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A note on orientations of the infinite random graph

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Abstract

We answer a question of Cameron's by giving examples of $2^{\aleph_0}$ many non-isomorphic acyclic orientations of the infinite random graph with a topological ordering that do not have the pigeonhole property. Our examples also embed each countable linear ordering.

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A graph is *n-existentially closed* or n-e.c. if for each n-subset S of vertices, and each subset T of S (possibly empty), there is a vertex not in S, joined to each vertex of T and no vertex of $S \setminus T$. The infinite random graph, written R, is the unique (up to isomorphism) countable graph that is n-e.c. for all $n \geq 1$. For more on the infinite random graph, the reader is directed to [2, 3].

The infinite random graph is intimately related to a certain vertex partition property. A graph G has the *pigeonhole property*, written $(\mathcal{P})$, if for every partition of the vertices of G into two non-empty parts, the subgraph induced by some one of the parts is isomorphic to G. This property was introduced by Cameron in [2], who in [3] classified the countable graphs with $(\mathcal{P})$; there are only four up to isomorphism: the graph with one vertex, the countably infinite clique and its complement, and R. In particular, R is the unique countable 1-e.c. graph that has $(\mathcal{P})$. The pigeonhole property may be easily generalized to any relational structure. The countable tournaments with $(\mathcal{P})$ were classified in [1];

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there are $\aleph_1$ many: the countable ordinal powers of ω and their reversals, and the countably infinite random tournament.

As proved in [1], a countable oriented graph with $(\mathcal{P})$ that is neither a tournament nor the infinite random oriented graph O, must be an orientation of R. Cameron [4] was the first to notice that any such orientation must be acyclic (that is, contains no directed cycles), have infinitely many sources or infinitely many sinks, and admits a homomorphism into a countable ordinal.

A *topological order* of the vertices of an oriented acyclic graph $D = (V, E)$ is a linear order $\preceq$ on V such that if $(x, y) \in E$, then $x \preceq y$. Cameron [4] posed the following problem.

Problem. Are there $2^{\aleph_0}$ (that is, cardinality of the real numbers) many non-isomorphic acyclic orientations of the infinite random graph with a topological ordering that do not have $(\mathcal{P})$?

We say that an orientation of R as described in the above Problem is *bad*.

The goal of this short note is to answer the above Problem affirmatively. We actually prove a stronger assertion, as stated in the following theorem, which we think is of interest in its own right.

Theorem 1. *There are $2^{\aleph_0}$ many non-isomorphic bad orientations of R that embed each countable linear ordering.*

We consider only countable simple graphs and oriented graphs, which we refer to as orgraphs. Directed edges are written (x, y) and we say that x is *joined to* y and y is *joined from* x. If (x, y) is a directed edge in an orgraph, then we forbid (y, x) to be a directed edge. If G is a graph or orgraph, then $V(G)$ is the set of vertices of G, and $E(G)$ is the set of edges of G if G is a graph, and the set of directed edges of G if G is an orgraph. If $B \subseteq V(G)$, then we write $G \restriction B$ for the subgraph or suborgraph induced by B; if H is an induced subgraph or suborgraph of G, then we write $H \leq G$. We write $G \cong H$ if G and H are isomorphic. We say that G *embeds* in H if G is isomorphic to an induced subgraph or suborgraph of H. If D is an orgraph, then the *graph* of D, written $G(D)$, is the graph with vertices $V(D)$ and with edge set the symmetric closure of $E(D)$. A vertex u is a *source* in D if for every $v \in V(D)$, we have that $(v, u) \notin E(D)$; a *sink* is defined dually.

Proof of Theorem 1. For each integer $i \geq 3$, let L_i be the i-vertex linear order, and let $\Phi = \{L_i : i \geq 3\}$. Fix X, an infinite, co-infinite subset of Φ, and enumerate $X = \{L_{i_1}, L_{i_2}, \ldots\}$ and $\Phi \backslash X = \{L_{j_1}, L_{j_2}, \ldots\}$.

