Available online at www.sciencedirect.com
ScienceDirect
ELSEVIER
Discrete Mathematics 306 (2006) 2327–2335
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

On an adjacency property of almost all tournaments

Anthony Bonato, Kathie Cameron

Department of Mathematics, Wilfrid Laurier University, Waterloo, Ont., Canada N2L 3C5

Received 3 March 2003; received in revised form 15 September 2003; accepted 15 September 2003
Available online 27 July 2006

In loving memory of Claude Berge

Abstract

Let n be a positive integer. A tournament is called n-existentially closed (or n-e.c.) if for every subset S of n vertices and for every subset T of S, there is a vertex $x \notin S$ which is directed toward every vertex in T and directed away from every vertex in $S\backslash T$. We prove that there is a 2-e.c. tournament with k vertices if and only if $k \geq 7$ and $k \neq 8$, and give explicit examples for all such orders k. We also give a replication operation which preserves the 2-e.c. property.

MSC: 05C35; 05C80; 05C75

Keywords: Adjacency property; n-Existentially closed; Tournament; Replication

1. Introduction

A *tournament* is a directed graph with exactly one arc between each pair of distinct vertices. Consider the following *adjacency property* for tournaments.

Definition 1. Let n be a positive integer. A tournament is called *n-existentially closed* or *n-e.c.* if for every n-element subset S of the vertices, and for every subset T of S, there is a vertex $x \notin S$ which is directed toward every vertex in T and directed away from every vertex in $S\backslash T$. (Note that T may be empty.)

Adjacency properties of tournaments were studied in [3,8,15,18,23]. Much of the research on such properties is motivated by the fact that while almost all tournaments (with arcs chosen independently and with probability p, where $0 < p < 1$ is a fixed real number) are n-e.c. for any fixed positive integer n (see [15]), few *explicit* examples of such tournaments are known.

Adjacency properties of graphs were studied by numerous authors; see [9] for a survey. A graph is called *n-existentially closed* or *n-e.c.* if it satisfies the following adjacency property: for every n-element subset S of the vertices, and for every subset T of S, there is a vertex not in S which is joined to every vertex of T and to no vertex of $S\backslash T$. The n-e.c. property is of

E-mail addresses: abonato@rogers.com (A. Bonato), kcameron@wlu.ca (K. Cameron).

0012-365X/$ - see front matter
doi:10.1016/j.disc.2005.12.030

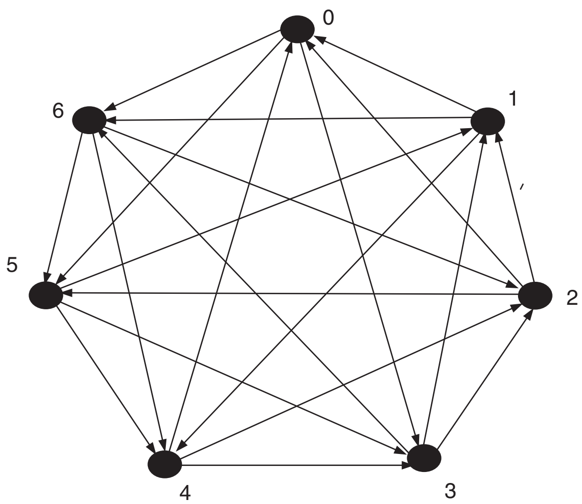

Fig. 1. The tournament D_7.

interest in part because the countable random graph is n-e.c. for all $n \geqslant 1$; in fact, the countable random graph is the unique (up to isomorphism) countable graph that is n-e.c. for all $n \geqslant 1$. The countable random tournament is the analogue of the random graph for tournaments; see [13]. The countable random tournament is the unique (up to isomorphism) countable tournament that is n-e.c. for all $n \geqslant 1$.

The cases $n = 1, 2$ for graphs were studied in [9,10,12]. For $n > 2$, few explicit examples of n-e.c. graphs are known other than large Paley graphs (see [2,8]). A prolific construction of n-e.c. graphs for all n was recently given in [14].

