The n-Ordered Graphs: A New Graph Class

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Abstract: For a positive integer n, we introduce the new graph class of n-ordered graphs, which generalize partial n-trees. Several characterizations are given for the finite n-ordered graphs, including one via a combinatorial game. We introduce new countably infinite graphs $R^{(n)}$, which we name the infinite random n-ordered graphs. The graphs $R^{(n)}$ play a crucial role in the theory of n-ordered graphs, and are inspired by recent research on the web graph and the infinite random graph. We characterize $R^{(n)}$ as a limit of a random process, and via an adjacency property and a certain folding operation. We prove that the induced subgraphs of $R^{(n)}$ are exactly the countable n-ordered graphs. We show that all countable groups embed in the automorphism group of $R^{(n)}$. J Graph Theory 60: 204–218, 2009

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1. INTRODUCTION

As is well known, a tree may be defined from K_1 by recursively adding vertices of degree one. A more general notion is an *n-tree*, $n \geq 1$, which is a graph recursively built from K_n by adding a new vertex of degree n joined to an existing n-clique. A *partial n-tree* is a spanning subgraph of an n-tree (see [4] for more on partial n-trees).

We say that a countable graph G is called *n-ordered* if there exists a well-ordering $(x_i : i \in I)$ of its vertices, where I is finite, or I has the order-type of $\mathbb{N}$ so that each x_j has at most n neighbors x_i, with $i < j$. (We may consider other countable order-types, but the restriction above is sufficient for our purposes.) In other words, a vertex is joined to at most n vertices appearing earlier in the ordering. The ordering $(x_i : i \in I)$ is an *n-ordering* of $V(G)$.

Each finite planar graph is 5-ordered, although K_5 is 5-ordered and not planar. Every partial n-tree is n-ordered, but the converse is false in general for $n \geq 3$. For example, the graph G in Figure 1 is 3-ordered, but is not a partial 3-tree. If G is n-ordered, then by the greedy algorithm $\omega(G)$, $\chi(G) \leq n + 1$.

Consider the following countably infinite graph, which is defined as a limit of a certain chain of finite graphs. Let $R_0 \cong K_n$. For some $t \geq 0$, assume that R_t is a finite graph containing R_0. For each subset S of cardinality n in $V(R_t)$, add a new vertex x_S joined only to the vertices of S. We say that x_S *extends* S. The graph R_t along with the new vertices x_S defines the graph R_{t+1}. Let $R^{(n)}$ be the graph with vertices $\bigcup_{t\in\mathbb{N}} V(R_t)$ and edges $\bigcup_{t\in\mathbb{N}} E(R_t)$. We will write $R^{(n)} = \lim_{t\to\infty} R_t$ for the graph formed as the limit of this chain of vertices and edges. We will call $R^{(n)}$ the *infinite random n-ordered graph*, for reasons that will become apparent as we proceed (see Corollary 9 and Theorem 10). This construction is reminiscent of one construction of the so-called *infinite random* or *Rado* graph, written R. For R, at time-step $t + 1$, for *all* subsets S of $V(R_t)$ (not just those of cardinality at most n) add a new vertex z_S joined only to S. Hence, R results by adding vertices joined in all possible ways to existing vertices.

The graph R is a well-studied example of a countably infinite limit graph; see the surveys [5,6] on R for additional background and references. Many new infinite limits have been recently discovered in relation to models for the *web graph*, whose vertices correspond to web pages, and edges represent links between pages. Several stochastic models for the web graph have been introduced (see [2]), and most are *on-line*, in the sense that new vertices appear over time. Hence, it is a natural

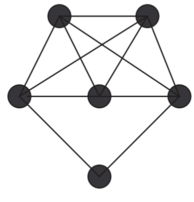

FIGURE 1. A 3-ordered graph that is not a partial 3-tree.

question to ask about the properties of limits of graphs generated by these models. For further reading on various types of infinite limit graphs corresponding to real-world networks, the reader is directed to [3,9].

The graph R satisfies the *n-existentially closed* or *n-e.c.* adjacency properties for all positive integers n. A graph is n-e.c. if for all disjoint finite sets of vertices A, B with $|A \cup B| = n$, there is a vertex $z \notin A \cup B$ joined to each vertex of A and to no vertex of B. We say that z is *correctly joined* (or *c.j.*) *to* A, B. It is easy to see that $R^{(n)}$ is n-e.c. A graph that is n-e.c. for all n we say is *e.c.* By a back-and-forth argument, a countably infinite graph is e.c. if and only if it is isomorphic to R. The graph R arises naturally via the following infinite random process which inspires its name. We add new vertices over countably many discrete time-steps. Fix $p \in (0, 1)$. At time $t = 0$ start with any fixed finite graph. At time-step $t + 1$, add in a new vertex x_{t+1}. For each of the existing vertices y, add the edge yx_{t+1} independently with probability p. Erdős and Rényi proved in [7] that with probability 1, a limit generated by this random process is e.c. and hence, isomorphic to R. This instance of a random process with a seemingly deterministic conclusion makes R the centre of much research activity.

