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A family of universal pseudo-homogeneous G-colourable graphs

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Abstract

For each finite core graph G there is a countable universal pseudo-homogeneous G-colourable graph $M(G)$ that is unique up to isomorphism. We investigate properties of $M(G)$ that are not unlike properties of the infinite random graph. In particular, we show that $M(G)$ has an independent dominating set and has one- and two-way hamiltonian paths when G is connected. We also investigate limits of the graphs $M(G_i)$, and we answer a question of Caicedo (Algebra Universalis 34 (1995) 314) on infinite antichains in the lattice of cores.

MSC: 05C15; 03C15

Keywords: Graph homomorphisms; Core graphs; Universal graphs

1. Introduction

The *infinite random graph R* is the unique (up to isomorphism) countable graph satisfying the following conditions, for all $n \geqslant 1$:
(n-e.c.) For every n-subset S of $V(R)$ and every partition of S into sets A, B, (one of which may be empty) there is a vertex $z \notin S$ that is joined to each vertex in A and to no vertex in B.

As first investigated by Henson in [7], R possesses many unusual properties. For example, R is countable *universal* (it embeds all countable graphs as induced subgraphs) and is *homogeneous* (an isomorphism between two finite induced subgraphs is induced by an automorphism). The random graph also has a hamiltonian path and an independent dominating set. For a recent excellent survey of results on R see Cameron [4].

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R exists because the class of finite graphs has the *amalgamation property*, (AP): if there are embeddings $f: A \to B$ and $g: A \to C$ between finite graphs A, B, and C, then we can find a finite graph D and embeddings $h: B \to D$ and $i: C \to D$ so that $ig = hf$. The class of finite graphs has (AP) as it is *closed under unions*: if B and C are graphs with either a common induced subgraph A in their intersection or empty intersection, we can form the graph $B \cup C$ with vertices $V(B) \cup V(C)$ and edges $E(B) \cup E(C)$. (We say that $B \cup C$ is the *union* of B and C *over* A.) Is there a graph similar to R for other classes of graphs, say for the class of n-colourable graphs for a given $n \geqslant 1$? As is well known, the latter classes are not amalgamation classes.

The results of the author in [1] prove that in the classes of n-colourable (and more generally, G-colourable) graphs there is a universal graph which is not homogeneous but pseudo-homogeneous (see Section 1.1). In the present article, we investigate these universal pseudo-homogeneous G-colourable graphs and compare and contrast their graph-theoretic properties with R (see Section 3). Using our techniques we supply new representations of R (see Section 2) and answer a question of Caicedo [2] on antichains in the lattice of cores (see Section 4).

Throughout, all graphs are countable and simple. If $S \subseteq V(G)$, then $G \restriction S$ is the *induced subgraph* of G on S. We write $G \leqslant H$ if G is an induced subgraph of H. (We also say that H is an *extension* of G.) For $x \in V(G)$, the *neighbour set* of x, denoted by $N(x)$, is the set of vertices joined to x; the *co-neighbour set of* x, denoted by $N^c(x)$, is the set of vertices not joined and not equal to x. The graph K_n is the *n-clique* or the *complete graph of order n*. We write $\omega(G)$ for the *clique number* of G (the order of a maximum clique in G) and $\chi(G)$ for the *chromatic number* of G (the minimum n so that G has a proper n-colouring).

A *homomorphism* between graphs G and H is a function $f: V(G) \to V(H)$ that is also a function from $E(G)$ to $E(H)$. If there is a homomorphism from H to G we write $H \to G$. A graph is *G-colourable* if it admits a homomorphism into G; the class of all G-colourable graphs is denoted $C(G)$. An injective homomorphism which also preserves non-edges is an *embedding*, and bijective embeddings from a graph onto itself are *automorphisms*. The group (under composition) of automorphisms of a graph is denoted by $\mathrm{aut}(G)$ with identity id_G. If $S \leqslant G$ and $f: G \to H$ is a map, we write $f \restriction S$ for the *restriction* of f to S. A finite graph G is a *core* if every homomorphism of G to itself is an automorphism. (Note that we restrict our attention to finite cores.) Each finite graph has a core graph as an induced subgraph. Each of the complete graphs and odd cycles are cores, as is the Petersen graph. A graph has a homomorphism into K_n if and only if it is n-colourable. For further information on homomorphisms the reader is directed to the survey of Hahn and Tardif [6].

