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A Pigeonhole Property for Relational Structures

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Abstract. We study those relational structures $\boldsymbol{S}$ with the property $(\mathcal{P})$ that each partition of S contains a block isomorphic to $\boldsymbol{S}$. We show that the Fraïssé limits of parametric classes $\mathcal{K}$ have property $(\mathcal{P})$; over a binary language, every countable structure in $\mathcal{K}$ satisfying $(\mathcal{P})$ along with a condition on 1-extensions must be isomorphic to this limit.

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1 Introduction

As proved first by HENSON in [4], the countable random graph $\boldsymbol{R}$ (also called the countable universal homogeneous graph) has the following unusual property: if the vertices of $\boldsymbol{R}$ are partitioned into finitely many sets $X_1, \dots, X_n$, then for some $1 \leq i \leq n$ the induced subgraph on X_i is isomorphic to $\boldsymbol{R}$. CAMERON (see [1] and [2]) has called this property the *pigeonhole property*. One can generalize the pigeonhole property from graphs to relational structures in the following way:

Definition 1.1. Let L be a relational language and let $\boldsymbol{S}$ be an L-structure. The structure $\boldsymbol{S}$ *has property* $(\mathcal{P})$ if $|S| > 1$ and for each $n \geq 2$, whenever $S = S_1 \uplus \cdots \uplus S_n$, then for some $1 \leq i \leq n$, $\boldsymbol{S} \restriction S_i \cong \boldsymbol{S}$, where $\uplus$ is disjoint union, and $\boldsymbol{S} \restriction S_i$ is the induced substructure on S_i in $\boldsymbol{S}$.

CAMERON asked the following question: what "nice" structures have property $(\mathcal{P})$? (see [1, p. 48]). We believe that Theorem 1.1 below gives a satisfactory answer to this question within many classes of relational structures. Before we can state Theorem 1.1, we must first supply some definitions. For background in basic model theory, the reader is referred to either [3] or [5]; see [5] for results on Fraïssé limits and e.c. structures. For L-structures $\boldsymbol{S}$, $\boldsymbol{T}$, we write $\boldsymbol{S} \leq \boldsymbol{T}$ if $\boldsymbol{S}$ is a substructure of $\boldsymbol{T}$. Throughout, we assume L is a relational language.

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Definition 1.2.

(1) Let $\mathcal{K}$ be a class of L-structures closed under isomorphisms. A structure $\boldsymbol{S} \in \mathcal{K}$ is 1-*e. c.* iff for all $\boldsymbol{T} \in \mathcal{K}$, $a \in S$ and $\theta(x, a)$ a quantifier-free L-formula, if $\boldsymbol{S} \leq \boldsymbol{T}$ and $\boldsymbol{T} \models \exists x\, \theta(x, a)$, then $\boldsymbol{S} \models \exists x\, \theta(x, a)$.

(2) For a first-order L-formula θ, $\forall\,(\text{distinct})x_1 \ldots x_n\theta$ abbreviates

$\forall x_1 \ldots x_n(\bigwedge_{1 \leq i < j \leq n} x_i \neq x_j \rightarrow \theta)$.

(3) $\mathcal{K}$ is a *parametric class over* L if the following conditions hold:

(a) $\mathcal{K}$ is axiomatized by a finite set of L-sentences of the form $\forall\,(\text{distinct})x_1 \ldots x_n\theta$, where θ is a Boolean combination of atomic L-formulas of the form $Ry_1 \ldots y_m$ with $\{y_1, \ldots, y_m\} = \{x_1, \ldots, x_n\}$;

(b) $\mathcal{K}$ contains infinite structures;

(c) there is only one isomorphism type of one element structure in $\mathcal{K}$.

Theorem 1.1. *Let* $\mathcal{K}$ *be a parametric class over a finite relational language* L *without unary predicate symbols.*

(1) $\mathcal{K}$ *has a Fraïssé limit* $\boldsymbol{F}(\mathcal{K})$ *and* $\boldsymbol{F}(\mathcal{K})$ *has property* $(\mathcal{P})$.

(2) *If* L *consists of only binary predicate symbols, then for a countable* $\boldsymbol{S} \in \mathcal{K}$ *the following are equivalent:*

(a) $\boldsymbol{S}$ *satisfies property* $(\mathcal{P})$ *and* $\boldsymbol{S}$ *is* 1-*e. c.*

(b) $\boldsymbol{S} \cong \boldsymbol{F}(\mathcal{K})$.

Remark 1.1.

(1) Parametric classes were first studied by W. Oberschelp in [6] in the context of 0-1 laws in finite model theory. See also [3, Chap. 3] for further details.