The goals of the present article are to present results on the new graph class of n-ordered graphs and on $R^{(n)}$. Our emphasis is on forging connections between properties of the infinite graph $R^{(n)}$ with the class of finite n-ordered graphs. Several characterizations of n-ordered graphs are given in Theorem 2. For example, we show that n-ordered graphs may be characterized by a certain two player combinatorial game. The induced subgraphs of $R^{(n)}$ are precisely the countable n-ordered graphs; see Theorem 10. We characterize the isomorphism type of $R^{(n)}$ in Theorem 7 as the unique n-ordered graph which satisfies the strongly n-e.c. adjacency property, and show that every graph with this adjacency property embeds $R^{(n)}$. Hence, the graph $R^{(n)}$ plays an intriguing role in the theory of n-ordered graphs: it is at once the maximal n-ordered graph (with respect to the induced subgraph relation), and the minimal graph satisfying the strongly n-e.c. adjacency property. We describe how $R^{(n)}$ arises naturally as the limit of a random process in Theorem 8. Using our characterizations of $R^{(n)}$, we study the automorphism group of $R^{(n)}$ and prove that it contains as subgroups isomorphic copies of all countable groups.

All graphs we consider are simple, undirected, and countable (that is, finite or countably infinite). We write $G \leq H$ if G is isomorphic to an induced subgraph of H, and $G \restriction S$ for the subgraph induced by $S \subseteq V(G)$. We write $G \cong H$ if G and H are isomorphic. The complement of G is written $\bar{G}$. The set of natural numbers, including 0, is denoted by $\mathbb{N}$. We write $\aleph_0$ for the cardinality of $\mathbb{N}$.

2. THE CLASS OF *n*-ORDERED GRAPHS

Throughout the rest of the section, we will assume that G is finite, unless otherwise stated. Let $\delta(G)$ and $\triangle(G)$ be the minimum and maximum degrees of G, respectively. Given a graph G, a *simple n-reduction* consists of deleting a single vertex of degree at most n. An *n-reduction* consists of a sequence of simple n-reductions. A graph G is an *n-core* if no n-reductions are possible in G. An *n-core of G* is an induced subgraph H such that H is obtained from G by an n-reduction and H is an n-core. If an n-core H of G is nontrivial, then $\delta(H) \geq n + 1$. For more on n-cores, see [1,11].

Each finite graph G is $| \triangle (G)|$-ordered. Hence, the parameter

$$\Theta(G) = \min\{n : G \text{ is } n\text{-ordered}\}$$

is well defined. We say that $\Theta(G)$ is the *orderability* of G. The $\Theta(G)$-core of G is always K_1, and the location of this K_1 in $V(G)$ need not be unique. Further, $\delta(G) \leq \Theta(G) \leq \triangle(G)$.

The following lemma is a part of folklore, and has a straightforward proof. Note that it holds for both finite and infinite graphs.

Lemma 1. *Let G be a countable graph. For all non-negative integers n, the n-core of G is unique up to isomorphism.*

The n-core of G and the orderability of G may be computed in polynomial time. The algorithm for computing the n-core is simple (and well known): iteratively delete vertices of degree at most n (in any order). The algorithm for computing $\Theta(G)$ is also straightforward: find the $r_0 = \delta(G)$-core (where $\delta(G)$ is the minimum degree of G) of G; call this G_1. Then find the $r_1 = \delta(G_1)$-core of G_1; iterate this process until K_1 is obtained. If the algorithm terminates in say k steps, then it is easy to see that $\Theta(G) = \max\{r_{0,} r_1, \ldots, r_{k-1}\}$.

In the following, we introduce a new combinatorial game played on a graph G called *n-deletion*. The game n-deletion is inspired by the well-known game of Cops and Robber; see [10]. There are two players: a *deleter* and *mover*. They move on alternate time-steps, with the deleter beginning the first round of play. The deleter's move consists of a simple n-reduction on G. The mover starts at any vertex of G, and a move for him consists of moving to an adjacent vertex. The mover can never remain on a vertex for more than one round or the deleter wins. In particular, if the mover's position is on an isolated vertex, then the mover loses. If the mover can move indefinitely, even just back-and-forth on an edge, then the deleter loses. A winning strategy for the mover is defined in the usual way (there is no strategy possible for the deleter).

Theorem 2. *Let G be a graph, and let n* ≥ 1 *be fixed. The following are equivalent.*

(1) *The graph G is n-ordered.*
(2) *The n-core of G is* K_1.
(3) *The deleter has a winning strategy for n-deletion played on G.*
(4) *There is an acyclic orientation of G so that each vertex has out-degree at most n.*

Proof. It is straightforward to see that items (1) and (2) are equivalent. To see that (1) implies (4), let $(x_i : 1 \leq i \leq r)$ be an n-ordering of G. Orient the edges so that (x_j, x_i) is directed edge whenever $i < j$ and $x_i x_j \in E(G)$. Then each vertex has at most n-out-neighbors. Since vertices may only point to vertices with smaller index, there are no directed cycles. For (4) implies (1), embed the given acyclic orientation of G into a linear order. The latter ordering is an n-ordering.

We now prove that (1) and (3) are equivalent. Suppose first that G is n-ordered. The deleter deletes vertices of degree at most n until the resulting graph is K_1. This is possible since G is n-ordered. Either the mover occupies an isolated vertex after one of the deleter's moves, or the mover resides on the K_1 after the deleter's last move. In either case, the deleter wins. Hence, (1) implies (3).

