Classes closed under unions and ne.c. structures

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Abstract

We discuss those classes $\mathcal{K}$ of relational structures closed under unions that are defined by excluding certain finite structures. We characterize such classes and show they contain an infinite family of pair-wise non-embeddable members. We investigate structures in $\mathcal{K}$ satisfying extension conditions and construct universal structures in $\mathcal{K}$ satisfying only finitely many of the extension conditions.

1 Introduction

In this article we investigate classes of combinatorial structures defined by excluding certain finite structures (so called "constrained classes") which satisfy the additional condition that they are closed under unions. Classes of relational structures closed under union were first studied by [9], and include the class of all graphs.

We characterize these classes via graph-theoretic properties of their constraints (see Proposition 19), and present various constructions possible in such classes (see Sections 4 and 5). Ne.c. structures are structures satisfying certain extension properties. Ne.c. graphs are precisely those graphs satisfying a certain adjacency property which has been actively studied by various authors for finite graphs (see [1], [2], [7], [4]); for example, it is known that almost all finite graphs are ne.c. [3]. In Section 6 we show that in classes $\mathcal{K}$ closed under unions, for each $n \geq 1$ there are countable structures which are ne.c. but not re.c. for some $r > n$, which embed all countable structures in $\mathcal{K}$.

2 Relational Structures

In this section we introduce the terminology needed to discuss relational structures. The reader who desires further information on relational structures is directed to [8].

Definition 1 *1. A signature is a set of symbols $L = \{R_i : i \in I\}$ along with a function $ar : L \to \mathbb{N} - \{0\}$.*

*2. An L-**structure** A is a pair consisting of a nonempty set $dom(A)$, the **domain** of A, and an operation $R \mapsto R^A$ defined on all $R \in L$ so that if $ar(R) = n$ then $R^A \subseteq dom(A)^n$. R^A is the **interpretation** of R in A.*

*3. The **order** of an L-structure A is the cardinality of its domain.*

Remark 2 *1.* We abuse notation and let A stand for both a structure and its domain. Furthermore, we suppress mention of the operation $R \mapsto R^A$.

2. We consider only non-empty structures.

Example 3 Let $L = \{E\}$, with $ar(E) = 2$.

1. A (simple) graph G is an L-structure with E^G irreflexive and symmetric. The domain of G is often called the set of vertices of G; the 2-tuples in E^G are the edges. In practice, we usually write E (without the superscript G) when G is clear from context.

2. An order P is an L-structure with E^P reflexive, anti-symmetric, and transitive. If $(a, b) \in E^P$ we usually write $a \leq b$.

3. Let $L = \{R\}$, $ar(R) = k$, for some $k \geq 2$.

   A k-uniform hypergraph H is an L-structure with R^H interpreted as all permutations of a set of k-element subsets of H.

Definition 4 *Let A, B be L-structures for a fixed signature L.*

1. *A* ***homomorphism*** *f from A to B is a map satisfying*

$$\bar{a} \in R^A \text{ implies } f(\bar{a}) \in R^B. \tag{1}$$

2. *f is an* ***embedding*** *if f is injective and the "implies" in (1) is replaced by an "if and only if". f is an* ***isomorphism*** *if it is a surjective embedding. We write that A and B are isomorphic as $A \cong B$.*

   An ***automorphism*** *of A is an isomorphism from A to itself; the automorphisms of A form a group, written $Aut(A)$.*

3. *S is a* ***substructure*** *of a structure A if $S \subseteq A$ and the inclusion map $S \hookrightarrow A$ is an embedding. The substructure relation is written $S \leq A$; we write $S < A$ if $S \leq A$ and $S \neq A$.*

4. *Let $S \subseteq A$. Then the* ***induced substructure*** *of A on S, in symbols $A \restriction S$, is the structure with domain S and relations $R^{A \restriction S} = R^A \cap (S^n)$, for $R \in L$ with $ar(R) = n$.*

If L is a signature, and $\mathcal{K}$ is a class of L-structures, we always assume that $\mathcal{K}$ is closed under isomorphism. The reason for this restriction is that we are interested only in the isomorphism types of structures.