(2) The hypothesis in Theorem 1.1 that L contain no unary predicate symbols is essential; further, the restriction on 1-element structures in parametric classes cannot be dropped. See Lemma 2.3 below.

(3) We do not know of an extension of Theorem 1.1(2) to languages with predicate symbols of arity ≥ 3.

2 Examples

We defer the proof of Theorem 1.1 to the end of the article and first consider some of its consequences.

2.1 Applications of Theorem 1.1

1. The class of k-uniform hypergraphs is axiomatized by the following parametric sentences:

$$\forall\,(\text{distinct})x_1 \ldots x_k\,(R(x_1 \ldots x_k) \rightarrow \bigwedge_{\sigma \in S_k}(R(x_{\sigma(1)} \ldots x_{\sigma(k)})),$$

where S_k is the symmetric group of order k, and for each $1 \leq j \leq k-1$,

$$\forall\,(\text{distinct})x_1 \ldots x_j\,(\bigwedge_{\{x_1,\ldots,x_j\}=\{y_1,\ldots,y_k\}} \neg R(y_1 \ldots y_k)).$$

By Theorem 1.1(1), the universal homogeneous k-uniform hypergraph has property $(\mathcal{P})$.

2. The following classes are parametric over a language with a single binary predicate symbol $\{E\}$; hence by Theorem 1.1(2) the countable universal homogeneous structure in each class is characterized by having property $(\mathcal{P})$ and by being 1-e. c. In each case, we give a parametric axiomatization:

(a) Digraphs: $\forall x\,(\neg xEx)$.

(b) Graphs: $\forall x\,(\neg xEx)$, $\forall\,(\text{distinct})xy(xEy \rightarrow yEx)$.

(c) Oriented (or asymmetric) graphs: $\forall x\,(\neg xEx)$, $\forall\,(\text{distinct})xy\,(xEy \rightarrow \neg yEx)$.

(d) Tournaments: $\forall x\,(\neg xEx)$, $\forall\,(\text{distinct})xy\,(xEy \leftrightarrow \neg yEx)$.

(e) "Undirected" tournaments: $\forall x\,(\neg xEx)$, $\forall\,(\text{distinct})xy\,(xEy \vee yEx)$.

2.2. Structures without property $(\mathcal{P})$

Definition 2.1. Let $\boldsymbol{S}$ be an L-structure. Define the *graph of* $\boldsymbol{S}$, written $\boldsymbol{G}(\boldsymbol{S})$, to be the graph with vertices S and edges

$$\{(x,y)\ :\ x,y \in S \text{ so that } x \neq y \text{ and there exists } R \in L \text{ of arity } n \geq 2 \text{ with } \{a_1,\ldots,a_n\} \subseteq S \text{ so that } x,y \in \{a_1,\ldots,a_n\} \text{ and } R^{\boldsymbol{S}}a_1\ldots a_n\}.$$

Lemma 2.1. *Let L consist of only binary predicate symbols, and let $\boldsymbol{S}$ be an L-structure satisfying property $(\mathcal{P})$. Then $\boldsymbol{G}(\boldsymbol{S})$ satisfies property $(\mathcal{P})$.*

Proof. Let $\boldsymbol{G}(\boldsymbol{S}) = S_1 \uplus \cdots \uplus S_n$. Then $S = S_1 \uplus \cdots \uplus S_n$, so for some $1 \leq i \leq n$, $\boldsymbol{S} \restriction S_i \cong \boldsymbol{S}$. Thus, $\boldsymbol{G}(\boldsymbol{S} \restriction S_i) \cong \boldsymbol{G}(\boldsymbol{S})$. But $\boldsymbol{G}(\boldsymbol{S} \restriction S_i) = \boldsymbol{G}(\boldsymbol{S}) \restriction S_i$. □

Corollary 2.2. *Let L be as in Lemma 2.1. Let $\boldsymbol{S}$ be a countable L-structure so that $\boldsymbol{G}(\boldsymbol{S})$ is not $\overline{\boldsymbol{K}_{\aleph_0}}$ or $\boldsymbol{K}_{\aleph_0}$. If $\boldsymbol{G}(\boldsymbol{S})$ does not have every countable graph as an induced subgraph, then $\boldsymbol{G}(\boldsymbol{S})$ does not have property $(\mathcal{P})$. In particular, the universal homogeneous order $\boldsymbol{P}$ does not satisfy property $(\mathcal{P})$.*

Proof. In [2], Cameron proved that the only countable graphs with property $(\mathcal{P})$ are $\boldsymbol{K}_{\aleph_0}$, $\overline{\boldsymbol{K}_{\aleph_0}}$, and $\boldsymbol{R}$. The first statement now follows by Lemma 2.1, as $\boldsymbol{G}(\boldsymbol{S})$ must be $\boldsymbol{R}$, and $\boldsymbol{R}$ embeds every countable graph as an induced subgraph. For the final statement of the corollary, note that $\boldsymbol{G}(\boldsymbol{P})$ is just the comparability graph of $\boldsymbol{P}$, and so has no induced subgraph isomorphic to the five cycle. □

Lemma 2.3.