For (3) implies (1), suppose the graph G is not n-ordered. In particular, since (1) and (2) are equivalent, the n-core H of G has more than one vertex. The mover's strategy is to always stay in H. No matter what move the deleter makes, the mover is safe: since vertices of the n-core H have degree at least $n + 1$, they are never deleted in any move of the deleter. As $n + 1 \geq 2$, the mover may always move in H. ■

A class of $\mathcal{C}$ graphs is *hereditary* if $G \leq H \in \mathcal{C}$ implies that $G \in \mathcal{C}$.

Corollary 3. *The class of n-ordered graphs is hereditary, and closed under the taking of countable disjoint unions.*

Proof. The proof follows by the equivalence of items (1) and (4) in Theorem 2. ■

The *Cartesian product* $G \square H$ of graphs G and H has vertices $V(G) \times V(H)$ and edges $(a, b)\ (c, d)$ if and only if $ac \in E(G)$ and $b = d$ or $bd \in E(H)$ and $a = c$. Orderability is additive with respect to the Cartesian product, as our next result demonstrates.

Theorem 4. *For graphs G and H,* $\Theta(G \square H) = \Theta(G) + \Theta(H)$.

Proof. Let $m = \Theta(G)$ and $n = \Theta(H)$. If m or n equal 0, then the result follows. For example, if $m = 0$, then G is independent. Then $G \square H$ is isomorphic to $|V(G)|$ many disjoint copies of H, and so $\Theta(G \square H) = \Theta(H)$.

Suppose that $m, n > 0$.We first give an $(m + n)$-ordering of $G \square H$ which will supply that $\Theta(G \square H) \leq \Theta(G) + \Theta(H)$. Let $\{x_1, \ldots, x_r\}$ be an m-ordering of G, and let $\{y_1, \ldots, y_s\}$ be an n-ordering of G. Order the pairs (x, y) of $G \square H$ lexicographically; that is, $(x, y) < (x', y')$ if and only if $x < x'$, or $x = x'$ and $y < y'$. It is straightforward to see that this is an $(m + n)$-ordering of $G \square H$.

For the lower bound on $\Theta(G \square H)$, we consider the $(m + n - 1)$-core of $G \square H$. The $(m - 1)$-core A of G is non-trivial, and the $(n - 1)$-core B of H is non-trivial. Then $A \square B$ is an induced subgraph of $G \square H$ with minimal degree $(m - 1) + 1 + (n - 1) + 1 = m + n > 0$. Hence, the $(m + n - 1)$-core of $G \square H$ is non-trivial (as it contains $A \square B$), and so by Theorem 2, $\Theta(G \square H) \geq \Theta(G) + \Theta(H)$. ■

It would be interesting to consider variants of the class of n-ordered graphs where at each time-step, different types of subsets are extended. A natural example (which results in graphs with chromatic number at most $n + 1$) extends all subsets with chromatic number at most n. We leave the consideration of this and analogous classes as an open-ended problem.

3. PROPERTIES OF $R^{(n)}$ AND RANDOM PROCESS

We consider various isomorphic ways (both deterministic and random) of representing $R^{(n)}$. We begin by supplying some structural information on the graph $R^{(n)}$ itself. A graph G is *strongly n-e.c.* if for all n-subsets A of vertices, and finite subsets B, there is a vertex $z \notin A \cup B$ correctly joined to A, B. Note that if G is strongly n-e.c., then G is infinite and strongly m-e.c. for all positive $m < n$ (vertices not in A or B can be added to A to form a set of cardinality n).

Let $J = \lim_{t\to\infty} H_t$ be a limit of a countable sequence $\mathcal{C} = (H_t : t \in \mathbb{N})$ of graphs, where $H_t \le H_{t+1}$ for all $t \in \mathbb{N}$. Define $\mathrm{age}_{\mathcal{C}} : V(J) \to \mathbb{N}$ by

$$\mathrm{age}_{\mathcal{C}}(x) = \begin{cases} t & \text{if } x \in V(H_t)\backslash V(H_{t-1}) \text{ where } t > 0; \\ 0 & \text{else.} \end{cases}$$

We will simply write age(x) if $\mathcal{C}$ is clear from context. The age of a finite subset, written age(S), is $\max\{\mathrm{age}(x) : x \in S\}$.

Theorem 5. *Fix a positive integer n.*

(1) *The graph $R^{(n)}$ is strongly n-e.c., but not $(n + 1)$-e.c.*
(2) *If G is a strongly n-e.c. graph, then $R^{(n)}$ is an induced subgraph of G.*
(3) $R^{(n)} \le R^{(n+1)}$.
(4) $\lim_{n\to\infty} R^{(n)} \cong R$.

Proof. For (1), fix A an n-subset and B a finite set in $V(R^{(n)})$, with A and B disjoint. Let $t_0 = \mathrm{age}(A \cup B)$. Then $z_A \in R_{t_0+1}$ is correctly joined to A and B.