1.1. Preliminaries

Let $\mathscr{C}$ be a class of countable graphs closed under isomorphisms. A countable graph $M \in \mathscr{C}$ is *universal pseudo-homogeneous* if there is a subclass $\mathscr{C}'$ of finite graphs from $\mathscr{C}$ such that:

(PH1) The graph M embeds each graph in $\mathscr{C}'$ as an induced subgraph.
(PH2) Each finite $S \leqslant M$ is contained in $T \leqslant M$ with $T \in \mathscr{C}'$.
(PH3) For each $G \leqslant M$ with $G \in \mathscr{C}'$ and for each graph $H \in \mathscr{C}'$ so that $G \leqslant H$, there is an $H' \leqslant M$ and an isomorphism $f: H \to H'$ such that $f \restriction G = \mathrm{id}_G$.

Pseudo-homogeneous graphs (and relational structures) were first investigated by Calais [3], and Fraïssé discussed them in Chapter 11 of his book [5]. For a more general model-theoretic approach, the reader is directed to Kuecker and Laskowski [9]. The graph M is the unique (up to isomorphism) countable graph with properties (PH1), (PH2), and (PH3). For a discussion of the existence and uniqueness of M, the reader is directed to the Pseudo Amalgamation Theorem of [5, Section 6.6]. Note that R is also universal pseudo-homogeneous with $\mathscr{C} = \mathscr{C}'$ the class of all finite graphs.

A graph H is *uniquely G-colourable* if $H \in C(G)$, each homomorphism from H into G is onto, and for any two homomorphisms $f, h: H \to G$ there is a $g \in \mathrm{aut}(G)$ so that $f = gh$. One of the main results of [1] is that for a finite core G, $C(G)$ has a universal pseudo-homogeneous graph $M(G)$ with $\mathscr{C}'$ the class of finite uniquely G-colourable graphs. From [1] it follows that $M(G)$ embeds each countable G-colourable graph and is uniquely G-colourable. The graph $M(G)$ also has the following additional property which we will use in several proofs.

(e.c. property) If $S \leqslant M(G)$ is finite and $S \leqslant T$ with T finite and $T \in C(G)$ so that the union of $M(G)$ and T over S is G-colourable, then there is a $T' \leqslant M(G)$ and an isomorphism $f: T \to T'$ such that $f \restriction S = \mathrm{id}_S$. (We say we can *amalgamate* T into $M(G)$ over S.)

e.c. is short for *existentially closed*. Each $A \in C(G)$ embeds in a uniquely G-colourable graph via the following construction. Assume A and G are disjoint. Fix a homomorphism $f: A \to G$ and define $A(f)$ to be the graph with vertices $V(A) \cup V(G)$ and edges: $E(A) \cup E(G) \cup \{xy\colon x \in A, y \in G, f(x)y \in E(G)\}$. The graph $A(f)$ is the *fixation* of A by f relative to G. We restate results from [1] that we use frequently below.

Theorem 1. *Fix G a core graph.*

1. *For every $A \in C(G)$ and $f: A \to G$ a homomorphism, $A(f)$ is uniquely G-colourable, and $f \cup \mathrm{id}_G: A(f) \to G$ is a homomorphism.*
2. *For all uniquely G-colourable graphs A, B, C so that $V(B) \cap V(C) = V(A)$, the union $B \cup C$ over A is uniquely G-colourable.*

2. Limits of the graphs $M(G_i)$

There are several known characterizations of R (see [4, Section 2]). In this section, we supply new characterizations of R via limits of the graphs $M(G_i)$.

Let $\{G_i\colon i \geqslant 1\}$ be a chain of countable graphs ordered by inclusion. Define $\lim G_i$ to be the graph whose vertices are $\bigcup_{i \geqslant 1} V(G_i)$ and edges $\bigcup_{i \geqslant 1} E(G_i)$; $\lim G_i$ is the *limit* of the graphs $\{G_i\colon i \geqslant 1\}$. If $\{G_i\colon i \geqslant 1\}$ is a family of graphs ordered by embedding, we form $\lim G_i$ by assuming that all embeddings are inclusions (after identifying each graph in the chain with its image). If there is a homomorphism from G into H we write $G \to H$; otherwise, we write $G \nrightarrow H$.