Definition 5 *Let L be a signature.*

1. *Let $\mathcal{K}(L)$ be the class of all L-structures.*

2. *Given $\mathcal{K} \subseteq \mathcal{K}(L)$, A is a $\mathcal{K}$-structure if $A \in \mathcal{K}$.*

3. *Given $\mathcal{K} \subseteq \mathcal{K}(L)$, let $\mathcal{K}_{fin}$ be the class of finite structures in $\mathcal{K}$.*

4. *Given a cardinal $\lambda > 0$, let $\mathcal{K}_\lambda$ be the class of $\mathcal{K}$-structures of order λ. $\mathcal{K}_\lambda$ is called the class of $\mathcal{K}$-structures of cardinality λ.*

We will be mainly interested in classes of structures defined by excluding certain finite L-structures, the so-called *constrained classes*. The main property of a constrained class $\mathcal{K}$ is that the induced substructures of $A \in \mathcal{K}$ are again in $\mathcal{K}$. Further, we will assume (unless otherwise stated) that L is finite. The key effect of the latter assumption is that for each $n \in \mathbb{N} - \{0\}$, $\mathcal{K}_n$ is finite (in fact, $|\mathcal{K}_n| \leq 2^{mn^\alpha}$, where $m = |L|$, and α is the maximum arity of a symbol of L as the reader can check).

Definition 6 *Let $A, B \in \mathcal{K}(L)$.*

1. *$A \hookrightarrow B$ if there is some embedding from A to B.*
2. *$A \sim B$ if $A \hookrightarrow B$ and $B \hookrightarrow A$.*

Remark 7

1. The relation $\hookrightarrow \subseteq \mathcal{K}(L) \times \mathcal{K}(L)$ is a pre-order; namely, it is a reflexive and transitive binary relation.
2. $\sim$ is an equivalence relation; the $\sim$-equivalence classes of $\mathcal{K}(L)_{fin}$ are isomorphism classes of finite $\mathcal{K}(L)$-structures. We will identify $\mathcal{K}(L)_{fin}$ with the isomorphism classes of $\mathcal{K}(L)_{fin}$. With this identification, $\mathcal{K}(L)_{fin}$ equipped with $\hookrightarrow$ becomes an order.

Definition 8 *Let $\mathcal{C} \subseteq \mathcal{K}(L)_{fin}$. Define*

$$\mathcal{K}(\neg\mathcal{C}) = \{B \in \mathcal{K} : C \not\hookrightarrow B, \text{ for each } C \in \mathcal{C}\}.$$

The members of $\mathcal{C}$ are called ***constraints*** *of $\mathcal{K}(\neg\mathcal{C})$.*

Remark 9 Given a class $\mathcal{K}(\neg\mathcal{C})$ we can always assume that $\mathcal{C}$ is an *antichain*; namely, the members of $\mathcal{C}$ are pairwise non-embeddable. If $\mathcal{C}$ is an antichain then each $M \in \mathcal{C}$ is *minimal;* that is, if $N < M$, then $N \in \mathcal{K}(\neg\mathcal{C})$.

Example 10

1. The class of directed graphs has a constraint the looped vertex. The class of simple graphs has as its contraints the looped vertex and the directed edge. The class of oriented graphs has the looped vertex and the undirected edge as its constraints.
2. The class of bipartite graphs has as its constraints those of the class of graphs and the odd cycles.
3. The class of orders has constraints as pictured in Figure **??**.

2.1 Unions of structures

In this subsection, we develop machinery to work with unions of structures.

Definition 11 *Let A, B be L-structures, so that A and B* ***agree****, that is,*

$$A \restriction A \cap B = B \restriction A \cap B$$

if $A \cap B \neq \emptyset$. The ***union*** *of A and B, $A \cup B$, is the L-structure with universe $A \cup B$, and with relations $R^{A \cup B} = R^A \cup R^B$, for $R \in L$. We also call $A \cup B$ the* ***free amalgam*** *of A and B over $A \cap B$.*

Remark 12

1. Let A,B, C, D be L-structures so that $C \leq A$ and $C \leq B$. By taking isomorphic copies we can assume that $A \cap B = C$. In this way, it makes sense to discuss the *free amalgam of A and B over C.*
2. Let $\mathcal{K} \subseteq \mathcal{K}(L)$. If $B, C \in \mathcal{K}$ agree, it does not necessarily follow that $B \cup C \in \mathcal{K}$. For example, let $\mathcal{K}$ be the class of orders, and let B and C be two-element chains that agree over the least element of B and the greatest element of C. However, $B \cup C$ fails to be transitive.

