

Proof of Lemma 2. Suppose that $B = R\restriction(\mathrm{Im}(f))$ has vertices $\{x^j : j \in \omega\}$, and let $\{B_i : i \in \omega\}$ be a set of $\aleph_0$ pairwise disjoint copies of B. Fix $i \in \omega$. Let $V(B_i) = \{x_i^j : j \in \omega\}$, so that $x_i^k x_i^l \in E(B_i)$ if and only if $x^k x^l \in E(B)$. By Lemma 1, there is a set of induced subgraphs of B, $\{A^j : j \in \omega\}$, with the properties (1)–(4) as stated in that lemma. The corresponding induced subgraphs of B_i are denoted $\{A_i^j : j \in \omega\}$.

We define a pair (R^*, f_i) inductively, so that $R^* \cong R$ and f_i retracts R^* to B_i. Let $R_0 = \biguplus_{j \in \omega} A_j^0$, with $f_i^0 : R_0 \to R_0$ given by

$$f_i^0(x) = \begin{cases} x & \text{if } x \in V(A_i^0); \\ x_i^r & \text{if } x = x_j^r \text{ for some } j \neq i, j, r \in \omega. \end{cases}$$

It is straightforward to check that f_i^0 is a retract of R_0 with image A_i^0.

For $n \in \omega$, assume that R_n has been defined so that:

1. R_n is countable;
2. for all $j \in \omega$, we have that $A_j^n \hookrightarrow R_n$, and the embeddings of the A_j^n are pairwise disjoint;

3. the map f_i^n is in $E(\mathrm{End}(R_n))$ with $\mathrm{Im}(f_i^n) = V(A_i^n)$;
4. the map f_i^n defined in (3) sends x_j^r to x_i^r for every $j \in \omega \backslash \{i\}$.

Let $\{S_k : k \in \omega\}$ be an enumeration of the n-vertex induced subgraphs of R_n. For a fixed $k \in \omega$, let $T_1^k, \ldots, T_{m_k}^k$ be a listing of the non-isomorphic graphs of order $n+1$ with S_k as an induced subgraph (there are only finitely many of these).

By taking isomorphic copies, we may assume that for $1 \leqslant r \leqslant m_k$, the $V(T_r^k)$ pairwise intersect in $V(S_k)$, and that each $V(T_r^k)$ intersects $V(R_n)$ in $V(S_k)$. (Therefore, the sets $V(T_r^k)$ may not be pairwise disjoint for distinct values of r.) Let

$$R'_{n+1,k} = R_n \cup \bigcup_{1 \leqslant r \leqslant m_k} T_r^k$$

over S_k. Let

$$R'_{n+1} = \bigcup \{R'_{n+1,k} : k \in \omega\}$$

over R_n.

By taking isomorphic copies we may assume that

$$V(R'_{n+1}) \cap V(A_j^{n+1}) = V(A_j^n)$$

for every $j \in \omega$. Define

$$R_{n+1} = R'_{n+1} \cup \bigcup_{j \in \omega} A_j^{n+1}$$

over $\bigcup_{j \in \omega} A_j^n$. Let $(n+1)_i$ be a vertex in A_i^{n+1} joined to every vertex of A_i^n (guaranteed by property (4) of Lemma 1).

Then R_{n+1} satisfies (1) and (2) of the inductive hypothesis. For items (3) and (4), define the mapping

$$f_i^{n+1}(x) = \begin{cases} f_i^n(x) & \text{if } x \in V(R_n); \\ (n+1)_i & \text{if } x \in V(R'_{n+1}) \backslash V(R_n); \\ x_i^r & \text{if } i \neq j, x = x_j^r, x_j^r \notin V(R'_{n+1}) \text{ and } j, r \in \omega; \\ x & \text{if } x \in V(A_i^{n+1}) \backslash V(A_i^n). \end{cases}$$

To verify (3) and (4) of the inductive hypothesis, it is enough to prove that f_i^{n+1} is a homomorphism. We consider edges xy with $x \in V(R_n)$ and $y \in V(R'_{n+1})$ (the other cases are straightforward to check). But then y is a vertex in some T_r^k extending some n vertices of an induced subgraph S_k of R_n, so $f_i^{n+1}(x) f_i^{n+1}(y) = f_i^n(x)(n+1)_i$, which is an edge of R_{n+1}, since $f_i^n(x) \in V(A_i^n)$ and $(n+1)_i$ is joined to each vertex of A_i^n.

For every $n \in \omega$, $R_n \hookrightarrow R_{n+1}$ and $f_i^{n+1} \restriction (R_n) = f_i^n$. If we let

$$R^* = \bigcup_{n \in \omega} R_n \quad \text{and} \quad f_i = \bigcup_{n \in \omega} f_i^n,$$

then R^* is n-e.c. for all $n \in \omega \backslash \{0\}$, and f_i is a retract of R^* onto B_i. To see that R^* is n-e.c. for a fixed $n \geqslant 1$, let S be a finite subset of $V(R^*)$ with r elements, and fix $T \subseteq S$. Then there is an $m \geqslant 1$ so that $S \subseteq V(R_m)$. A vertex joined to each vertex of T and to no vertex of $S \backslash T$ may be found in R_{m+r+1}. Note that $f_i \neq f_j$ if $i \neq j$, since

$\mathrm{Im}(f_i) \cap \mathrm{Im}(f_j) = \emptyset$. Hence, $R^* \cong R$, and the set $\{f_i : f_i \in E(\mathrm{End}(R^*))\backslash\{1\}\}$ satisfies properties (1) and (2) in the statement of the lemma. □

Proof of Theorem 6. We consider the map

$$F : (\mathscr{C}^*\backslash\{1\}, \leqslant) \to (E(\mathrm{End}(R)), \preceq)$$

from the proof of Theorem 4. For $G \in \mathscr{C}^*\backslash\{1\}$, let

$$\tilde{G} = R\restriction(\mathrm{Im}(F(G))).$$

The same proof as before shows that F is injective, and if $G, H \in \mathscr{C}^*\backslash\{1\}$ and $G \leqslant H$, then $\tilde{G}$ embeds in $\tilde{H}$ via the identity map.