Definition 2. 1. A sequence $\mathscr{C} = (G_i\colon i \geqslant 1)$ of graphs is *increasing* if $G_i \to G_j$ and $G_j \nrightarrow G_i$ if and only if $i < j$.

2. An increasing sequence $\mathscr{C} = (G_i\colon i \geqslant 1)$ of graphs is *cofinal* if the set $\{\omega(G_i)\colon i \geqslant 1\}$ is unbounded.

For example, the increasing sequence $(K_i\colon i \geqslant 1)$ is cofinal; however, using the density of cores (see [11, Theorem 5.1]) there are $2^{\aleph_0}$ increasing non-cofinal sequences of cores. (Density also implies the existence of $2^{\aleph_0}$ many increasing cofinal sequences of finite cores.) If G and H are cores, then $G \to H$ implies that $M(G) \leqslant M(H)$ by universality. Therefore, an increasing sequence $\mathscr{C} = (G_i\colon i \geqslant 1)$ of finite cores naturally gives rise to the limit $\lim M(G_i)$.

We prove the following.

Theorem 3. *An increasing sequence of cores $\mathscr{C} = (G_i\colon i \geqslant 1)$ is cofinal if and only if $\lim M(G_i) \cong R$.*

Before we prove Theorem 3, we prove the following lemma.

Lemma 4. *If an increasing chain $\mathscr{C} = (G_i\colon i \geqslant 1)$ of cores is cofinal then there is an increasing cofinal subsequence $(G_{r_i}\colon i \geqslant 1)$ of $\mathscr{C}$ and an increasing sequence of complete graphs $(K_{r_i}\colon i \geqslant 1)$ so that*

$$G_{r_1} \to K_{r_1} \to G_{r_2} \to K_{r_2} \to \cdots.$$

Proof. Let $G_{r_1} = G_1$ and if $\chi(G_1) = s$, let $K_{r_1} = K_s$. Assume G_{r_n} and K_{r_n} are defined. By the cofinality of $\mathscr{C}$, we can find a G_t so that $\omega(G_t) \geqslant r_n$; set $G_{r_{n+1}} = G_t$, and choose $K_{r_{n+1}}$ to have index $r_{n+1} = \chi(G_{r_{n+1}})$. $\square$

Using Lemma 4 and previous remarks we see that any increasing cofinal sequence of cores gives rise to the following commuting diagram (where all arrows are inclusions).

$$\begin{array}{ccccccccc}
 & M(K_{r_1}) & & & M(K_{r_2}) & & & \cdots & \\
\nearrow & & \searrow & \nearrow & & \searrow & \nearrow & & \\
M(G_1) & \to \cdots & & M(G_{r_2}) & \to \cdots & & M(G_{r_3}) & \to \cdots &
\end{array}$$

From the diagram it follows that

$$\lim M(G_i) = \lim M(K_{r_i}). \tag{1}$$

Proof of Theorem 3. For necessity, suppose the set $\{\omega(G_i)\colon i \geqslant 1\}$ is bounded above by n. If $K_{n+1} \leqslant \lim M(G_i)$ then for some $j, K_{n+1} \leqslant M(G_j)$. But then $K_{n+1} \to G_j$ so that $K_{n+1} \leqslant G_j$, which is a contradiction. However, by universality, $K_{n+1} \leqslant R$, so $\lim M(G_i) \ncong R$.

For sufficiency, fix an increasing cofinal chain $\mathscr{C} = (G_i\colon i \geqslant 1)$, and let $G = \lim M(G_i)$. We show that G satisfies each of the n-e.c. properties stated at the beginning of the introduction. By (1), without loss of generality, we may assume that $G = \lim M(K_r)$ for some increasing sequence of complete graphs. Fix S an n-subset of $V(G)$, and fix a partition of S with sets A and B. Let j be the least index so that $S \subseteq V(M(K_j))$. Define $l = |S| + j$. Then

$$S \subseteq V(M(K_j)) \subseteq V(M(K_l)).$$

Let C be the graph containing $G \restriction S$ as an induced subgraph formed by adding a new vertex x joined only to vertices in A. We claim that $C \to K_l$. To see this, note that we can colour $G \restriction S$ using at most $|S| < l$ colours, and so we can colour x with one of any of the remaining unused colours from $\{1, \ldots, l\}$.