(1) *Let $\boldsymbol{S}$ be a structure with more than one isomorphism type of one element structure. Then $\boldsymbol{S}$ does not have property $(\mathcal{P})$.*

(2) *For every finite nonempty language L, the random L-structure (the Fraïssé limit of the class of finite L-structures) does not have property $(\mathcal{P})$.*

(3) *If L contains unary predicate symbols and $\boldsymbol{S}$ interprets these predicate symbols nontrivially (that is, for some $P \in L$, $\emptyset \neq P^{\boldsymbol{S}} \neq S$), then $\boldsymbol{S}$ does not have property $(\mathcal{P})$.*

Proof. The proofs of (1) and (3) are left to the reader. For (2), note that the random L-structure contains more than one isomorphism type of one element structure. For example, by universality, there is a one element substructure with empty relations, and for $R \in L$ there is a one element structure $\boldsymbol{S}$ with domain $\{a\}$ whose only relation is $R^{\boldsymbol{S}}a\ldots a$. □

3 Proof of Theorem 1.1

(1) The proof of the existence of $\boldsymbol{F}(\mathcal{K})$ in $\mathcal{K}$ is implicit in the discussion of [3, Section 3.2], and so we omit the details. We now show that $\boldsymbol{F}(\mathcal{K})$ has property $(\mathcal{P})$. Fix $n \geq 2$. Let $F(\mathcal{K}) = S_1 \uplus \cdots \uplus S_n$. If $\boldsymbol{F}(\mathcal{K})$ does not have property $(\mathcal{P})$, then there are finite sets $X_i \subseteq S_i$ so that $\boldsymbol{F}(\mathcal{K}) \restriction X_i$ has a one element extension $\boldsymbol{Y}_i$ with domain $X_i \cup \{d_i\}$ not realized in $\boldsymbol{F}(\mathcal{K}) \restriction S_i$ (we are using the fact that $\boldsymbol{F}(\mathcal{K})$ is the unique countable e. c. model of $\mathcal{K}$). Let $\boldsymbol{A} = \boldsymbol{F}(\mathcal{K}) \restriction \bigcup_{i=1}^{n} X_i$. By hypothesis, we may identify the $\boldsymbol{Y}_i \restriction \{d_i\}$'s; let the isomorphism type of $\boldsymbol{Y}_i \restriction \{d_i\}$ have domain $\{d\}$. Define a structure $\boldsymbol{B}$ with domain $A \cup \{d\}$ and with relations those of $\boldsymbol{A}$, the relations in $\boldsymbol{Y}_i$ induced by the extension by d of each X_i and the set of relations defined as follows: By hypothesis, there is a $\mathcal{K}$-structure $\boldsymbol{C}$ of infinite cardinality. Let $m = \max(\text{arity}\,(R) : R \in L)$ and fix a tuple $\bar{c}$ of length m from C. For each tuple $\bar{a}$ from B of length $\leq m$, so that d occurs in $\bar{a}$ and for each $1 \leq i \leq n$, $\bar{a}$ does not list the elements of $X_i \cup \{d\}$, let $\bar{c}'$ be a sub-tuple of $\bar{c}$ of length $|\bar{a}|$, and for each $R \in L$, $\bar{a} \in R^{\boldsymbol{B}}$ if and only if $\bar{c}' \in R^{\boldsymbol{C}}$. With the definition of the relations of $\boldsymbol{B}$ just given, each tuple of distinct elements of appropriate length from B will satisfy the axioms of $\mathcal{K}$, so $\boldsymbol{B} \in \mathcal{K}$. Hence, $\boldsymbol{B}$ is an $(|A|+1)$-element $\mathcal{K}$-extension of $\boldsymbol{A} \leq \boldsymbol{F}(\mathcal{K})$, and so may be realized in $\boldsymbol{F}(\mathcal{K})$. If d is realized in $\boldsymbol{F}(\mathcal{K}) \restriction S_i$, for some $1 \leq i \leq n$, then by construction $\boldsymbol{Y}_i$ is realized in $\boldsymbol{F}(\mathcal{K}) \restriction S_i$, contrary to hypothesis.