Let $S_1, S_2, \ldots, S_{n+1} \subseteq V(R_t)$ be $n + 1$ disjoint n-subsets, and, for each S_i, $1 \le i \le n + 1$, let z_{S_i} be the new vertex in R_{t+1} that extends S_i. Take $S = \{z_{S_i} : 1 \le i \le n + 1\}$. No vertex of R_{t+1} is joined to all of S. Assume that no vertex of R_j is joined to all of S, where $j \ge t + 1$ is fixed. No vertex of R_{j+1} is joined to all of S. Hence, $R^{(n)}$ is not $(n + 1)$-e.c.

For item (2), we proceed by induction on t to embed R_t in G. For the inductive step, let R'_t be the copy of R_t in G. For $k \le n$, by the strongly n-e.c. property for G, for each k-subset S of $V(R'_t)$, there is a vertex z_S of $V(G)\backslash V(R'_t)$ joined to S and to no vertex of $V(R'_t)\backslash S$. Add each of these vertices z_S to $V(R'_t)$ successively, in such a way that they are all pairwise non-joined, and let R'_{t+1} be the resulting subgraph. Then clearly, $R'_{t+1} \cong R_{t+1}$ and $R'_t \le R'_{t+1}$. Hence, the chain $(R'_t : t \in \mathbb{N})$ has a limit isomorphic to $R^{(n)}$, which is in turn isomorphic to an induced subgraph of G.

Item (3) follows from (2), since $R^{(n+1)}$ is strongly n-e.c. For item (4), first note that the limit is defined with respect to the chain $(R^{(n)} : n \in \mathbb{N}\backslash\{0\})$, where $R^{(n)}$ is an induced subgraph of $R^{(n+1)}$ (as in (3)). Let $G = \lim_{n\to\infty} R^{(n)}$. It is sufficient to prove that G is e.c. For this, let A, B be disjoint finite subsets of G. Suppose that $m_1 = |A \cup B|$. Further, suppose that m_2 is such that $A \cup B \subseteq V(R^{(m_2)})$. Let $m = m_1 + m_2$. Then $A \cup B \subseteq V(R^{(m)})$, and $R^{(m)}$ is m-e.c. Therefore, there is a vertex correctly joined to A, B in $R^{(m)}$, hence in G. ■

Let G and H be countable graphs such that $H \leq G$ and $|V(H)| \geq n$. We say that $H \preccurlyeq_{(n,v)} G$ if there is a vertex $v \in V(G)$ such that $H = G - v$ and $\deg_G(v) = n$. We write $H \preccurlyeq_{(n)} G$ if there is countable chain $(G_t : t \in I)$ so that $G_0 \cong H$ and $G_t \preccurlyeq_{(n,v)} G_{t+1}$ for some $v \in G_{t+1}$, and if I is finite, then G equals the G_t with maximum index; if I is infinite, then $G = \lim_{t\to\infty} G_t$. For example, $K_n \preccurlyeq_{(n)} R^{(n)}$ for all positive integers n.

For another illustrative example, consider the unique isomorphism type T_∞ of a countable tree such that each vertex has infinite degree. It is straightforward to check that $K_1 \preccurlyeq_{(1)} T_\infty$, T_∞ is strongly 1-e.c., and $R^{(1)}$ is isomorphic to T_∞.

It is clear that $\preccurlyeq_{(n)}$ is an order relation when restricted to finite graphs. Our next theorem demonstrates that it is an order relation also on countable graphs.

Theorem 6. *The relation $\preccurlyeq_{(n)}$ is transitive on countable graphs.*

Proof. Let G, H, and J be countable graphs such that $G \preccurlyeq_{(n)} H \preccurlyeq_{(n)} J$. We consider the only nontrivial case where $V(H)\backslash V(G)$ and $V(J)\backslash V(H)$ are both countably infinite.

Suppose that $H = \lim_{n\to\infty} G_t$, where $G_0 \cong G$, and $G_t \preccurlyeq_{(v_{t+1},n)} G_{t+1}$ for all $t \in N$. That is, for $t \geq 1$, v_t is the unique vertex in $V(G_t)\backslash V(G_{t-1})$, and v_t has degree exactly n in G_t. Similarly, suppose that $J = \lim_{n\to\infty} H_t$, where $H_0 \cong H$, and $H_{t-1} \preccurlyeq_{(u_t,n)} H_t$ for all $t \in \mathbb{N}\backslash\{0\}$. Thus, for $t \geq 1$, u_t is the unique vertex in $V(H_t)\backslash V(H_{t-1})$. Note that the two chains $(v_t : t \geq 1)$ and $(u_t : t \geq 1)$ enumerate all vertices in $V(J)\backslash V(G)$.

Define the binary relation $\ll$ on $V(J)\backslash V(G)$ as follows. We have that $v_i \ll v_j$ if and only if $i \leq j$, $u_i \ll u_j$ if and only if $i \leq j$, and $v_i \ll u_j$ if and only if $u_i v_j \in E(J)$. Let $<$ be a linear extension of the transitive closure of the relation $\ll$, and let $(w_t : t \geq 1)$ be a strictly increasing chain listing all the elements of $V(J)\backslash V(G)$ according to this linear order.

We claim that $G \preccurlyeq_{(n)} J$ via the chain $(w_t : t \geq 1)$. Namely, let $J_0 = G$, and for $t \geq 1$, let J_t be the subgraph of J induced on $\{w_1, \ldots, w_t\}$. We will show that $J_t \preccurlyeq_{(w_{t+1},n)} J_{t+1}$ for all $t \geq 0$. For this, it is enough to show that for each $t \geq 1$, w_t has exactly n neighbors in J_t.