In the present article, we concentrate on the 2-e.c. adjacency property. Note that a tournament is 2-e.c. if the following adjacencies hold: for every pair of vertices, u and v, there are four other vertices: one directed toward both u and v, one directed away from both u and v, one directed toward u and away from v, and one directed toward v and away from u. In Section 3, we prove that there is a 2-e.c. tournament with k vertices if and only if $k \geqslant 7$ and $k \neq 8$, and give explicit examples for all such orders k.

We consider only finite and simple tournaments. For a tournament G, $V(G)$ denotes its vertex-set and $E(G)$ denotes its arc-set. The order of G is $|V(G)|$. We denote an arc directed from x to y by (x, y). For a vertex $x \in V(G)$, we define $N_{\text{out}}(x) = \{y : (x, y) \in E(G)\}$, and $N_{\text{in}}(x) = \{y : (y, x) \in E(G)\}$. As usual, a vertex x with $N_{\text{in}}(x) = \emptyset$ is called a source and a vertex x with $N_{\text{out}}(x) = \emptyset$ is called a sink. If $U \subseteq V(G)$, $G\restriction U$ is the subgraph of G induced by U; for $x \in V(G)$, $G - x = G\restriction(V(G)\backslash\{x\})$. For basic information on graphs and tournaments, see [4,11].

The *Paley tournament* of order q, written D_q, where q is a prime power congruent to 3 (mod 4), is the tournament with vertices the elements of $GF(q)$, the finite field with q elements, and $(x, y) \in E(D_q)$ if and only if $x - y$ is a nonzero quadratic residue. For D_7, see Fig. 1. As discussed above for Paley graphs, for a fixed positive n, sufficiently large Paley tournaments are n-e.c. (see [18]); however, no other explicit families of tournaments with these adjacency properties are known.

The next lemma follows from the definitions.

Lemma 1. *Let G be an n-e.c. tournament for some $n > 1$. For a fixed $v \in V(G)$, the tournaments $G - v$, $G\restriction N_{\text{in}}(v)$, and $G\restriction N_{\text{out}}(v)$ are each $(n-1)$-e.c.*

Definition 2. A tournament G is *n-e.c. minimal* if G has the smallest number of vertices among all n-e.c. tournaments. An n-e.c. tournament is *critical* if deleting any vertex leaves a tournament which is not n-e.c.

Clearly, an n-e.c. minimal tournament is n-e.c. critical. In Section 2, we show that there are exactly two 1-e.c. critical tournaments up to isomorphism. In Section 4, we give examples of 2-e.c. critical tournaments of all possible orders $k \geqslant 7$ and $k \neq 8$. Vertex-criticality for various properties has been studied by many authors, including Berge [6,5,7,1,17,20–22,24,25].

2. The 1-e.c. critical tournaments

We make the following trivial observations.

Remark 1. A tournament is 1-e.c. if and only if it has no source or sink.

Remark 2. A tournament with a directed hamilton cycle is 1-e.c.

The tournament D_3 is the directed circuit on three vertices. It is easy to see that D_3 is the unique (up to isomorphism) 1-e.c. minimal tournament, and thus, it is 1-e.c. critical. Define T_6 to be the tournament consisting of two copies of D_3, with arcs oriented from the first copy to the second. It is straightforward to check that T_6 is 1-e.c. critical.

Theorem 2. *The only* 1-*e.c. critical tournaments* (*up to isomorphism*) *are* D_3 *and* T_6.

Proof. Let G be a 1-e.c. critical tournament. We first observe that a strongly connected component S of G has exactly three vertices. To see this, suppose that S has at least $k \geqslant 4$ vertices. By a theorem of Moon [19], S has a directed circuit C of length $k-1$. Deleting the vertex that C misses in S leaves a 1-e.c. tournament, which is a contradiction.

We claim that if G has exactly one or two strongly connected components, then G is isomorphic to D_3 or T_6, respectively. Assume to the contrary that G has $r \geqslant 3$ strongly connected components. From G we construct an auxiliary tournament G', whose vertices are the strongly connected components of G with the induced adjacencies. Note that G' is isomorphic to the r-element linear order. Let u be a vertex of G' that is neither a least nor greatest element. If we delete a vertex x in the strongly connected component of G corresponding to u, then the remaining graph $G-x$, is 1-e.c., which is a contradiction. □

3. Examples of 2-e.c. tournaments

In this section, our main theorem is the following.