We must prove that if $\tilde{G}$ embeds in $\tilde{H}$, then $G \leqslant H$. Suppose that $h : \tilde{G} \to \tilde{H}$ is an embedding. The graph $\tilde{G}$ is isomorphic to one which consists of the join of $K'_{\aleph_0}$ with a graph Z which consists of the disjoint union of each $J \in \mathscr{C}'\backslash\{1\}$ along with the disjoint union of 1. Let X be the copy of G in Z from $\tilde{G}$. We consider cases depending on the location of $h(V(X))$ in $V(\tilde{H})$.

Case 1. The set $h(V(X)) \cap V(K'_{\aleph_0})$ is not empty. In this case, every vertex of $h(V(X)) \cap V(K'_{\aleph_0})$ is universal in $\tilde{H}$. (A vertex is *universal* if it is joined to all vertices distinct from itself.) But then that would imply that there are vertices of X that are universal in $\tilde{G}$. But no vertex of X is joined to a vertex of 1 in $\tilde{G}$. This is a contradiction.

Case 2. The set $h(V(X)) \cap V(1)$ is not empty. For similar reasons stated in Case 1, we have $h(V(1)) \cap V(K'_{\aleph_0}) = \emptyset$. Since 1 is an infinite clique, the copy of 1 in $\tilde{G}$, call it $1_{\tilde{G}}$, embeds via h into the copy of 1 in $\tilde{H}$, call it $1_{\tilde{H}}$. But then in this case, each vertex of $h(V(X)) \cap V(1_{\tilde{H}})$ is joined to each vertex of $h(V(1_{\tilde{G}}))$, which would imply there are edges between $1_{\tilde{G}}$ and X in $\tilde{G}$, which is a contradiction.

Case 3. The set $h(V(X))$ is a subset of $\biguplus_{J \in \downarrow H} V(J)$. Define $\hat{X} = \tilde{H}\restriction h(V(X))$. Suppose that $\hat{X} = \biguplus_{i=1}^{n} G_i$, where the G_i are finite connected graphs. Then every G_i embeds in some $J_i \leqslant H$, so $\hat{X} \leqslant \biguplus_{i=1}^{n} J_i$. Since $\biguplus_{i=1}^{n} J_i \leqslant \biguplus_{i=1}^{n} H_i$, where every $H_i = H$, we have $\biguplus_{i=1}^{n} J_i \leqslant H$. The following chain of homomorphisms gives the desired conclusion

$$G \leqslant \tilde{G}\restriction(V(X)) \leqslant \hat{X} \leqslant \biguplus_{i=1}^{n} J_i \leqslant \biguplus_{i=1}^{n} H_i \leqslant H.$$

We proved in Theorem 4 that every countable order (considered as a quasi-order with singleton $\sim$-classes) embeds in $(E(\mathrm{End}(R)), \preceq)$. In fact, since $\tilde{G} \not\cong R$ for every $G \in \mathscr{C}$, we have embedded every countable order into $P' = (E(\mathrm{End}(R))\backslash\{1\}, \preceq)$.

Let B be a countable quasi-order. First embed $B/\sim$ (considered as a quasi-order with singleton $\sim$-classes) into $\mathscr{C}$, and then into P' using the map F defined above. To simplify notation, we will identify $B/\sim$ with its image in $\mathscr{C}$. Since each $\sim$-block of B is at most countable, we can embed all of B into P' as follows. For every $f \in E(\mathrm{End}(R))\backslash\{1\}$, choose a set $\{f_i : f_i \in E(\mathrm{End}(R))\}$ as in Lemma 2. For every $x \in B$, enumerate the set $\{y : y \sim x\}$ as $\{y_i(x) : 1 \leqslant i < n\}$, where $1 \leqslant n \leqslant \omega$. Define $F' : B \to P'$ by $F'(y_i(x)) = f_i$ where F maps the $\sim$-class of x to f. Then F is well-defined and injective (since F is injective and $f_i \neq f_j$ if $i \neq j$). Fix $x_1, x_2 \in B$. Suppose that $x_1 = y_i(x_1)$, $x_2 = y_j(x_2)$, and F maps the $\sim$-class of x_1 to f, and the $\sim$-class of x_2 to g. If $x_1 \sim x_2$, then $F(x_1) = f_i \sim f_j = g_j = F(x_2)$. Suppose that $x_1 \not\sim x_2$. If $x_1 \leqslant x_2$, then $f \leqslant g$ since F is an embedding, and so $f_i \leqslant g_j$. If $x_1 \not\leqslant x_2$, then $f \not\leqslant g$ since F is an embedding, and so $f_i \not\leqslant g_j$. □

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