Definition 13 *Let $\mathcal{K}$ be a class of L-structures.*

1. *$\mathcal{K}$ is* ***closed under disjoint union*** *if for $A, B \in \mathcal{K}$ with $A \cap B = \emptyset$,*

   $$A \cup B \in \mathcal{K}.$$

   In this case we write $A \uplus B$ for $A \cup B$.

2. *$\mathcal{K}$ has the* ***free amalgamation property (FAP)*** *if for A, $B \in \mathcal{K}$ with $A \cap B \neq \emptyset$ and so A and B agree then*

   $$A \cup B \in \mathcal{K}.$$

3. *$\mathcal{K}$ is* ***closed under unions*** *if $\mathcal{K}$ is closed under disjoint union and has the free amalgamation property*

Remark 14 Closure under disjoint union and FAP are independent notions. For example, the class of trees (connected acyclic graphs) has FAP but is not closed under disjoint union; the class of orders is closed under disjoint union but does not have FAP.

We now introduce the notion of unions of several structures.

Definition 15 *Let $\{A_i : 1 \leq i \leq n\} \subseteq \mathcal{K}(L)$. For $n \geq 2$, we define $A = \bigcup_{i=1}^{n} A_i$ inductively.*

1. *For $n = 2$, if A_1 and A_2 agree, let $A = A_1 \cup A_2$.*
2. *For $k \geq 2$, assume $\bigcup_{i=1}^{k} A_i$ has been defined, so that $\bigcup_{i=1}^{k} A_i$ and A_{k+1} agree.*

   Define
$$A = \bigcup_{i=1}^{k+1} A_i = (\bigcup_{i=1}^{k} A_i) \cup A_{k+1}.$$

Remark 16 1. Let $\mathcal{K}$ be a free amalgamation class. If $\bigcup_{i=1}^{k+1} A_i$ in (2) of Definition 15 is defined, then it is in $\mathcal{K}$.

2. Let A_i, C be L-structures for $1 \leq i \leq n$, so that $C \leq A_i$, for all $i \in \{1, \ldots, n\}$. By taking isomorphic copies assume that for each $k \geq 1$,
$$\bigcup_{i=1}^{k} A_i \cap A_{k+1} = C.$$
   In this way we may define inductively the *free amalgam of* $\{A_i : 1 \leq i \leq n\}$ *over* C as in Definition 15.

A crucial tool we use is the notion of the graph of a structure.

Definition 17 *Let A be an L-structure. Define the* ***graph*** *of A, denoted by $G(A)$, to be the graph with vertices A, and edges $\{(x, y) : x, y \in A$ so that $x \neq y$ and there exists $R \in L$ and $\bar{a} \subseteq A$ so that $x, y \in \bar{a}$ and $\bar{a} \in R^A\}$.*

Example 18 1. If A itself is a graph, then $G(A) = A$.

2. If A is a digraph, then $G(A)$ results by forming the symmetric closure of the directed edges of A.
3. If A is an order, then $G(A)$ is the comparability graph of $A : xE^{G(A)}y$ iff $x < y$ or $y < x$.

3 Structural characterization of constrained classes closed under unions

Classes closed under unions are intimately linked to complete graphs via the following theorem.

Proposition 19 *Let $\mathcal{K}$ be a constrained L-class, with $\mathcal{K} = \mathcal{K}(\neg\{M_i : i \in I\})$ so that the M_i are an minimal (see Remark 9). Then the following are equivalent.*

1. *$\mathcal{K}$ is closed under unions.*

2. *For each $i \in I$, $G(M_i)$ is complete.*

Using Proposition 19, we list some examples of classes closed under union, and a few classes not closed under union.

(1) Classes closed under unions.

1. $\mathcal{K}(L)$ for L an arbitrary finite relational language; there are no minimal constraints.

2. Constrained classes of graphs closed under unions:

   (a) The class of all graphs: the minimal constraints have graphs K_1 and K_2 (see Example 10 (1)).