Fix $t \geq 1$. We consider two cases. First, suppose $w_t = v_i$ for some $i \geq 1$. By construction, we know that

$$|N_J(v_i) \cap (V(G) \cup \{v_1, v_2, \ldots, v_{i-1}\})| = n. \tag{3.1}$$

It follows from the ordering of the w_t that

$$\{w_{i'} : 1 \leq i' \leq t-1\} \cap \{v_{i'} : i' \geq 1\} = \{v_1, \ldots, v_{i-1}\}. \tag{3.2}$$

Hence, (3.1) and (3.2) imply that w_t has exactly n neighbors in $V(G_{t-1})\backslash\{u_j : j \geq 1\}$. Suppose that w_t is joined to a vertex $u_j \in V(J_{t-1})$. Then $u_j = w_{t'}$ for some $t' < t$, and $u_j v_i \in E(J)$. Therefore,

$$w_t = v_i \ll u_j = w_{t'},$$

which contradicts our assumption about the ordering of the w_t.

For the second case, suppose that $w_t = u_j$. In this case

$$|N_J(u_j) \cap (V(H) \cup \{u_1, \ldots, u_{j-1}\})| = n. \tag{3.3}$$

Also,

$$\{w_{i'} : 1 \leq i' \leq t-1\} \cap \{u_{i'} : i' \geq 1\} = \{u_1, \ldots, u_{j-1}\}. \tag{3.4}$$

Since $V(H)$ includes all the $\{v_i : i \geq 1\}$, (3.3) and (3.4) implies that w_t has exactly n neighbors in $V(J_{t-1}) \cup \{v_i : i \geq 1\}$. Suppose that w_t is joined to a vertex v_i so that $v_i \notin V(J_{t-1})$, so $v_i = w_{t'}$ with $t' > t$. Then

$$w_{t'} = v_i \ll u_j = w_t,$$

which contradicts our assumption about the ordering of the w_t. ■

Our next result uses the strongly n-e.c. properties and the relations $\preccurlyeq_{(n)}$ to characterize the isomorphism type of $R^{(n)}$.

Theorem 7. *Let n be a fixed positive integer and let G be a countable graph. Then $G \cong R^{(n)}$ if and only if G is strongly n-e.c. and $K_n \preccurlyeq_{(n)} G$.*

Proof. As the forward direction is immediate, we prove the reverse direction. Suppose, without loss of generality, that $G = \lim_{n\to\infty} G_t$, where $G_0 \cong K_n$, and $G_t \preccurlyeq_{(n)} G_{t+1}$ for all $t \in N$. Enumerate $V(G)\backslash V(G_0)$ as $\{v_t : t \in \mathbb{N}\backslash\{0\}\}$ so that v_t is the unique vertex of G_t not in G_{t-1}.

Let $f_0 : G_0 \to R_0$ be any fixed isomorphism. As the induction hypothesis, suppose that for a fixed $t \geq 0$, there is a finite induced subgraph J_t of G containing G_t along with an isomorphism $f_t : J_t \to R_t$ extending f_0.

The graph R_{t+1} is formed from R_t by extending all n-sets of vertices. If we can find analogous vertices in G for all n-subsets of J_t, including the vertex v_{t+1}, then this will define J_{t+1}.

The vertex v_{t+1} extends some fixed n-set S of G_t. List the n-sets of R_t as $(X_i : 1 \leq i \leq r)$, with $X_1 = f_t(S)$. By induction, we may find vertices a_i extending the n-sets

$f_t^{-1}(X_i)$, so that none of the a_i are pairwise joined. More explicitly, let $a_1 = v_{t+1}$. Assuming $a_1, \ldots, a_i$ have been chosen such that $age(a_j) < age(a_{j+1})$ for all $1 \leq j \leq i$, choose a_{i+1} to be the first v_k extending $f_t^{-1}(X_{i+1})$ so that $age(v_k) > age(a_i)$. Note that v_k is not joined to any v_j, where $j < k$, unless $v_j \in f_t^{-1}(X_{i+1})$. The vertex v_k exists by the strongly n-e.c. property for G. Define J_{t+1} to be the subgraph induced by $V(G_t) \cup \{a_i : 1 \leq i \leq r\}$. It is straightforward to see that $f_{t+1} : J_{t+1} \to R_{t+1}$ is an isomorphism.

Define $f : G \to R^{(n)}$ by $f = \bigcup_{t \in N} f_t$. Then f is an isomorphism by construction. ■

We now introduce a new random graph process which we name *Model n* which with high probability will generate $R^{(n)}$. The model has some similarities to the Erdős–Rényi model for R, but there are important differences. In Model n new vertices are added so that older vertices have a higher probability of acquiring new neighbors than their younger counterparts.