Let $D = C \cup M(K_l)$ over $M(K_l) \restriction S$. Then $M(K_l) \leqslant D$, and in D we can find a vertex joined to each vertex in A but to no vertex in B. Further, $\chi(D) = l$. To see this fix a proper l-colouring of $M(K_l)$. By the choice of l we can colour x by a colour not used in S to obtain a proper l-colouring of D. By the e.c. property of $M(K_l)$ we can find a vertex z in $M(K_l)$ joined to vertices in A and not joined to vertices in B. As $M(K_l) \leqslant G$ the proof follows. $\square$

Theorem 3 raises the following question for which we have no answer: what can be said about graphs which are limits of increasing *non-cofinal* sequences of cores?

3. Independent dominating sets, co-neighbour sets, and spanning subgraphs

Since $K_{\aleph_0} \nleqslant M(G)$ for any core G, $\overline{K_{\aleph_0}} \leqslant M(G)$ by Ramsey's theorem. As proven first in Theorem 3.2 of [7], R has an *independent dominating set* (a set of vertices S with no edges between them so each $x \notin S$ is joined to some vertex in S). The original proof in [7] gave an explicit construction of such a set; it did not appeal to the Axiom of Choice (AC). We have the following analogous result for each of the graphs $M(G)$. We observe that the following theorem provides a construction of $M(G)$ and at the same time yields a *recursive* independent set.

Theorem 5. *For each core graph G, $M(G)$ has an independent dominating set.*

Proof. If $G = K_1$ then $M(G) = \overline{K_{\aleph_0}}$. We therefore assume G is non-trivial. We inductively construct a G-colourable graph A with an independent dominating set A^0, and then show that $A \cong M(G)$.

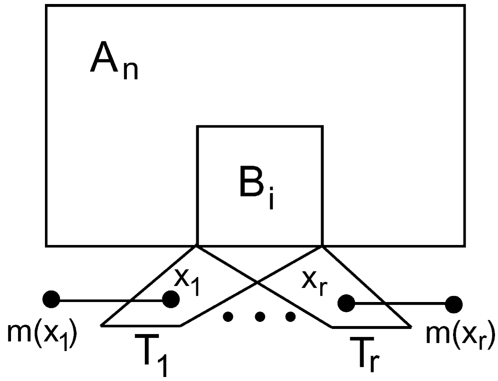

Fig. 1. The graph A''_{n+1}.

Enumerate all finite G-colourable graphs as $\{G_n\colon n \geqslant 1\}$ with $G_1 = K_1$. Define $A_0 = A_0^0 = G_1$. Assume A_n and A_n^0 have been defined with the following properties.

1. A_n is a finite uniquely G-colourable graph.
2. $G_1, \ldots, G_n \leqslant A_n$.
3. $A_0^0 \subseteq A_n^0 \subseteq A_n$.
4. A_n^0 is an independent dominating set in A_n.

We now define A_{n+1} in stages.

Define A'_{n+1} as follows. List all n-vertex uniquely G-colourable induced subgraphs of A_n as $B_1, \ldots, B_j$. For each $1 \leqslant i \leqslant j$, list the distinct isomorphism types of uniquely G-colourable one-element extensions of B_i as $T_1, \ldots, T_r$. Without loss of generality, we may assume that $V(T_k) \cap V(A_n) \in V\{B_i\colon 1 \leqslant i \leqslant j\}$ for $1 \leqslant k \leqslant r$. Take the unions of $T_1, \ldots, T_r$ and A_n over B_i to form $A_{n+1}(i)$. Take the unions of $A_{n+1}(1), \ldots, A_{n+1}(j)$ over A_n to form A'_{n+1}. The graph A'_{n+1} is G-colourable by Theorem 1(2).

For each extension T_k of B_i match the vertex of T_k not in $V(B_i)$ with a new vertex disjoint from A'_{n+1}. Perform this matching for all vertices in $V(A'_{n+1}) \setminus V(A_n)$; we name the resulting graph A''_{n+1}. See Fig. 1. Our notation is: $x \in V(T_k) \setminus V(B_i)$ is matched with $m(x)$.