   (b) For $n \geq 3$, the classes of K_n-free graphs: the minimal constraints are the minimal constraints of graphs and K_n.

   By Proposition 19 there are no other constrained classes of graphs closed under union.

3. Constrained classes of directed graphs closed under unions.

   (a) The class of all directed graphs: the minimal constraint has graph K_1 (see Example 10 (1)).

   (b) The class of all oriented graphs: the minimal constraints have graphs K_1 and K_2 (see Example 10 (1)).

(c) A complete oriented graph is precisely a tournament. Hence, by Proposition 19, an constrained class of oriented digraphs closed under union must have constraints which are tournaments.

(d) The Henson classes of digraphs, defined by excluding a countable set of pairwise non-embeddable tournaments; see [9] (this gives $2^{\aleph_0}$ many examples).

(2) Classes not closed under unions.

(a) Orders: as in Example 10 (3) above, there is a minimal constraint of order 3 whose graph is the two path, P_2.

(b) Tournaments: $\overline{K_2}$ is the graph of one of the minimal constraints.

Proof of Proposition 19 ($1 \Rightarrow 2$)

We first show that for each $i \in I$, $G(M_i)$ is connected.

Fix $i \in I$. If $G(M_i)$ is not connected, then $G(M_i) = A \uplus B$. Then $M_i = (M_i \restriction A) \uplus (M_i \restriction B)$: for $R \in L$, $\bar{a} \in R^M$ iff $\bar{a} \in R^{M_i \restriction A}$ or $\bar{a} \in R^{M_i \restriction B}$. However, M_i is minimal so $M_i \restriction A$ and $M_i \restriction B$ are in $\mathcal{K}$. This is a contradiction as $\mathcal{K}$ is closed under disjoint unions. This proves Claim 1.

Fix $i \in I$ so that $M = M_i$ is not complete. Let $x, y \in M$ so that $\neg x E^{G(M_i)} y$ with x and y distinct.

Define $A = M \restriction M - \{x, y\}$, $B = M \restriction M - \{y\}$, and $C = M \restriction M - \{x\}$.

Then B and C agree: as $B \cap C = A$, it is enough to check that for each $R \in L$, $R^{B \restriction A} = R^{C \restriction A}$. But this is immediate as $B, C \leq M$.

But then $M = B \cup C$, since for each $R \in L$, $\bar{a} \in R^M$ iff $\bar{a} \in R^B$ and $y \notin \bar{a}$, or $\bar{a} \in R^C$ and $x \notin \bar{a}$. Since $\neg x E^{G(M)} y$ this is in turn equivalent to $\bar{a} \in R^{B \cup C}$.

Finally, as $G(M)$ is connected, $A \neq \emptyset$: x and y are not adjacent and so there must be some path connecting x to y. Hence, we may realize M as a free amalgam of proper substructures A, B, and C, all of which are in $\mathcal{K}$ by the minimality of M. This is a contradiction, as $\mathcal{K}$ is closed under union.

($2 \Rightarrow 1$) We first show $\mathcal{K}$ is closed under disjoint union.

Let B, $C \in \mathcal{K}$ so that $B \uplus C \notin \mathcal{K}$. Then for some $i \in I$, M_i embeds in $B \uplus C$; without loss of generality, we assume that $M_i \leq B \uplus C$. As B, C are in $\mathcal{K}$, $M_i \cap B \neq \emptyset$ and $M_i \cap C \neq \emptyset$.

Let $M_B = M_i \restriction M_i \cap B$ and $M_C = M_i \restriction M_i \cap C$. Then $M_i = M_B \uplus M_C$. But then $G(M_i)$ is disconnected, contradicting that $G(M_i)$ is complete.

We next show that $\mathcal{K}$ is closed under FAP. If not then there are $A, B, C \in \mathcal{K}$ so that $A = B \cap C$ and B and C agree, and there is some minimal constraint M_i of $\mathcal{K}$ that embeds in $B \cup C$.

Since B, $C \in \mathcal{K}$, $M_i \cap B$ and $M_i \cap C$ are nonempty.

Case i) $M_i \cap A = \emptyset$.

Then as above, $M_i = M_B \uplus M_C$, with the same contradiction as before.