For Model n, the single parameter of the model is $n \in \mathbb{N}\backslash\{0\}$. Start with $G_0 \cong K_n$, with vertices labeled $v_1, \ldots v_n$. For $t \geq 0$ fixed, assume that G_{t-1} has been defined and there are finitely many vertices in G_t. At time t, add a new vertex v_{n+t}, and choose a set S of n distinct vertices from $V(G_{t-1})$, where the probability that a vertex v_i is included in the set is exponentially proportional to its age. More precisely, for each $S = \{v_{i_1}, \ldots, v_{i_n}\}$, define $\mu(S) = 2^{-(i_1+\cdots+i_n)}$. Define

$$C_t = \sum_{1 \leq j_1 < j_2 < \cdots < j_n \leq t+n-1} 2^{-(j_1+j_2+\cdots+j_n)}.$$

In particular, C_t is the sum of all the $\mu(S)$, where S a subset of cardinality n from $V(G_{t-1})$. The probability that S is chosen from $V(G_{t-1})$ equals $\mu(S)/C_t$. If S is chosen, then join v_{n+t} to each vertex of S. The probability of an event A in a probability space is written $\mathbb{P}(A)$.

Theorem 8. *Let $G = \lim_{t\to\infty} G_t$, where G_t was generated by Model n. With probability 1, G is strongly n-e.c.*

Proof. Fix disjoint finite subsets of $V(G)$, A and B, so that $|A| = n$. We prove that the probability that there is no vertex correctly joined to A and B in G is 0. As there are only countably many choices for A and B and a countable union of measure 0 sets is measure 0, the proof will follow. Let $A = \{v_{i_1}, \ldots, v_{i_n}\}$, so that $age(v_{i_j}) \leq age(v_{i_{j+1}})$, for all $1 \leq j \leq n$. Let $t_0 = age(A \cup B)$. For each $t \geq t_0 + 1$, let V_t be the event that v_{n+t} is joined to all vertices in A and to no vertex in $V(G_{n+t-1})$. Then

$$\mathbb{P}(V_t) = 2^{-(i_1+\cdots+i_n)}/C_t.$$

Note that $C_t \leq \left(\sum_{j=1}^{t+n-1} 2^{-j}\right)^n \leq 1$ for all t. Therefore, if we let t' be the minimum age of a vertex in $A \cup B$, then

$$\mathbb{P}(V_t) \geq 2^{-(i_1+\cdots+i_n)} \geq 2^{-nt'} \tag{3.5}$$

for all $t \geq t_0$.

Therefore, the probability that there exists no vertex in G that is c.j. to A and B is at most

$$\mathbb{P}\left(\bigcap_{t=t_0}^{\infty} \bar{V}_t\right) = \prod_{t=t_0}^{\infty}(1 - \mathbb{P}(V_t)) \leq \lim_{t\to\infty}(1 - 2^{-nt'})^t = 0.$$

■

In Model n, exactly n edges are added at each time-step from the new vertex to existing vertices. Hence, by Theorems 7 and 8 we have the following.

Corollary 9. *With probability* 1*, a limit generated by Model n is isomorphic to* $R^{(n)}$.

Corollary 9 gives some insight into the finite graphs generated by Model n. For example, as we will prove in Theorem 10, each finite partial (in fact, countable) n-tree embeds in $R^{(n)}$. Hence, with probability 1 as $t \to \infty$, each finite partial n-tree embeds in the graph G_t generated by Model n.

4. THE n-ORDERED GRAPHS AND $R^{(n)}$

The n-ordered graphs play an important role in the structure of $R^{(n)}$. Our first result characterizes the class of isomorphism types of countable induced subgraphs of $R^{(n)}$ (sometimes referred to as the *age* of the graph).

Theorem 10. *A countable graph G is an induced subgraph of* $R^{(n)}$ *if and only if G is n-ordered.*

Proof. For the forward direction, suppose that $G \leq R^{(n)}$. List the vertices of G according to their age in $R^{(n)}$ from youngest to oldest: $V(G) = (x_i : i \in I)$, where I is finite or $I = \mathbb{N}$. It is not hard to see that this gives an n-ordering of G.

For the reverse direction, let $(x_i : i \in I)$ be a fixed n-ordering of G, so that $G = \lim_{t\to\infty} G_t$, where G_t is the graph induced on $\{x_i : i \leq t\}$. We embed G_t into $R^{(n)}$ inductively. Let G_0' be the graph induced on a fixed vertex x of $R^{(n)}$. For $t \geq 0$, assume that $G_t \cong G_t' \leq R^{(n)}$, and that G_t' contains x. Let $age(G_t') = s$. The vertex x_{t+1} is joined to at most n vertices S in G_t. Let $T = V(G_t)\backslash S$, and let S' and T' be the corresponding subsets of $V(G_t')$. Let X' be a set in $R^{(n)}$ with cardinality $n - |S'|$ that is disjoint from S' and T'. Suppose that $age(X') = t'$; without loss of generality, $t' > t$.

Then $z_{S' \cup X'} \in R_{t'+1}$ is joined to S' in R_t and no vertex of T'. Hence,

$$G'_{t+1} = R_{t'+1} \upharpoonright \left(V(G'_t) \cup \{z_{S'}\}\right) \cong G_{t+1},$$

and $G'_t \leq G'_{t+1}$.