Theorem 3. *There is a* 2-*e.c. tournament with k vertices if and only if* $k \geqslant 7$ *and* $k \neq 8$.

To prove Theorem 3, we first prove the following theorem.

Theorem 4. *There is a unique* (*up to isomorphism*) 2-*e.c. minimal tournament*, *the Paley tournament* D_7.

Proof. Let G be a 2-e.c. tournament. Then since the unique minimal 1-e.c. tournament has three vertices, $|V(G)| \geqslant 7$ by Lemma 1. Suppose now $|V(G)|=7$, say $V(G)=\{1,2,3,4,5,6,7\}$. Say $N_{\text{in}}(7)=\{1,2,3\}$, $(1,2), (2,3), (3,1) \in E(G)$; $N_{\text{out}}(7)=\{4,5,6\}$; $(4,6), (6,5), (5,4) \in E(G)$. See Fig. 2(a). Vertex 1 currently has outdegree two, but needs outdegree three, so without loss of generality, assume that $(1,4) \in E(G)$. Then by considering the degrees of 1 and 4, we get $(5,1), (6,1) \in E(G)$ and $(4,2), (4,3) \in E(G)$. See Fig. 2(b). Since $N_{\text{in}}(1)=\{3,5,6\}$ and $(6,5) \in E(G)$, it follows that $(5,3), (3,6) \in E(G)$. See Fig. 2(c). Then, for degree of 5, $(2,5) \in E(G)$, and then for degree of 2, $(6,2) \in E(G)$. See Fig. 2(d). Then $f: V(G) \to V(D_7)$ is an isomorphism, where $f(1)=0$, $f(2)=5$, $f(3)=4$, $f(4)=6$, $f(5)=1$, $f(6)=2$ and $f(7)=3$. □

Given a 2-e.c. tournament, another 2-e.c. tournament with two more vertices can be constructed using a "tournament version" of the replication operation which was instrumental in [9].

Definition 3. Let G be a tournament and let $(a,b) \in E(G)$. Add two new vertices a', b' such that a' has the same adjacencies to vertices of G other than b as a does, b' has the same adjacencies to vertices of G other than a as b does,

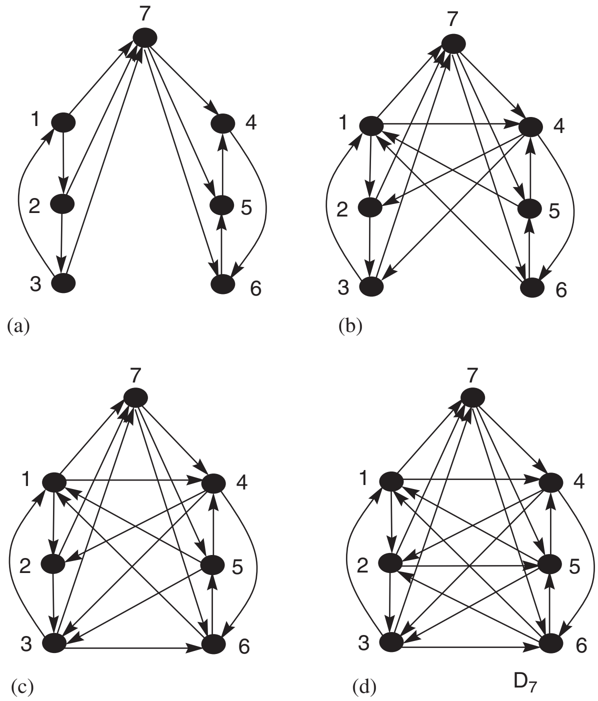

Fig. 2. The proof of Theorem 4.