We claim that A''_{n+1} is G-colourable. To see this say $f: A'_{n+1} \to G$ is a homomorphism. As G is a non-trivial core, no vertex in G is isolated. For $x \in V(A'_{n+1})$ if $f(x) = a$, then fix some $a(x) \in V(G)$ joined to a.

Define a map $f^*: A''_{n+1} \to G$ by

$$f^*(x) = \begin{cases} a(z) & \text{if } x = m(z) \in V(A''_{n+1}) \setminus V(A'_{n+1}) \text{ and } f(z) = a, \\ f(x) & \text{else.} \end{cases}$$

Then f^* is a homomorphism (by the choice of $a(x)$.)

Add disjoint copies of $\overline{K_{|G|}}$ and G_{n+1} to A''_{n+1} to form $A^{(3)}_{n+1}$. Then $A^{(3)}_{n+1} \in C(G)$. Let $h: A^{(3)}_{n+1} \to G$ be some fixed homomorphism whose restriction to $\overline{K_{|G|}}$ is injective.

Define $A_{n+1}=A^{(3)}_{n+1}(h)$; by Theorem 1(1), A_{n+1} is uniquely G-colourable. Define A^{0*}_{n+1} to be A^0_n along with vertices $\{m(x)\colon x\in V(A'_{n+1})\setminus V(A_n)\}$ added in A''_{n+1} and $\overline{K_{|G|}}$ added in $A^{(3)}_{n+1}$. Define D_{n+1} to be an independent dominating set in the copy of G_{n+1} added in $A^{(3)}_{n+1}$. Define A^0_{n+1} to be $A^{0*}_{n+1}\cup D_{n+1}$.

Then A_{n+1} and A^0_{n+1} satisfy items (1), (2), and (3).

To see that (4) holds, observe that we only need to consider the case of vertices in $V(A_{n+1})\setminus(V(A^{(3)}_{n+1})\cup A^0_{n+1})$, the other cases begin clear by construction and induction hypothesis. In this case we consider a vertex x in the copy of $G\nleqq A^{(3)}_{n+1}$ added in the fixation $A^{(3)}_{n+1}(h)$. As x is not isolated in G, there is some z joined to x in G. Since $h\restriction\overline{K_{|G|}}$ is injective, there is some $y\in V(\overline{K_{|G|}})\subseteq V(A^0_{n+1})$ so that $h(y)=z$. But then $yx\in E(A_{n+1})$ from the definition of fixation.

Let $A=\lim A_i$ and let $A^0=\lim A^0_i$. Then $A\cong M(G)$ as A satisfies conditions (PH1) $-$(PH3) of Section 1.1. The set A^0 is an independent dominating set as each A^0_i is in A_i. □

Another unusual property of R is its "indestructibility" (see [4, Section 3]). In particular,

1. (a) For each $x\in V(R)$, $R\restriction N(x)\cong R$; (b) $R\restriction N^{\mathrm{c}}(x)\cong R$; and
2. $\bar{R}\cong R$.

A property of R that implies (1) is the *pigeonhole property*:

3. For each partition of $V(R)$ into sets $A_1,\dots,A_n$ at least one of $R\restriction A_i$ is isomorphic to R (see Corollary 1.5 of [7]).

Properties (1) and (3) may be called *fractal properties* since they ensure that a graph G satisfying them contains proper induced subgraphs isomorphic to G. Properties (1) and (3) hold in $M(G)$ if $G=K_1$ (since then $M(G)=\overline{K_{\aleph_0}}$). However, for any G, $K_{\aleph_0}\leqslant\overline{M(G)}$ so (2) fails. Given $G\neq K_1$, $\omega(M(G)\restriction N(x))<\omega(M(G))$ and so $M(G)\restriction N(x)$ cannot be isomorphic to $M(G)$, thus (1a) fails. (3) fails for each $M(G)$ with $G\neq K_1$ since we may take the A_i to be the $|G|$-many independent sets partitioning $V(M(G))$. Despite these negative results we do have the following theorem. Given a graph G and $x\in V(G)$, $G-x$ is the subgraph formed by deleting x and all edges incident with x.