It follows that

$$G = \lim_{t \to \infty} G_t \cong \lim_{t \to \infty} G'_t \leq R^{(n)}.$$

■

Corollary 11. *The graph $R^{(n)}$ embeds all countable n-ordered graphs.*

Hence, the graph $R^{(n)}$ is a countable *universal* n-ordered graph (that is, embeds all countable n-ordered graphs), analogous to the infinite random graph, which is a countable universal graph. Furthermore, every strongly n-e.c. graph embeds $R^{(n)}$ by Theorem 5 (2). The graph $R^{(n)}$ is then both a maximal n-ordered graph and a minimal strongly n-e.c. graph (with respect to the embedding order on countable graphs).

We may vary the definition of $R^{(n)}$ somewhat and obtain an isomorphic graph. For example, for a countable graph G, define $R^{(n,G)}$ by the same limit process used to define $R^{(n)}$ but with the initial graph $R_0 \cong G$. A graph *generates* $R^{(n)}$ if $R^{(n,G)} \cong R^{(n)}$. We do not know exactly which graphs generate $R^{(n)}$. We partially characterize graphs which do and do not generate $R^{(n)}$ in the following corollary.

Corollary 12. *Let G be a countable graph, and let n be a positive integer.*

(1) *If $K_n \preccurlyeq_{(n)} G$, then $R^{(n,G)} \cong R^{(n)}$.*
(2) *If either*
   (a) *G has a nontrivial n-core, or*
   (b) *$|V(G)| = n$ and $|E(G)| < \binom{n}{2}$,*
   then $R^{(n,G)} \ncong R^{(n)}$.

Proof. For item (1), since $K_n \preccurlyeq_{(n)} G$ and $G \preccurlyeq_{(n)} R^{(n,G)}$, we have that $K_n \preccurlyeq_{(n)} R^{(n,G)}$ by Theorem 6. It is straightforward to see that $R^{(n,G)}$ is strongly n-e.c. Hence, the proof follows from Theorem 7.

For item (2a), assume G has a nontrivial n-core C. Without loss of generality, assume that $G = C$. Let $H = R^{(n)}$, and $J = R^{(n,G)}$, and assume for a contradiction that f is an isomorphism from H to J. Let X be the vertices of H that correspond to the initial copy of K_n with age 0, and let Y be the vertices of J that correspond to the initial copy of G of age 0.

Construct a set $S \subseteq V(H)$ as follows. Let S_0 be the set with elements $f^{-1}(Y)$. For $i > 0$, form S_{i+1} from S_i as follows: for each $v \in S_i$, add all neighbors of v with lower age than v. Stop when no new vertices can be added, and let $S = \bigcup_i S_i$. The set S contains X. The n-core of $H \upharpoonright S$ is K_1.

Let $T = f(S)$, so that $Y \subseteq T$. The n-core of T is G. However, the n-core of a countable graph is unique by Lemma 1, which contradicts that $H \restriction S \cong J \restriction T$.

For item (2b), we proceed in a similar fashion as the proof of (2a), defining the sets X, Y, S, and T as before. Note that S and T are finite. Let $|S| = r$. Since all vertices in S except those in X have exactly n lower age neighbors in S, $H \restriction S$ has exactly $n(r-n) + \binom{n}{2}$ edges.

Assume that all edges in $J \restriction T$ are directed from higher age to lower age vertices. Then, each vertex has out-degree at most n, while vertices in Y have out-degree zero. This implies that $J \restriction T$ has at most $n(r-n) + |E(G)|$ edges. But then $|E(J \restriction T)| < |E(G \restriction S)|$ by hypothesis, which contradicts the fact that f is an isomorphism. ■

As an application of Corollary 12, note that by (2b), $R^{(2,\bar{K}_2)} \ncong R^{(2)}$. Corollary 12 is also useful to prove various properties of $R^{(n)}$. For example, let $K_n(\omega)$ be the graph formed from K_n by adding $\aleph_0$-many pairwise non-joined vertices joined to each vertex of K_n. Then $K_n \preccurlyeq_{(n)} K_n(\omega)$ and by Corollary 12, $K_n(\omega)$ generates $R^{(n)}$. See Theorem 14 for an application of the corollary to the automorphism group of $R^{(n)}$.

The graph $R^{(1)}$ is the unique countable tree with each vertex of infinite degree. Hence, $R^{(1)}$ has infinite diameter. However, for $n \geq 2$, $R^{(n)}$ has diameter 2 (since any 2-e.c. graph has diameter 2). The spanning subgraphs of R are well known; see [5]. A *ray* is an infinite path that extends indefinitely in one direction; a *double ray* is an infinite path that extends indefinitely in two directions. A *one-way Hamiltonian path* is a spanning subgraph that is a ray, while a *two-way Hamiltonian path* is a spanning subgraph that is a double ray. Henson [8] proved first that R contains one- and two-way Hamiltonian paths, and we obtain a similar result for $R^{(n)}$.

Theorem 13. *If G is a strongly 2-e.c. graph, then G has one and two-way Hamilton paths. In particular, for $n \geq 2$, $R^{(n)}$ has one and two-way Hamilton paths.*

Proof. We prove that G has one-way Hamilton path; the existence of a two-way Hamilton path is similar. Without loss of generality, let $V(G) = \mathbb{N}$.