a, b, a', b', a is a directed circuit, a and a' are joined either way and b and b' are joined either way; that is, a *replicate* $R = R(G, e)$ is a tournament with $V(R) = V(G) \cup \{a', b'\}$ and

$$\begin{aligned} E(R) = E(G) &\cup \{(a', v) : v \in N_{\mathrm{out}}(a)\backslash\{b\}\} \cup \{(v, a') : v \in N_{\mathrm{in}}(a)\} \\ &\cup \{(b', v) : v \in N_{\mathrm{out}}(b)\} \cup \{(v, b') : v \in N_{\mathrm{in}}(b)\backslash\{a\}\} \\ &\cup \{(b, a'), (a', b'), (b', a)\} \cup \{\text{exactly one of } (a, a'), (a', a)\} \\ &\cup \{\text{exactly one of } (b, b'), (b', b)\}. \end{aligned}$$

We observe that for each arc e, there are four nonidentical replicates $R(G, e)$ that we may construct (depending on how we orient the edges aa', bb').

Definition 4. Let G be a tournament, and let $n \geqslant 1$ be fixed.

(1) An *n-e.c. tournament problem* is a $2 \times n$ matrix

$$\begin{pmatrix} x_1 & \ldots & x_n \\ i_1 & \ldots & i_n \end{pmatrix},$$

where $\{x_1, \ldots, x_n\}$ is an n-element subset of $V(G)$, and for $1 \leqslant j \leqslant n$, $i_j \in \{\uparrow, \downarrow\}$.

(2) A *solution* to an n-e.c. tournament problem is a vertex $z \in V(G)$ such that $z \in N_{\mathrm{in}}(x_j)$ if $i_j = \uparrow$ and $z \in N_{\mathrm{out}}(x_j)$ if $i_j = \downarrow$.

Note that a tournament G is n-e.c. if and only if each n-e.c. tournament problem in G has a solution.

Theorem 5. *If G is a 2-e.c. tournament, then for every $e \in E(G)$, each replicate $R = R(G, e)$ is 2-e.c.*

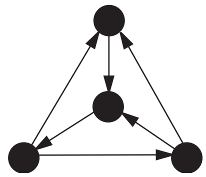

Fig. 3. The unique 1-e.c. tournament of order 4.

Proof. Fix $e=(a,b)\in E(G)$. Fix distinct $x,y\in V(R)$. We show that each problem $\begin{pmatrix} x & y \\ i & j \end{pmatrix}$, $i,j\in\{\uparrow,\downarrow\}$ has a solution in R.

Case 1: $|\{a',b'\}\cap\{x,y\}|=0$. A solution to the problem in G is a solution to the problem in R.

Case 2: $|\{a',b'\}\cap\{x,y\}|=1$.

Assume that $x=a'$ and $y\neq b'$. First, suppose $y=a$. If $(i,j)=(\uparrow,\uparrow)$, an in-neighbour of a in G solves the problem; if $(i,j)=(\downarrow,\downarrow)$, an out-neighbour of a in G other than b solves the problem. The vertex b solves $\begin{pmatrix} a' & a \\ \uparrow & \downarrow \end{pmatrix}$ and b' solves $\begin{pmatrix} a' & a \\ \downarrow & \uparrow \end{pmatrix}$.

If $y\neq a$, first solve $\begin{pmatrix} a & y \\ i & j \end{pmatrix}$ by say, c, in G. If $c\neq b$, then c also solves $\begin{pmatrix} a' & y \\ i & j \end{pmatrix}$. If $c=b$, then $i=\downarrow$ and $y\neq b$, so b' solves $\begin{pmatrix} a' & y \\ \downarrow & j \end{pmatrix}$.

The case when $x=b'$ and $y\neq a'$ follows by a similar argument.

Case 3: $|\{a',b'\}\cap\{x,y\}|=2$.

Where z is a solution of $\begin{pmatrix} a & b \\ i & j \end{pmatrix}$ in G, z is a solution of $\begin{pmatrix} a' & b' \\ i & j \end{pmatrix}$ in R. $\square$

Using tournament replication on D_7, we obtain 2-e.c. tournaments for any odd order k, $k\geqslant 7$. Now we work on finding 2-e.c. tournaments of all possible even orders.

Theorem 6. *There is no* 2-*e.c. tournament of order* 8.