Define an orgraph $D(X)$ inductively, as follows. As we proceed with the induction, each vertex will be assigned exactly one colour, red or blue. Once a vertex has been assigned a colour, it will have that colour throughout the induction. If an induced suborgraph H has each of its vertices red (blue), then we say that H is red (blue). The orgraph $D(X)_1$ consists of two disjoint vertices with no directed edges between them, with one vertex red and the other blue. The suborgraph Blue(1) is this blue vertex. Assume that $D(X)_n$ is defined, finite, and if $n \geq 2$, then $D(X)_{n-1} \leq D(X)_n$. We will also assume that there is at least one blue vertex in $D(X)_n$, and the suborgraph Blue(n) induced by the set of all the blue vertices of $D(X)_n$ is a finite linear order. The orgraph $D(X)_{n+1}$ is defined in four stages.

If M is a linear order, then a vertex is called *special for M* if x is joined to the first and last vertices of M only, and x is not joined from any vertex of M. For each red copy L of any one of $L_{i_1}, L_{i_2}, \ldots, L_{i_n}$ in $D(X)_n$ whose initial vertex is *not a source* in $D(X)_n$, add a new red vertex x_L that is special for L. The resulting orgraph is called $D(X)'_{n+1}$.

For each pair $u, v \in \mathrm{Blue}(n)$ so that $u < v$ in the linear order $\mathrm{Blue}(n)$, if no vertex of $\mathrm{Blue}(n)$ is lying between u and v, then add a new blue vertex b_{uv} so that $u < b_{uv} < v$. Further, b_{uv} is greater than all predecessors of u and less than all successors of v, and $b_{uv} < b_{u'v'}$ if $v \leq u'$. The vertices b_{uv} are not directed to or from any red vertex. Observe that there are at most $\binom{|\mathrm{Blue}\,(n)|}{2}$ new blue vertices of the form b_{uv} added at this stage. After these blue vertices are added, add a new blue vertex a strictly less all the blue vertices, and a new blue vertex z strictly greater than all the blue vertices; the vertices a and z are not joined to or from any red vertex. The orgraph with these additional blue vertices $\mathrm{Blue}(n+1) = \{a, z\} \cup \{b_{uv} : u, v \in \mathrm{Blue}(n), u < v\}$ is called $D(X)'_{n+1,\mathrm{blue}}$.

To $D(X)'_{n+1,\mathrm{blue}}$ add a disjoint red copy of $L_{i_{n+1}}$ with an additional red vertex x_{n+1} that is special for $L_{i_{n+1}}$, and add a disjoint red copy of $L_{j_{n+1}}$ that has an additional red vertex y_{n+1} that is joined to the initial vertex of $L_{j_{n+1}}$ only and is not joined from any vertex of $L_{j_{n+1}}$. The resulting orgraph is called $D(X)''_{n+1}$.

Consider the finite undirected graph $G = G(D(X)''_{n+1})$. For each subset S of $V = V(G)$, and each subset T of S (possibly empty), add a new red vertex which is joined only to vertices of T. Orient the edges so that if a vertex $x \notin V$ is joined to $y \in V$, then (y, x) is a directed edge (in other words, edges are directed *out of* G). Give G back its orientation from $D(X)''_{n+1}$. This new orgraph is called $D(X)_{n+1}$. Observe that $\mathrm{Blue}(n+1) \leq D(X)_{n+1}$ forms a linear order.

Define

$$D(X) = \bigcup_{n \geq 1} D(X)_n.$$

Hence, $D(X)$ is the union of the chain

$$D(X)_1 \leq D(X)_2 \leq D(X)_3 \leq \cdots.$$

It follows that $H = G(D(X)) \cong R$, since H is n-e.c. for all $n \geq 1$. To see this, fix $S \subseteq V(H)$ and T a subset of S. Then $S \subseteq V(D(X)_m)$ for some positive integer m. A vertex not in S joined to vertices of T but not $S \backslash T$ may be found in $V(D(X)_{m+1})$ (by construction of $D(X)_{m+1}$).

Let $\mathrm{Blue}(X)$ be the suborgraph induced by the blue vertices of $D(X)$. The orgraph $\mathrm{Blue}(X)$ is then the union of the chain

$$\mathrm{Blue}(1) \leq \mathrm{Blue}(2) \leq \mathrm{Blue}(3) \leq \ldots$$

and is a linear order. It is not hard to see that $\mathrm{Blue}(X)$ is dense and has no endpoints. Therefore, $\mathrm{Blue}(X)$ is isomorphic to the order type of the rational numbers (see Theorem 2.8 of [5]). From the fact that the order type of the rational numbers embeds all countable linear orders (see Theorem 2.5 of [5]), it follows that $D(X)$ embeds each countable linear order.