(2) By (1), $\boldsymbol{F}(\mathcal{K})$ satisfies property $(\mathcal{P})$. The limit $\boldsymbol{F}(\mathcal{K})$ is 1-e. c. as it is e. c.

For the converse, we show that $\boldsymbol{S}$ is e. c. The proof is by induction on $n \geq 1$. The induction hypothesis is as follows: If $\boldsymbol{S} \leq \boldsymbol{T}$, $\boldsymbol{T} \in \mathcal{K}$ and for $\bar{a} \in S^n$, $\theta(x, \bar{a})$ is a quantifier-free L-formula, then $\boldsymbol{T} \models \exists x\, \theta(x, \bar{a})$ implies $\boldsymbol{S} \models \exists x\, \theta(x, \bar{a})$ (note that we only require a single variable in $\exists x\, \theta(x, \bar{a})$; the general case of a formula of the form $\exists \bar{x}\, \theta(\bar{x}, \bar{a})$ with $\bar{x}$ a finite tuple of variables will follow by induction).

The case $n = 1$ holds as $\boldsymbol{S}$ is 1-e. c. Let $\boldsymbol{S} \leq \boldsymbol{T}$ and $\boldsymbol{T} \in \mathcal{K}$ so that for an $(n+1)$-element subset $\{a_1, \ldots, a_{n+1}\} \subseteq S$ and for a quantifier-free L-formula $\theta(x, a_1, \ldots, a_{n+1})$, $\boldsymbol{T} \models \exists x\, \theta(x, a_1, \ldots, a_{n+1})$. Without loss of generality, we can assume that $\theta(x, a_1, \ldots, a_{n+1})$ has the form $\theta'(x, a_{n+1}) \wedge \theta''(x, a_1, \ldots, a_n)$ with suitable quantifier-free L-formulas $\theta'(x, a_{n+1})$ and $\theta''(x, a_1, \ldots, a_n)$ (since L contains only binary symbols and a subformula containing no x is automatically true in $\boldsymbol{S}$). Furthermore, we can assume that each witness to $\exists x\, \theta(x, a_1, \ldots, a_{n+1})$ in $\boldsymbol{T}$ is distinct from each element of $\{a_1, \ldots, a_{n+1}\}$ (otherwise, there is a witness to $\exists x\, \theta(x, a_1, \ldots, a_{n+1})$ in $\boldsymbol{S}$). In particular, we may assume that $\theta(x, a_1, \ldots, a_{n+1})$ contains the conjunction $\bigwedge_{1 \leq i \leq n+1} x \neq a_i$, $\theta'(x, a_{n+1})$ contains $x \neq a_{n+1}$, and $\theta''(x, a_1, \ldots, a_n)$ contains $\bigwedge_{1 \leq i \leq n} x \neq a_i$. Define the following subsets of S:

$$\Omega = \{c \in S : \boldsymbol{S} \models \theta(c, a_1, \ldots, a_{n+1})\},$$
$$\Omega' = \{c \in S : \boldsymbol{S} \models \theta'(c, a_{n+1})\},$$
$$\Omega'' = \{c \in S : \boldsymbol{S} \models \theta''(c, a_1, \ldots, a_n)\}.$$

We show that $\Omega \neq \emptyset$. By inductive hypothesis, both $\Omega' \neq \emptyset$ and $\Omega'' \neq \emptyset$. Note that $\Omega = \Omega' \cap \Omega''$. By hypothesis, $a_{n+1} \notin \Omega'$ and $\{a_1, \ldots, a_n\} \cap \Omega'' = \emptyset$. Let $A = \{a_{n+1}\} \cup \Omega''$ and let $B = S - A$. If $\boldsymbol{S} \restriction B \cong \boldsymbol{S}$, as $\{a_1, \ldots, a_n\} \subseteq B$, $\boldsymbol{S} \restriction B \not\models \exists x\, \theta''(x, a_1, \ldots, a_n)$, as all the witnesses to $\exists x\, \theta''(x, a_1, \ldots, a_n)$ in $\boldsymbol{S}$ are in A. But $\Omega'' \neq \emptyset$ in B (as $\Omega'' \neq \emptyset$ in $\boldsymbol{S}$ by inductive hypothesis). Contradiction.

Hence, as $\boldsymbol{S}$ satisfies property $(\mathcal{P})$, $\boldsymbol{S} \restriction A \cong \boldsymbol{S}$. It follows by inductive hypothesis that $\Omega'_A = \{c \in A : \boldsymbol{S} \restriction A \models \theta'(c, a_{n+1})\} \neq \emptyset$ so that $\Omega'_A \cap \Omega'' \neq \emptyset$. But then $\Omega' \cap \Omega'' \neq \emptyset$. □

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