Theorem 6. *Fix $G\neq K_1$ a core graph, and fix $x\in V(M(G))$. The following hold.*

1. $M(G)\restriction N^{\mathrm{c}}(x)\cong M(G)$.
2. If $G=K_n$, *for a fixed* $n>1$, $M(G)\restriction N(x)\cong M(K_{n-1})$.
3. $M(G)-x\cong M(G)$.

Item (3) is usually referred to as *inexhaustability* (see [5, Chapter 10]).

Proof. (1) Let x be the vertex fixed in the statement of the theorem. We show that $H=M(G)\restriction N^{\mathrm{c}}(x)$ has properties (PH1)–(PH3) described in Section 1.1. We first show property (PH2) holds.

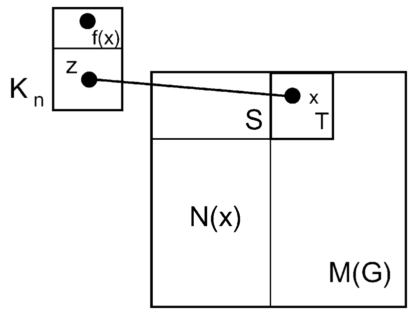

Fig. 2. The extension T'.

Fix $S \leqslant H$ with S finite. Let $f : M(G) \to G$ be a fixed homomorphism. Let $A = M(G) \restriction (S \cup \{x\})$, and consider the extension B formed by first taking $A(f \restriction A)$ and then deleting all edges between x and the copy of G in $A(f \restriction A)$. Then $B \in C(G)$. Let C be the union of B and $M(G)$ over A. The map $f \cup \mathrm{id}_G : C \to G$ is a homomorphism by Theorem 1(1), and so by the e.c. property of $M(G)$, we can amalgamate B into $M(G)$ over A as B'. Note that $A(f \restriction S) \leqslant B'$ and that no vertex of the copy of G in B' is joined to x. The copy of $A(f \restriction S) - x$ in B' is the desired uniquely G-colourable extension of S in H (by Theorem 1(1)).

For property (PH1), let A be a finite G-colourable graph. Form $C = M(G) \uplus A$ (the disjoint union of $M(G)$ and A). Then C is G-colourable, and no vertex of A in C is joined to x. By the e.c. property, we can find a copy A' of A in $M(G)$ so that no vertex of A' is joined to x. But then A' is in H.

For property (PH3), let A be a uniquely G-colourable finite induced subgraph of H, and let B be an extension of A that is also a uniquely G-colourable finite graph. Without loss of generality, we can assume that $V(B) \setminus V(A)$ is disjoint from $M(G)$. By Theorem 2 since A, B, and $M(G)$ are uniquely G-colourable, we can form $B \cup M(G)$. Note that each vertex in $V(B) \setminus V(A)$ is not joined to x. By the e.c. property, we can amalgamate B into $M(G)$ as B'. Observe that B' is in H.

(2) We show that $H = M(G) \restriction N(x)$ satisfies properties (PH1)–(PH3) relative to $C(K_{n-1})$. For property (PH1), fix a finite $(n-1)$-colourable graph A. Extend $M(G)$ by a disjoint copy of A so that each vertex of A is joined to x (and x only). Then $M(G) \cup A$ over $M(G) \restriction \{x\}$ is n-colourable, so by the e.c. there is a copy of A inside $M(G)$ and hence in H.

For property (PH2), fix S a finite subset of $N(x)$, and let $f : M(G) \to G$ be a fixed homomorphism. Let T be the induced subgraph of $M(G)$ on $\{x\} \cup S$. Form $T' = T(f \restriction T)$. Then for each $y \in S$, $f(y) \neq f(x)$ in the copy of K_n in T', as x is joined to y. Further, if $z \in V(K_n) \setminus \{f(x)\}$ then z is joined to x (by the definition of fixation). See Fig. 2.

By Theorem 1, $T' \cup M(G)$ over T is n-colourable, so we may amalgamate T' into $M(G)$ and therefore, find a copy T'' of T' extending T; let the image of $f(x)$ in T'' be named b. The induced subgraph on $S \cup (V(T'') \setminus \{b\})$ is uniquely $(n-1)$-colourable, as it is isomorphic to $S(f \restriction S)$ relative to K_{n-1}. By previous remarks, it is a subset of $N(x)$.