We prove that $D(X)$ is acyclic by induction on n. Clearly, $D(X)_1$ is acyclic. Assume that $D(X)_n$ is acyclic. In $D(X)'_{n+1}$, the addition of no vertex x_L creates a directed cycle.

No blue vertex added in $D(X)'_{n+1,\text{blue}}$ is joined to or from a red vertex; further, Blue$(n+1)$ is a linear order and hence, acyclic. Therefore, there is no directed cycle in $D(X)'_{n+1,\text{blue}}$. In $D(X)''_{n+1}$, the addition of neither x_{n+1} nor y_{n+1} creates a directed cycle. Since each z added in $V(D(X)_{n+1})\backslash V(D(X)''_{n+1})$ is a sink in $D(X)_{n+1}$, there are no directed cycles in $D(X)_{n+1}$.

Define the subset A of the vertices of $D(X)$ to be all the "primed vertices"; in particular, A_1 is the unique red vertex of $D(X)_1$ that we name v, and $A_n = V(D(X)'_n)\backslash V(D(X)_{n-1})$ with $A = \bigcup_{n\geq 1} A_n$. The set A is *stable*, that is, there are no directed edges between vertices of A. To see this, we prove the following stronger claim.

Claim 1. *The vertex v, and each vertex x_L added in at stage $D(X)'_n$, for each $n > 1$, are sources in $D(X)$.*

To prove the claim, we prove by induction on s that v is a source in each $D(X)_s$ for $s \geq 1$, and each vertex x_L is a source in $D(X)_s$ for every $s \geq n$. Since the arguments used for the cases of v and x_L are similar, we consider only the argument for x_L.

If $s = n$, there is nothing to prove. Assume that x_L is a source in $D(X)_s$. If there is a vertex joined to x_L in $D(X)'_{s+1}$, then this vertex is of the form $x_{L'}$, where L' is a linear order in $D(X)_s$ whose initial vertex is not a source. But then x_L is a vertex of L' and so must be the initial vertex of L', which gives a contradiction. Clearly, x_L is a source in $D(X)'_{n+1,\text{blue}}$, $D(X)''_{s+1}$, and $D(X)_{s+1}$.

For $n \geq 2$, define B_n to be the red vertices in $V(D(X)_n)\backslash V(D(X)'_n)$, and let $B = \bigcup_{n\geq 2} B_n$. Then A, Blue(X), and B form a partition of $V(D(X))$. We linearly order the vertices of $D(X)$ by first enumerating A with v as the first vertex, then listing the vertices of A_2 in some linear order, and then listing the vertices of A_3 in some order, and so on. Observe that each A_i is finite. Now include Blue(X) in the linear order, respecting the linear ordering of Blue(X). Hence, each vertex of A is less every vertex of Blue(X). Next, list the vertices of B by first listing $B_1, B_2, \ldots$. Within each B_n, list first x_n, and then list vertices of L_{i_n}, respecting the linear order of L_{i_n}. Next list y_n and then the vertices of L_{j_n}, respecting the linear order of L_{j_n}. Finally, list the vertices of $V(D(X)_n)\backslash V(D(X)''_n)$ in any way. Hence, each vertex of $A \cup$ Blue(X) is less than every vertex of B. Name this linear ordering of $V(D(X))$ by $\mathcal{L} = (V(D(X)), \preceq)$.

The linear order $\mathcal{L}$ is topological. To see this, note that this holds for vertices in A, since there are no directed edges between vertices of A. Since Blue(X) is itself a linear order, the ordering there is topological. In B, the topological property is satisfied by our choice of ordering of the vertices of B. As the vertices of A are all sources in $D(X)$, and since A forms an initial segment of $\mathcal{L}$, we need only check the topological property for directed edges (c, d), where either $c \in$ Blue(X) and $d \in B$ or $c \in B$ and $d \in$ Blue(X). However, since there are no directed edges (c, d) with c red and d blue, this follows immediately.

The linear order $\mathcal{L}$ has order type $\omega + \eta + \omega$, where ω is the order type of the natural numbers, and η is the order type of the rational numbers. We now prove the following claim.