Case ii) $M_i \cap A \neq \emptyset$.

Let $M_A = M_i \restriction M_i \cap A$; M_B and M_C are as above. Then $M_\triangle \leq \triangle$, for $\triangle \in \{A, B, C\}$.

We show that M_i is a free amalgam of M_B and M_C over M_A. M_B and M_C agree over M_A as $M_B, M_C \leq M_i$.

We show that $M_i = M_B \cup M_C$. We use the fact that $M_i \leq B \cup C$ and that for $R \in L$, $R^{B \cup C} = R^B \cup R^C$.For $R \in L$, $\bar{a} \in R^{M_i}$ iff $\bar{a}$ is from $M_i \cap B$ and $\bar{a} \in R^{M_i}$ or $\bar{a}$ is from $M_i \cap C$ and $\bar{a} \in R^{M_i}$. In turn, this is equivalent to $\bar{a} \in R^{M_B}$ or $\bar{a} \in R^{M_C}$, which itself is equivalent to $\bar{a} \in R^{M_B \cup M_C}$.

Hence, we can realize M_i as a free amalgam of M_B and M_C over M_A. But then in $G(M_i)$ there is no edge from some element of $M_B - M_A$ to any element of $M_C - M_A$, contradicting that $G(M_i)$ is complete. $\square$

4 Antichains in classes closed under unions

Definition 20 *Let $\mathcal{K}$ be a constrained class of L-structures.*

1. *$\mathcal{K}$ **has edges** if there is some $A \in \mathcal{K}$ so that $G(A)$ has edges.*

2. *If $\mathcal{K}$ has edges, suppose $A \in \mathcal{K}$ is such that $G(A)$ has edges. Assume $\bar{a} \subseteq A$ is such that $\bar{a}$ contains at least two distinct elements and there is some $R \in L$ with $\bar{a} \in R^A$. Then $A \restriction \bar{a} \in \mathcal{K}$ is called an **edge**. The **length** of $A \restriction \bar{a}$ is $|\bar{a}|$.*

3. *A class $\mathcal{C}$ of $\mathcal{K}$-structures is an* antichain *if distinct members of $\mathcal{C}$ are pairwise non-embeddable.*

The chordless cycles form an antichain in the class of graphs. In our next proposition, we generalize this fact to classes closed under unions with edges.

Proposition 21 *Let $\mathcal{K}$ be a class closed under unions and with edges. Then $\mathcal{K}$ contains a countably infinite antichain of finite structures.*

Proof. Fix A an edge, and let a, b be distinct elements of A. Define a set of finite L-structures

$$\{C_n(A) : n \geq 4 \text{ and } n \text{ even}\}$$

as follows. Fix $n \geq 4$ even. List n copies of A as $A_1, \ldots, A_n$. By taking isomorphic copies we may assume that for $1 \leq i < j \leq n$, $A_i \cap A_j \neq \emptyset$ if and only if $j = i + 1$, and for $1 \leq i \leq n - 1$,

$$A_i \cap A_{i+1} = \begin{cases} \{a\} & \text{if } i \text{ is odd,} \\ \{b\} & \text{else.} \end{cases}$$

Define $P_2(A)$ to be the free amalgam of A_1 and A_2 over $A_1 \restriction \{a\}$. Assume $P_n(A)$ has been defined for $n \geq 2$, so that the domain of $P_n(A)$ is $A_1 \cup \cdots \cup A_n$. Define $P_{n+1}(A)$ to be the free amalgam of $P_n(A)$ and A_{n+1} over

$$P_n(A) \restriction A_n \cap A_{n+1}.$$

Then $P_n(A) \in \mathcal{K}$, for each $n \geq 1$.

List copies of A_1 and $P_{n-1}(A)$ as B, C, respectively, so that

$$B \cap C = \{a\} \cup \{b\},$$

where $b \in A_1$ and $a \in A_{n-1}$. Define $C_n(A)$ be the free amalgam of B and C over $B \restriction \{a\} \cup \{b\}$. Then $C_n(A) \in \mathcal{K}$ (note that if $\mathcal{K}$ is the class of graphs with $A = K_2$, $C_n(A) = C_n$, the chordless n-cycle).

Claim $\{C_n(A) : n \geq 4$ and n even$\}$ is an antichain.