We construct such a path by induction. Let X_0 be the subgraph induced by $\{0\}$. Suppose that X_n contains the vertices $\{0, 1, \ldots, n-1\}$, and further suppose that X_n has a Hamilton path P. If n is contained in $V(X_n)$, then let $X_{n+1} = X_n$. Suppose that n is not in X_n, and let u be an endvertex of P. By the strongly 2-e.c. property, there is a vertex z joined to n and u, and joined to none of the other vertices in X_n. Let X_{n+1} be the graph formed by taking the subgraph induced by $V(X_n) \cup \{z, n+1\}$. Then X_{n+1} contains a Hamilton path. A Hamilton path in G is then $X = \lim_{n\to\infty} X_n$. ■

A. Symmetries of $R^{(n)}$

A mapping $f : G \to H$ with the property that $xy \in E(G)$ implies that $f(x)f(y) \in E(H)$ is a *homomorphism*. We write fg for the composition of two mappings. If $S \subseteq V(G)$, then we write $f \restriction S$ for the restriction of f to S. An embedding $f : G \to H$ is an injective homomorphism with the property that $xy \notin E(G)$ implies that

$f(x)f(y) \notin E(H)$. An *automorphism* is a bijective embedding. The group of all automorphisms of G under composition is written Aut(G). We write id_G for the identity mapping on G.

The automorphism group of R has been actively studied (see [5]). For example, the e.c. property and a back-and-forth argument together imply that each countable group is isomorphic to a subgroup of the automorphism group of R; we say that Aut(R) *embeds* each countable group. We prove an analogous result for $R^{(n)}$.

Theorem 14. *The automorphism group of $R^{(n)}$ embeds all countable groups. In particular, the countably infinite symmetric group S_ω is a subgroup of* Aut($R^{(n)}$).

Proof. For a positive integer r, let $K_n(\omega)$ be the graph formed from K_n by adding infinitely many pairwise non-joined vertices joined to each vertex of K_n. Then $K_n \preccurlyeq_{(n)} K_n(\omega)$ and by Corollary 12, $K_n(\omega)$ generates $R^{(n)}$. Let $G = R^{(K_n(\omega),n)}$. The countably infinite symmetric group S_ω is a subgroup of Aut($K_n(\omega)$), since we may permute in all ways the elements of the independent set outside K_n, leaving K_n fixed. By Cayley's theorem on symmetric groups, each countable group is a subgroup of S_ω. Hence, to prove the theorem it is enough to prove that the Aut($K_n(\omega)$) = Aut(R_0) is isomorphic to a subgroup of Aut(G).

Let f_0 be an automorphism of R_0. Assume that f_t is an automorphism of R_t such that $f_t \restriction X_0 = f_0$. For each vertex z_S of $V(R_{t+1})\backslash V(R_t)$, where S is a subset of $V(R_t)$ satisfying $|S| = n$, define $f_{t+1}(z_S) = z_{f_t(S)}$. Otherwise, define f_{t+1} to be f_t on R_t. It follows that f_{t+1} is an automorphism of R_{t+1} satisfying $f_{t+1} \restriction R_t = f_t$.

Define $F \in$ Aut(G) by $F = \bigcup_{t\in\mathbb{N}} f_t$, and let α : Aut(R_0) $\to$ Aut(G) be defined by $\alpha(f) = F$. Since $F \restriction R_0 = f$, it follows that α is injective. To prove that S_ω is isomorphic to a subgroup of Aut(G), it is sufficient to show that α is a group homomorphism.

As α clearly preserves the identity automorphism, we show that for all $x \in V(G)$,

$$\alpha(fg)(x) = \alpha(f)\alpha(g)(x). \tag{4.1}$$

Fix $x \in V(G)$ with age(x) = t. We prove (4.1) by induction on t. If $t = 0$, then (4.1) clearly holds. Suppose that (4.1) holds for all vertices x with age(x) $\leq t$. Let age(x) = $t + 1$. Then x is of the form z_S, where S is a finite subset of X_t such that $|S| \leq n$. Now,

$$\begin{aligned} \alpha(fg)(x) &= (fg)_{t+1}(x) \\ &= z_{(fg)_t(S)} \\ &= z_{f_t g_t(S)} \\ &= \alpha(f)\alpha(g)(x), \end{aligned}$$

where the third equality follows by induction hypothesis. Hence (4.1) holds by induction on t. ∎

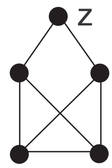

FIGURE 2. A graph G with $K_2 \preceq G$.

The tree $R^{(1)}$ is easily seen to be vertex-transitive. However, for $n > 1$, the graph $R^{(n)}$ is not in general vertex-transitive. This is in sharp contrast to R, which possesses a large amount of symmetry; see [5]. For example, if $n = 2$, then consider the 2-ordered graph G in Figure 2. Since every vertex but z has degree 3, z may only appear as the last vertex in any 2-ordering of G. As $K_2 \preceq G$, by Theorem 12 we may generate $R^{(2)}$ by G. If $R^{(2)}$ were vertex-transitive, then we may automorphically map z in G to a vertex u of the copy of K_2 with age 0. In particular, there is an isomorphic copy G' of G in $R^{(2)}$, containing u, and with u acting as z. But then the 2-ordering of $R^{(2)}$ induces a 2-ordering L of G' with z as the first or second vertex in L, which is a contradiction.

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