For property (PH3), fix a finite uniquely $(n-1)$-colourable graph S such that $V(S) \subseteq N(x)$, and suppose S is extended by a finite uniquely $(n-1)$-colourable graph T. The graph T' formed by adding all edges between x and $V(T) \setminus V(S)$ is n-colourable and so we can amalgamate T' into $M(G)$ as T''. T'' is then an induced subgraph of $M(G) \restriction N(x)$.

The proof of item (3) is similar to the proofs of (1) and (2). □

As mentioned in the Introduction, R has one- and two-way infinite hamiltonian paths and has 1-factors. If G is not connected, then $M(G)$ is not connected, so we cannot hope for hamiltonian paths. However, we do have the following theorem.

Theorem 7. *If G is connected nontrivial core, then the following hold.*

1. *$M(G)$ has infinite one-way and two-way hamiltonian paths.*
2. *$M(G)$ has a 1-factor.*

Proof. (1) We find a one-way hamiltonian path; the argument for the two-way path is similar. Enumerate $V(M(G)) = \{x_i\colon i \geqslant 1\}$. Define $Q_1 = M(G) \restriction \{x_1\}$. Assume that Q_n is defined and is a subgraph of $M(G)$ that is a path containing $\{x_1, \ldots, x_n\}$ with terminal vertices x_1 and z.

We can assume $x_{n+1} \notin V(Q_n)$. Fix a homomorphism $f: M(G) \to G$ and let

$$B = M(G) \restriction (Q_n \cup \{x_{n+1}\}).$$

If z is joined to x_{n+1} we may let $Q_{n+1} = B$. Otherwise, consider $C = B(f \restriction B)$. In the copy of G in C we claim there is a vertex $j(z)$ joined to z and a vertex $j(x_{n+1})$ joined to x_{n+1}.

Case (i): $f(z) = f(x_{n+1})$.

Since G has no isolated vertex, there is a $b \in V(G)$ joined to $f(z) = f(x_{n+1})$. Let $j(z) = j(x_{n+1}) = b$.

Case (ii): $f(z) \neq f(x_{n+1})$.

There is a $b_1 \in V(G)$ joined to $f(z)$ and a $b_2 \in V(G)$ joined to $f(x_{n+1})$. Let $b_1 = j(z)$ and $b_2 = j(x_{n+1})$.

By Theorem 1(1), $C \cup M(G)$ (over Q_n) is G-colourable by $f \cup \mathrm{id}_G$.

If $j(z) = j(x_{n+1})$ then we use the e.c. property to find a vertex z' (not in B) joining z and x_{n+1} in $M(G)$; we then let Q_{n+1} be Q_n along with $\{x_{n+1}, z'\}$.

If $j(z) \neq j(x_{n+1})$, then since G is connected, there is a path P in G connecting $j(z)$ to $j(x_{n+1})$. Using the e.c. property we can find a path P' (not in B) connecting z and x_{n+1} in $M(G)$. We then let Q_{n+1} be Q_n along with $\{x_{n+1}\} \cup P'$.

The graph $\lim Q_n$ is the desired one-way hamiltonian path in $M(G)$.

For (2), we may use the hamiltonian path constructed in (1), or proceed as follows. Define $R_1 = M(G) \restriction \{x_1\}$. Assume that R_n is defined, and is a matching containing

$\{x_1,\ldots,x_n\}$. We can assume $x_{n+1} \notin V(R_n)$. We claim that x_{n+1} is joined to some vertex not in $V(R_n)$. To see this, fix $f : M(G) \to G$, and let $A = M(G) \restriction V(R_n) \cup \{x_{n+1}\}$. Then $B = A(f \restriction A)$ is a G-colourable extension of A, and $B \cup M(G)$ over A is G-colourable. By the e.c. property, we can find a copy B' of B in $M(G)$. Observe that B' contains a copy of G disjoint from R_n. Now, $f(x_{n+1})$ is not isolated in G, so x_{n+1} is joined to some neighbour z of $f(x_{n+1})$ in B' not in R_n. Let R_{n+1} be R_n along with the edge zx_{n+1}, so that R_{n+1} is a matching containing $\{x_1,\ldots x_{n+1}\}$.