Proof. It is straightforward to see that there is a unique 1-e.c. tournament of order 4; see Fig. 3. Let G be a 2-e.c. tournament of order 8. Then G has a vertex of degree 4. In fact, the outdegree sequence of G is completely determined.

Claim. *G has exactly four vertices of indegree* 3 *and four vertices of indegree* 4.

Let $v\in V(G)$. Since both $G\restriction N_{\text{in}}(v)$ and $G\restriction N_{\text{out}}(v)$ are 1-e.c., it follows that $3\leqslant|N_{\text{in}}(v)|\leqslant 4$. Let x be the number of vertices of indegree 3, and let y be the number of vertices of indegree 4. Then since the sum of all indegrees is the number of arcs,

$$x+y=8,$$
$$3x+4y=28.$$

Solving the system establishes the claim.

Now suppose $V(G)=\{1,\dots,8\}$. For each vertex v of G, one of the subgraphs induced by $N_{\text{in}}(v)$ and $N_{\text{out}}(v)$ is D_3 and the other is the tournament of Fig. 3.

Without loss of generality, suppose vertex 1 has indegree 4 and the subgraphs induced by $N_{\text{in}}(1)$ and $N_{\text{out}}(1)$ are as in Fig. 4.

Case 1: Vertex 8 has indegree 3.

Without loss of generality, by the symmetry of 2, 3, and 4 in the directed graph in Fig. 4, $(2,8),(3,8),(8,4)\in E(G)$. $N_{\text{in}}(8)=\{2,3,7\}$ and $(2,3)\in E(G)$, so for $G\restriction N_{\text{in}}(8)\cong D_3$, also $(3,7),(7,2)\in E(G)$. $N_{\text{out}}(8)=\{1,4,5,6\}$ and $(5,1),(5,6)\in E(G)$, so $(4,5)\in E(G)$.

Now $|N_{\text{out}}(7)|=4$, so all remaining arcs meeting 7 must be directed toward 7, so $(4,7)\in E(G)$. Then $N_{\text{in}}(7)=\{3,4,6\}$ and $(3,4)\in E(G)$, so $(4,6),(6,3)\in E(G)$. The vertices 4, 5, and 8 are in $N_{\text{in}}(6)$, but $(8,5),(4,5)\in E(G)$ so $|N_{\text{in}}(6)|=4$, so $(2,6)\in E(G)$. Now $N_{\text{in}}(6)=\{2,4,5,8\}$ and $(4,5),(8,5)\in E(G)$ so $(5,2)\in E(G)$.

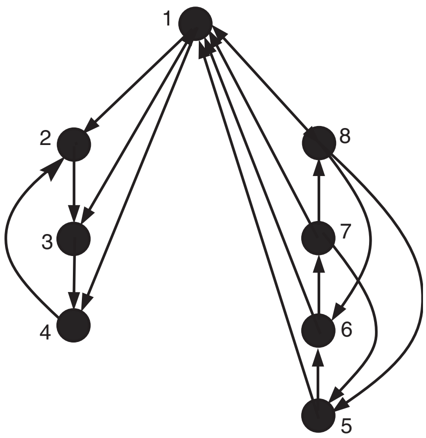

Fig. 4. The in- and out-neighbours of 1.

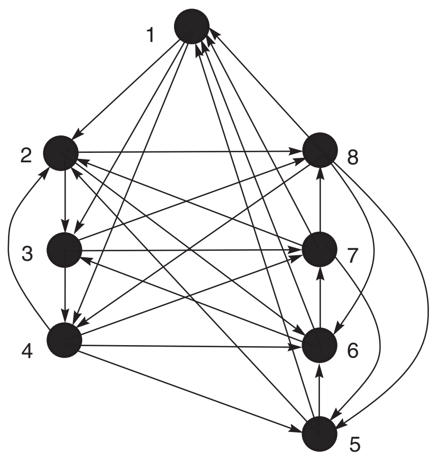

Fig. 5. G missing one arc.