Claim 2. *The orgraph $D(X)$ does not have $(\mathcal{P})$.*

Let $L \in X$ be fixed, and let L' be the copy of L added in disjointly to $D(X)''_m$ for some m. Recall that in $D(X)''_m$, the vertex x_m is special for L'. Let the initial vertex of L' be named a'. Let $C = \{x \in B \backslash \{x_m\} : x \preceq a'\}$. Observe that C is finite, since B has order type ω in $\mathcal{L}$. Furthermore, all the vertices of $D(X)$ different from x_m which are joined to a' are in $A \cup C$.

Let $A' = A \cup C \cup \text{Blue}(X)$, and let $B' = V(D(X)) \backslash A'$. The orgraph A' cannot be isomorphic to $D(X)$. Otherwise, $G = G(D(X) \restriction (A'))$ would be isomorphic to R. Since A is a stable set and $\text{Blue}(X)$ is a linear order, G consists of the union of an infinite empty graph (the subgraph induced by the vertices of A), an infinite complete graph (the subgraph induced by the vertices of $\text{Blue}(X)$) and a finite graph (the subgraph induced by the vertices of C). But then G does not have property $(\mathcal{P})$, which is a contradiction.

We show that $B'' = D(X) \restriction B'$ cannot be isomorphic to $D(X)$ using the *back-and-forth* game or method, which in our case is a two player game of perfect information played in countably many steps on two countable orgraphs D_0 and D_1. The players are named the *duplicator* and the *spoiler*. (The names come from the facts that the duplicator is trying to show the structures are alike, while the spoiler is trying to show they are different.) A move consists of a choice of a vertex from either structure, and the spoiler makes the first move. The players take turns choosing vertices from the $V(D_i)$, so that if one player chooses a vertex from $V(D_i)$, the other must choose a vertex of $V(D_{i+1})$ (the indices are mod 2). Players cannot choose previously chosen vertices. After n rounds, this gives rise to a list of vertices $U_n = \{a_i : 1 \leq i \leq n\}$ from D_0 and $V_n = \{b_i : 1 \leq i \leq n\}$ from D_1. The duplicator wins if for every $n \geq 1$, the suborgraph induced by U_n is isomorphic to the suborgraph induced by V_n. Otherwise, the spoiler wins. From this it follows that the duplicator has a winning strategy if and only if D_0 and D_1 are isomorphic. See [2] for more on the back-and-forth method.

Now the spoiler chooses in B'' the vertex x_m and the vertices of L' in succession. The duplicator must respond with $|V(L')| + 1$ corresponding vertices in $D(X)$ that give rise to a linear order L'' and a vertex x''_m which is joined to the first vertex of L'', which we name α. Since (x''_m, α) is a directed edge in $D(X)$, α is not a source and so there is a $z \in V(D(X)) \backslash \{x''_m\}$ that is joined to α. The spoiler can win in the next round by choosing z. To see this, note that the duplicator cannot now choose an appropriate vertex of B'', since the spoiler already has chosen all the vertices of B'' which are joined to α. Claim 2 follows.

It is not hard to show that there are $2^{\aleph_0}$ many distinct infinite, co-infinite subsets of the natural numbers. To finish the proof of the theorem, we use this fact in conjunction with the following claim.

Claim 3. *If $X \neq Y$, then $D(X) \ncong D(Y)$.*

Without loss of generality, there is an $L_i \in X \backslash Y$; name the first vertex of L_i a and the last vertex z. In $D(X)$, for *every* copy of L_i so that a is not a source, there is a vertex that is joined to *both a and z*. This is clear for the red vertices by construction, and for the blue vertices by the fact that $\text{Blue}(X)$ is a dense linear order without endpoints.

We show that this property fails for $D(Y)$, which will prove Claim 3. Consider a fixed copy of L_i added at some stage $D(Y)''_m$. Since (y_m, a) is a directed edge, the vertex a is not a source. However, there is no vertex $x' \in V(D(Y))$ which is joined to both a and z, although there is in $D(X)$. If there were such an x' in $D(Y)$, then x' must be a vertex added in at some stage $D(Y)'_{r+1}$. But the vertices of $D(Y)'_{r+1}$ not among the vertices of $D(Y)_r$ are special only for linear orders isomorphic to those in Y. Hence, if (x', a) is a directed edge, then (x', z) is *not* a directed edge, otherwise, $L_i \in Y$. (We are tacitly using here the fact that there is, up to isomorphism, exactly one linear order on n vertices, if n is a positive integer.) □

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