The graph $\lim R_n$ is the desired 1-factor. □

We remark that we do not know if $M(G)$ has a 1-factorization or decomposes into hamiltonian paths, even if $G = K_2$.

4. Infinite antichains

The relation $G \to H$ and $H \to G$, denoted by $G \leftrightarrow H$, is an equivalence relation on the class of all graphs. The equivalence classes, denoted by $\mathscr{H}(G)$, are ordered by $\mathscr{H}(G) \preccurlyeq \mathscr{H}(H)$ if $G \to H$. For ease of notation, we will identify $\mathscr{H}(G)$ with G and write $G \preccurlyeq H$ if $\mathscr{H}(G) \preccurlyeq \mathscr{H}(H)$. If G is strictly less than H in this order, we then write $G \prec H$. As is well known, this order is a distributive lattice L of infinite height with a 0 $(= K_1)$ and a single atom $(= K_2)$. The lattice L is also dense in each interval above K_2 (see [11, Theorem 5.1]). In [2] Caicedo asked if for every $n \geqslant 3$ the interval $[0, K_n]$ has infinite width. We answer this question positively in a strong fashion in Theorem 9. We use a result of Zhu [12]. Another solution to this question, using different methods, appears in [10]. Recall that the *girth* of G is the order of a smallest cycle in G (or ∞ if G acyclic), and is denoted by $g(G)$; the *odd-girth* of G, denoted by $\mathrm{og}(G)$, is the order of a smallest odd-cycle in G (or ∞ if G is bipartite).

Theorem 8 (Zhu [12]). *If G and H be non-trivial graphs with $H \prec G$ and $n \geqslant 4$ then there is a graph J such that $g(J) = n$ and $J \preccurlyeq G$ but $J \not\preccurlyeq H$.*

Theorem 9. *In each interval $[0, G]$ in L with $K_2 \prec G$ there exist the following families.*

1. *An infinite $\prec$-antichain of finite graphs.*
2. *An infinite $\preccurlyeq$-antichain of countable universal pseudo-homogeneous graphs.*

Proof. We note that (2) follows from (1) since $M(G) \leq M(H)$ if and only if $G \preccurlyeq H$.

Fix $G \succ K_2$. Using the density of L, we can find an infinite increasing chain $(G_i \colon i \geqslant 1)$ in $[0, G]$. Using Theorem 8, we can find a graph $H_1 \preccurlyeq G_2$ so that $H_1 \not\preccurlyeq G_1$ with $\mathrm{og}(H_1) = 5$. In general, for each $n \geqslant 2$ using Theorem 8 we choose H_n to be a graph $\preccurlyeq G_{n+1}$ and $\not\preccurlyeq G_n$ with

$$\mathrm{og}(H_n) = 2n + 3.$$

We claim that $\{H_n: n \geqslant 1\}$ is an $\leqslant$-antichain in $[0, G]$. Observe first that $H_n \leqslant G_{n+1} \leqslant G$ so $H_n \in [0, G]$.

Fix $i < j$. As

$$\mathrm{og}(H_i) < \mathrm{og}(H_j),$$

we must have $H_i \nleqslant H_j$. If $H_j \leqslant H_i$ then since $H_i \leqslant G_{i+1}$ we have that $H_j \leqslant G_{i+1}$. But $j \geqslant i+1$ so that $G_{i+1} \leqslant G_j$ so $H_j \leqslant G_j$, which is a contradiction. □

5. Comments and open problems

As is well known the first-order theory of the infinite random graph has the *finite model property*: every first-order sentence true in R is true in some finite graph (see [4]). In Corollary 2.6 of [8], Kolaitis et al. found a countable universal n-colourable graph $D(n)$ with the finite model property. The existence of $D(n)$ follows from the proof in [8] that the class of finite n-colourable graphs has a 0-l law. The graph $D(n)$ is the unique countable n-colourable graph satisfying a certain set of axioms (see [8]); using the universal and e.c. properties of $M(K_n)$, it can be shown that $M(K_n)$ satisfies these axioms, so that $D(n) \cong M(K_n)$. In particular, the theory of $M(K_n)$ has the finite model property. If G is not complete, does the theory of $M(G)$ have the finite model property?

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