Now we have all but one arc of G, either (3, 5) or (5, 3). See Fig. 5. If that arc were (3, 5), then $N_{\mathrm{out}}(3) = \{4, 5, 7, 8\}$ and $(4, 5), (7, 5), (8, 5) \in E(G)$ which is a contradiction. Otherwise, if that arc were (5, 3), then $N_{\mathrm{out}}(5) = \{1, 2, 3, 6\}$ and $(1, 3), (2, 3), (6, 3) \in E(G)$, which is a contradiction.

Case 2: Vertex 8 has indegree 4.

In this case $(2, 8), (3, 8), (4, 8) \in E(G)$. Then $N_{\mathrm{out}}(8) = \{1, 5, 6\}$, but $(5, 1), (6, 1) \in E(G)$, which is a contradiction. □

To find 2-e.c. tournaments of all possible even orders as described in Theorem 3, it is sufficient to give an example of a 2-e.c. tournament of order 10, and then use replication. For this, see the tournament R' in Fig. 6. It is straightforward to verify that R' is 2-e.c.: one need only check the vertices 1, 2, and 10 versus each of the other vertices. The details are tedious and are therefore omitted.

In [12] it was proved that whenever there is a 2-e.c. graph of order m, then there is an 2-e.c. graph of order $m + 1$, and the question of this type of monotonicity was raised in general for n-e.c. graphs. We remark that the "gap" for 2-e.c. tournaments supplies the first example of nonmonotonicity of a 2-e.c. property.

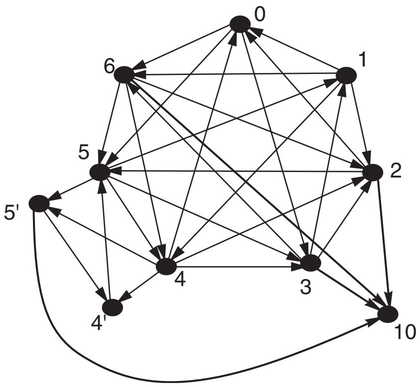

Fig. 6. The tournament R'. Reverse the arc (2, 1) in $R(D_7, (5, 4))$ (where $(4, 4')$ and $(5, 5')$ are arcs), and add a new vertex 10 so that $N_{\text{in}}(10) = \{2, 3, 5', 6\}$. Note that not all arcs are shown.

4. Examples of 2-e.c. critical tournaments

Definition 5. An arc $e = (a, b)$ of tournament G is *good* if every vertex $v \neq a, b$ is the unique solution to some 2-e.c. tournament problem not involving a or b.

Lemma 7. *Let G be a 2-e.c. critical tournament and let arc $e = (a, b)$ be good. Then each replicate $R = R(G, e)$ is a 2-e.c. critical tournament.*

Proof. Note that: the unique solution of $\begin{pmatrix} b & b' \\ \uparrow & \downarrow \end{pmatrix}$ is a, of $\begin{pmatrix} b & b' \\ \downarrow & \uparrow \end{pmatrix}$ is a, of $\begin{pmatrix} a & a' \\ \uparrow & \downarrow \end{pmatrix}$ is b', and of $\begin{pmatrix} a & a' \\ \downarrow & \uparrow \end{pmatrix}$ is b. Now let $x \in V(R) - \{a, a', b, b'\}$. By hypothesis, x is the unique solution to some 2-e.c. tournament problem in G. If a' were a solution to this problem in R then a would be a solution to it in G, and if b' were a solution to this problem in R, then b would be a solution to it in G. Therefore, x is the unique solution to this problem in R. □

Definition 6. In the definition of replication of the arc $e = (a, b)$ in tournament G, we insist that the arc between a and a' be (a, a'), and the arc between b and b' be (b', b), then we call the replication a *type*-1 *replication*, and use a subscript 1 to indicate the resulting tournament, $R_1(G, e)$.

Lemma 8. *Let G be a 2-e.c. critical tournament and let $e = (a, b) \in E(G)$ be good. Repeatedly replicating e using type-1 replication gives a 2-e.c. critical tournament.*

Proof. Define $G_0 = G$. For $k \geqslant 0$, define $G_{k+1} = R_1(G_k, e)$, and call the replication arc $e_{k+1} = (a_{k+1}, b_{k+1})$. Then by Lemma 5, G_{k+1} is a 2-e.c. tournament of order $|V(G)| + 2k$. We need to show that G_{k+1} is 2-e.c. critical.