To see this fix m and n even positive integers ≥ 4, so that $m < n$. It is enough to show that $C_m(A)$ does not embed in $C_n(A)$.

Assume otherwise. List the copies of A in $C_m(A)$ as $A_1, \ldots, A_m$, and the copies of A in $C_n(A)$ as $B_1, \ldots, B_n$. A_1 must embed as some B_i by freeness; without loss of generality, we may assume A_1 embeds as B_1. Proceeding inductively, assume A_i embeds as B_i for each $1 \leq i \leq j$. Again by freeness, A_{j+1} must embed as B_{j-1} or B_{j+1}. Hence, A_{j+1} must embed as B_{j+1}.

Now, fix $x \in A_m - A_1$; in particular, x is adjacent (in $G(C_m(A))$ to some element of A_1. But then the image of x in $C_n(A)$, as an element of B_m, is not adjacent (in $G(C_n(A))$ to any element of B_1, by freeness and choice of m and n. This is a contradiction. □

5 Further constructions in classes closed under union

If $\mathcal{K}$ is closed under unions another consequence is that we may "delete edges" from $\mathcal{K}$-structures and remain in $\mathcal{K}$.

Definition 22 *Let $A \in \mathcal{K}(L)$, with $|A| \geq 2$. Let $x, y \in A$. Define A_{-xy} to be the L-structure with domain A and for $R \in L$,*

$$R^{A_{-xy}} = \{\bar{a} : \bar{a} \in R^A \text{ and } \{x, y\} \nsubseteq \bar{a}\}.$$

Lemma 23 *Let $\mathcal{K} \subseteq \mathcal{K}(L)$ be an constrained free amalgamation class, and let $A \in \mathcal{K}$. Then for all $x, y \in A$, $A_{-xy} \in \mathcal{K}$.*

Proof. Let $B = A \restriction A - \{y\}$, $C = A \restriction A - \{x\}$. Then B and C agree over $A \restriction A - \{x, y\}$. Further, $B \cup C = A_{-xy}$, as $B \cup C$ contains all of the relations of A except those involving x, y. $\square$

A further consequence of closure under union is that, under certain restrictions on $\mathcal{K}$, the class of graphs of $\mathcal{K}$-structures contains all triangle free-graphs.

Definition 24 *Let $\mathcal{K} \subseteq \mathcal{K}(L)$. Define*

$$G(\mathcal{K}) = \{G(A) : A \in \mathcal{K}\}.$$

Definition 25 *Let $\mathcal{K} \subseteq \mathcal{K}(L)$*

1. *$A \in \mathcal{K}$ is a 2-**edge** if A is a two-element $\mathcal{K}$-structure with graph K_2.*
2. *$\mathcal{K}$ **has a 2-edge** if there is some 2-edge $A \in \mathcal{K}$.*

Let $\mathcal{C}$ be the class of all triangle-free graphs.

Lemma 26 *Let $\emptyset \neq \mathcal{K} \subseteq \mathcal{K}(L)$ be an constrained free amalgamation class with 2-edge A and assume that there is a unique isomorphism type of one-element structure in $\mathcal{K}$. Then $\mathcal{C}_{fin} \subseteq G(\mathcal{K})$.*

PROOF. We proceed by induction; the induction hypothesis is that $\mathcal{C}_n \subseteq \mathcal{K}$, for $n \geq 1$.

$\mathcal{C}_1 = \{K_1\}$ can be realized as the graph of a one-element substructure of A. $\mathcal{C}_2 = \{\overline{K_2}, K_2\}$. $\overline{K_2}$ can be realized as the disjoint union of a structure in $\mathcal{K}$ realizing K_1 with itself; K_2 can be realized by A itself.

Let $B \in \mathcal{C}_{n+1}$. Then B is a 1-element extension of a $\mathcal{C}_n$-structure B', which, by induction, is realized as the graph of a $\mathcal{K}$-structure C'. Let $a \in B - B'$.

B is determined by the vertices X that a is adjacent to in B'. Note that if $X \neq \emptyset$, then X is independent: an edge in X will result in a triangle in B. Let $|X| = m \geq 1$.