We proceed by induction on k. Assume G_k is 2-e.c. critical and that for $1 \leqslant j \leqslant k$, vertex a_j uniquely solves $\begin{pmatrix} b_j & b \\ \uparrow & \downarrow \end{pmatrix}$ and vertex b_j uniquely solves $\begin{pmatrix} a_j & a \\ \downarrow & \uparrow \end{pmatrix}$. Consider G_{k+1}.

Since (a, b) is good in G, each vertex $v \in V(G) \backslash \{a, b\}$ is the unique solution to some 2-e.c. tournament problem in G not involving a or b. In G_{k+1}, no vertex a_j or b_j, $1 \leqslant j \leqslant k + 1$ can solve this problem, because otherwise, a or b would have solved it in G.

Vertices a_{k+1} and b_{k+1} cannot solve the problems that a_j and b_j $(1 \leqslant j \leqslant k)$ uniquely solve in the induction hypothesis: for the a_j problem, (b_j, a_{k+1}) and (b_{k+1}, b) are arcs of G_{k+1}; for the b_j problem, (b_{k+1}, a_j) and (a, a_{k+1}) are arcs of G_{k+1}.

Vertex a_{k+1} uniquely solves $\begin{pmatrix} b_{k+1} & b \\ \uparrow & \downarrow \end{pmatrix}$ since every vertex except a_{k+1} and a (and b) is directed the same way with respect to b and b_{k+1}, and a is directed toward b. By a similar argument, vertex b_{k+1} uniquely solves $\begin{pmatrix} a_{k+1} & a \\ \downarrow & \uparrow \end{pmatrix}$, since every vertex except b_{k+1} and b (and a) is directed the same way with respect to a and a_{k+1}, and a is directed toward b.

Finally, a uniquely solves $\begin{pmatrix} b_{k+1} & b \\ \downarrow & \uparrow \end{pmatrix}$ and b uniquely solves $\begin{pmatrix} a_{k+1} & a \\ \uparrow & \downarrow \end{pmatrix}$. □

Using Lemmas 7 and 8, we may construct 2-e.c. critical tournament for all the possible orders k, where $k \geqslant 7$ and $k \neq 8$ as follows. For the odd orders, we note that in D_7, (4, 3) is a good arc, as demonstrated by the following table.

Vertex	0	1	2	5	6
Uniquely solves	$\begin{pmatrix} 2 & 5 \\ \downarrow & \uparrow \end{pmatrix}$	$\begin{pmatrix} 0 & 6 \\ \uparrow & \uparrow \end{pmatrix}$	$\begin{pmatrix} 0 & 5 \\ \uparrow & \uparrow \end{pmatrix}$	$\begin{pmatrix} 0 & 2 \\ \downarrow & \downarrow \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ \downarrow & \downarrow \end{pmatrix}$

For the even orders, the following tables demonstrate that the tournament R' (introduced at the end of Section 3) is 2-e.c. critical, and that (0, 6) is a good arc.

Vertex	0	1	2	3	4
Uniquely solves	$\begin{pmatrix} 1 & 2 \\ \downarrow & \downarrow \end{pmatrix}$	$\begin{pmatrix} 5 & 4 \\ \downarrow & \uparrow \end{pmatrix}$	$\begin{pmatrix} 4 & 3 \\ \downarrow & \downarrow \end{pmatrix}$	$\begin{pmatrix} 5' & 4' \\ \downarrow & \downarrow \end{pmatrix}$	$\begin{pmatrix} 5 & 5' \\ \downarrow & \uparrow \end{pmatrix}$

Vertex	5	6	4′	5′	10
Uniquely solves	$\begin{pmatrix} 4 & 4' \\ \uparrow & \downarrow \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ \downarrow & \downarrow \end{pmatrix}$	$\begin{pmatrix} 5' & 3 \\ \downarrow & \uparrow \end{pmatrix}$	$\begin{pmatrix} 4 & 4' \\ \downarrow & \uparrow \end{pmatrix}$	$\begin{pmatrix} 2 & 5' \\ \downarrow & \downarrow \end{pmatrix}$

We close with the following problem: find examples of n-e.c. tournaments, where $n \geqslant 3$, that are not Paley tournaments.

Acknowledgements

The authors gratefully acknowledge the support of the Natural Sciences and Engineering Research Council of Canada (NSERC).

References

[1] R. Aharoni, E. Berger, The number of edges in critical strongly connected graphs, Discrete Math. 234 (2001) 119–123.
[2] W. Ananchuen, L. Caccetta, On the adjacency properties of Paley graphs, Networks 23 (1993) 227–236.
[3] W. Ananchuen, L. Caccetta, On tournaments with a prescribed property, Ars Combin. 36 (1993) 89–96.
[4] C. Berge, Graphs and Hypergraphs, North-Holland, Amsterdam, 1973.
[5] C. Berge, Some Common properties for regularizable graphs, edge-critical graphs and B-graphs, Theory and Practice of Combinatorics, North-Holland Mathematical Studies, vol. 60, North-Holland, Amsterdam, 1982, pp. 31–44.
[6] C. Berge, About vertex-critical nonbicolorable hypergraphs, Austral. J. Combin. 9 (1994) 211–219.
[7] C. Berge, Some properties of non-bicolorable hypergraphs and the four-color problem, First International Colloquium on Graphs and Optimization (GOI), 1992 (Grimentz), Discrete Appl. Math. 65 (1996) 73–79.
[8] B. Bollobás, Random Graphs, Academic Press, London, 1985.
[9] A. Bonato, K. Cameron, On an adjacency property of almost all graphs, Discrete Math. 231 (2001) 103–119.
[10] A. Bonato, K. Cameron, On 2-e.c. line-critical graphs, J. Combin. Math. Combin. Comput. 38 (2001) 111–121.
[11] J.A. Bondy, U.S.R. Murty, Graph Theory with Applications, North-Holland, New York, 1976.
[12] L. Caccetta, P. Erdős, K. Vijayan, A property of random graphs, Ars Combin. 19 (1985) 287–294.
[13] P.J. Cameron, The random graph, in: R.L. Graham, J. Nešetřil (Eds.), Algorithms and Combinatorics, vol. 14, Springer, New York, 1997, pp. 333–351.
[14] P.J. Cameron, D. Stark, A prolific construction of strongly regular graphs with the n-e.c. property, Electron. J. Combin. 9 (1) (2002), Research Paper 31.
[15] P. Erdős, On a problem in graph theory, Math. Gaz. 47 (1963) 220–223.
[17] T. Gallai, Kritische Graphen I, Magyar Tud. Akad. Mat. Kutató Int. Közl. 8 (1963) 165–192.
[18] R.L. Graham, J.H. Spencer, A constructive solution to a tournament problem, Canad. Math. Bull. 14 (1971) 45–48.
[19] J.W. Moon, On subtournaments of a tournament, Canad. Math. Bull. 9 (1966) 297–301.
[20] V. Neumann-Lara, Vertex critical 4-dichromatic circulant tournaments, Discrete Math. 170 (1997) 289–291.

[21] J. Plesník, Diametrically critical tournaments, Časopis Pěst. Mat. 100 (1975) 361–370.
[22] J.H. Schmerl, W.T. Trotter, Critically indecomposable partially ordered sets, graphs, tournaments and other binary relational structures, Discrete Math. 113 (1993) 191–205.
[23] E. Szekeres, G. Szekeres, On a problem of Schütte and Erdős, Math. Gaz. 49 (1965) 290–293.
[24] C. Thomassen, Color-critical graphs on a fixed surface, J. Combin. Theory Ser. B 70 (1997) 67–100.
[25] B. Toft, An investigation of colour-critical graphs with complements of low connectivity, Advances in Graph Theory (Cambridge Combinatorial Conference, Trinity Colloquium, Cambridge, 1977); Ann. Discrete Math. 3 (1978) 279–287.