Let $A = \{x, y\}$. Using closure under union in $\mathcal{K}$ we can form a $\mathcal{K}$-structure S_X, with domain $X \cup \{x\}$, so that $G(S_X)$ is the following rooted tree with m-leaves:

Let $A_1, \ldots, A_n$ be n copies of A.

Let $S_X(1) = A_1$.

Assume $S_X(k) \in \mathcal{K}$ has been defined for some $1 \leq k < n$.

Assume, by taking isomorphic copies, that $G(S_X(k))$ is a rooted tree with root x and that

$$S_X(k) \cap A_{k+1} = \{x\}.$$

Define

$$S_X(k+1) = S_X(k) \cup A_{k+1}.$$

Define

$$S_X = S_X(n).$$

As there is only one isomorphism type of one-element structures in $\mathcal{K}$, S_X and C' agree; hence, we can form the free amalgam of S_X and C' over $S_X \restriction X = C' \restriction X$. Then

$$G(S_X \cup C') = B.$$

□

6 Ne.c. structures

Definition 27 *Let $\mathcal{K}$ be a class of L-structures. Fix $n \geq 1$.*

1. *Let A, B, C be in $\mathcal{K}$ with $A \leq B$ and $A \leq C$. C is* ***realized in*** *A if there is an embedding f of C into B so that f is the identity on A.*

2. *$A \in \mathcal{K}$ is n**e.c.** if*

    (a) *A embeds each isomorphism type of one element structure in $\mathcal{K}$;*

    (b) *Let $B \leq A$ and let $B \leq C$ with $|B| = n$, $|C| = n + 1$, then C is realized in B.*

3. *$A \in \mathcal{K}$ is **strictly** n**e.c.** if A is ne.c. but there is some $r > n$ so that A is not re.c..*

Ne.c. structures have been studied primarily in the class of finite graphs. Caccetta et al [4] refer to such graphs as having property $P(n)$: for any two sets A and B with $A \cap B = \emptyset$ and $|A \cup B| = n$ there is a vertex $u \notin A \cup B$ that is joined to every vertex of A and not joined to any vertex of B. For a fixed $n \geq 1$, a sufficiently large Paley graph is ne.c. (see [1]).

Definition 28 *$A \in \mathcal{K}$ is **universal** if A embeds each countable structure in $\mathcal{K}$.*

Remark 29 For $\mathcal{K}$ a constrained class of L-structures closed under unions, if a countable structure $A \in \mathcal{K}$ is ne.c. for all $n \geq 1$, then A is universal and unique up to isomorphism. A is sometimes called the *Fraïssé limit of* $\mathcal{K}$ (see [5] for more on Fraïssé limits) or an e.c. graph. The Fraïssé limit of the class of graphs is known as the *random graph* (see [6] for a survey of results on the random graph).

In our next theorem we show that in constrained classes $\mathcal{K}$ closed under union, there exist ne.c. structures that are universal but not isomorphic to the Fraïssé limit of $\mathcal{K}$.

Theorem 30 *Let $\mathcal{K} \subseteq \mathcal{K}(L)$ be an constrained classes closed under unions with edges. Then for each $n \geq 1$, there is a strictly ne.c. universal structure in $\mathcal{K}$*

In the following proof we make use of the following observation about unions that we call "freeness".

Freeness If A and B agree over $A \cap B \neq \emptyset$, then in $G(A \cup B)$, there is no edge between a vertex of $A - A \cap B$ and a vertex of $B - A \cap B$.

Proof of Theorem 30. By hypothesis, we can choose an “edge” $A \in \mathcal{K}$ (see Definition 20)

Fix $a \in A$.

Define

$$1_a = A \restriction a \in \mathcal{K}.$$

Inductively define

$$(n+1)_a = (n)_a \uplus 1_a.$$

As $\mathcal{K}$ is closed under unions, $(n)_a \in \mathcal{K}$, for all $n \geq 1$.

Note that $(n)_a$ is an n-element structure so that

$$G((n)_a) = \overline{K_n}.$$

We define $M_n \in \mathcal{K}_{fin}$, $n \geq 1$ as the union of a chain of finite $\mathcal{K}$-structures M_n^k, $k \geq 0$ (and hence, $M_n \in \mathcal{K}$ as $\mathcal{K}$ is constrained).

Let

$$M_n^0 = (n+1)_a.$$

Assume that for $k \geq 0$, $M_n^k \in \mathcal{K}$ with M_n^0 a substructure of M_n^k.

Define M_n^{k+1} as follows. List the substructures S of M_n^k so that either

1. $M_n^0 \cap S \neq \emptyset$ and $|S| \leq n$; or
2. $M_n^0 \cap S = \emptyset$ and $|S| \leq k+1$;

as $S_1, \ldots, S_i$.

For each $1 \leq r \leq i$, list the isomorphism types of extensions of S_r to a $\mathcal{K}_{fin}$-structure of order $\leq n+1$ as $T_1, \ldots, T_j$ (there are only finitely many as L is finite).

Freely amalgamate $T_1, \ldots, T_j$ and M_n^k over S_r to obtain $M_{n,r}^k \in \mathcal{K}_{fin}$. Freely amalgamate $M_{n,1}^k, \ldots, M_{n,i}^k$ over M_n^k to obtain $(M_n^{k+1})' \in \mathcal{K}_{fin}$.

Form $M_n^{k+1} \in \mathcal{K}_{fin}$ by taking the disjoint union of $(M_n^{k+1})'$ with all isomorphism types of $\mathcal{K}_{k+1}$-structures (there are only finitely many).

Define

$$M_n = \bigcup_{k \geq 0} M_n^k.$$

By construction, M_n is ne.c. (every one-element extension of a $\mathcal{K}_n$-structure that embeds in M_n is realized in M_n). $M_n \restriction M_n - M_n^0$ is ke.c. for all $k \geq 1$ (each finite extension of a substructure of $M_n \restriction M_n - M_n^0$ is realized in $M_n \upharpoonright M_n - M_n^0$); by Remark 29, $M_n \upharpoonright M_n - M_n^0$, and hence, M_n, are universal.

We now form an extension $C \in \mathcal{K}_{fin}$ of M_n^0 that is not realized in M_n. Of course, by "C not realized in M_n" we mean that there is no isomorphism β from C onto a substructure of M_n so that $\beta \upharpoonright M_n^0 = id_{M_n^0}$.

Let $A_1, \ldots, A_{n+1}$ be $n+1$ copies of A.

Let $C \in \mathcal{K}_{fin}$ be the free amalgam of $A_1, \ldots, A_{n+1}$ over $A \upharpoonright A - \{a\}$.

It can be arranged that $C \geq M_n^0$. See Figure .... Observe that each vertex of $C - M_n^0$ is adjacent with each vertex of M_n^0 in $G(C)$.

We show that C is not realized in M_n by showing that C is not realized in M_n^k for all $k \geq 0$.

C is not realized in M_n^0 as

$$|C| > |M_n^0|.$$

Assume C is not realized in M_n^k, for $k \geq 0$.

(1) C is not realized in $(M_n^{k+1})' - M_n^k$.

To see this, we must check two cases.

Case 1 C is realized by an extension T of S so that $M_n^0 \cap S \neq \emptyset$ and $|S| \leq n$.

In this case, by freeness, each vertex of T in $G(M_n)$ is adjacent to at most n of the vertices of M_n^0, which is contrary to the choice of C.

Case 2 C is realized by an extension T of S so that $M_n^0 \cap S = \emptyset$ and $|S| \leq n$.

In this case, by freeness, each vertex of T in $G(M_n)$ is adjacent to none of the vertices of M_n^0.

(2) C is not realized in $M_n^{k+1} - (M_n^{k+1})'$.

(2) follows as no element of $M_n^{k+1} - (M_n^{k+1})'$ is adjacent (in the graph of M_n^{k+1}) to an element of M_n^0.

(1) and (2) together show that C is not realized in M_n, so that M_n does not satisfy $(|A| + n - 1)e.c.$: note that

$$|C| = |A| - 1 + n + 1 = |A| + n.$$

□

The results of Theorem 30 are in stark contrast to some other known results on ne.c. structures for constrained classes not closed under union. In particular, in was shown in [10] that every 4e.c. order is ne.c. for all $n \geq 1$. It is easy to show that every 2e.c. linear order is dense and without endpoints, so that every 2e.c. order is ne.c., for all $n \geq 2$.

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