Large Families of Mutually Embeddable Vertex-Transitive Graphs

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Abstract: For each infinite cardinal κ, we give examples of 2^κ many non-isomorphic vertex-transitive graphs of order κ that are pairwise isomorphic to induced subgraphs of each other. We consider examples of graphs with these properties that are also universal, in the sense that they embed all graphs with smaller orders as induced subgraphs.
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1. INTRODUCTION

All the graphs we consider are undirected and simple. We use the notation of [6] for graph theory, and [10] for basic facts on ordinals and cardinals. We work within ZFC; no additional set-theoretic axioms will be assumed unless stated otherwise.

The countably infinite random graph R is the unique countable graph, up to isomorphism, satisfying the *existentially closed* or *e.c. property*: for each pair of finite disjoint sets S and T of vertices, there is a vertex $z \notin S \cup T$ adjacent to each vertex of S, and not adjacent to any vertex of T. Furthermore, R is *homogeneous*: every isomorphism between finite induced subgraphs extends to an automorphism. The proof of these results uses the *back and forth* method, where a bijection and its inverse are constructed recursively at the same time; see [3]. To a neophyte, this is perhaps surprising: one's initial instinct is to first construct a function and then prove that it is bijective. The Cantor-Schröder-Bernstein theorem (see [1]) proves that mutual embeddings of sets imply the existence of a bijection between the sets. The analogous property is false for graphs, where the role of bijections is played by graph isomorphisms. We write $G \leq H$ if G is isomorphic to an induced subgraph of H; we then say that G *embeds* in H. We write $G \sim H$ if $G \leq H$ and $H \leq G$, and we write $G \cong H$ if G is isomorphic to H. A family $\{G_i\}_{i \in I}$ of graphs is called *mutually embeddable* if for every $i, j \in I$, $G_i \sim G_j$.

There exist many families of graphs containing non-isomorphic mutually embeddable members. For example, let X be the set of all infinite subsets of positive integers. Given $s \in X$, define G_s to be the disjoint union of each of the graphs P_{s_i}, where $s = \{s_i : i \in \mathbb{N}\}$. Then for distinct s and s' in X, $G_s \sim G_{s'}$ but $G_s \ncong G_{s'}$. Hence, $\mathcal{G}_X = \{G_s : s \in X\}$ is a mutually embeddable family of countable graphs, and $\mathcal{G}_X$ contains $2^{\aleph_0}$ many non-isomorphic graphs.

In the present article, we consider an open-ended question motivated in part by the examples in the previous paragraph: which graph properties guarantee that the existence of mutual embeddings implies the existence of an isomorphism? At the other extreme, we may also ask how many properties can graphs in a family share with R and still contain uncountable families of mutually embeddable graphs. For instance note that R is vertex-transitive while none of the graphs G_s of the previous paragraph are. Does there also exist examples of mutually embeddable non-isomorphic vertex-transitive graphs? We note that mutually embeddable *locally finite* vertex-transitive graphs are necessarily isomorphic. Indeed, if G and H have these properties, then G and H are regular, and the existence of mutual embeddings implies that they must have the same degree. Hence, an embedding of G into H acts locally as an isomorphism, and so it maps the connected components of G to isomorphic connected components of H. The Cantor-Schröder-Bernstein theorem then implies that G and H are isomorphic.

We will show that in contrast, for every infinite cardinal κ there are 2^κ many vertex-transitive mutually embeddable non-isomorphic graphs of order κ; see Theorems 3.1 and 3.2. Note that for an infinite cardinal κ, there are 2^κ many non-isomorphic graphs of order κ, so Theorems 3.1 and 3.2 are the best possible results in this direction. In general it is not a simple matter to prove that two graphs are not isomorphic. Our construction uses the weak Cartesian product of graphs (see [6]), and we use the unique factorization property of the product to show that the graphs are indeed non-isomorphic. To the authors knowledge, this is the first application of the unique factorization theorem for weak Cartesian products with infinitely many factors. In Theorems 4.1 and Corollary 4.1 we strengthen Theorem 3.1 to include graphs G that have the additional property, shared by R, of *universality*: G embeds each graph of order at most the order of G. However, our proofs of these results use the Generalized Continuum Hypothesis in the uncountable case.

2. THE WEAK CARTESIAN PRODUCT OF GRAPHS

The *Cartesian product* of a family $\{G_i\}_{i\in I}$ of graphs is the graph $\prod_{i\in I} G_i$ defined by

$$V\left(\prod_{i\in I} G_i\right) = \left\{ f : I \mapsto \bigcup_{i\in I} V(G_i) : f(i) \in V(G_i) \text{ for all } i \in I \right\},$$

$$E\left(\prod_{i\in I} G_i\right) = \{fg : \text{there exists } j \in I \text{ such that } f(j)g(j) \in E(G_j)$$
$$\text{and } f(i) = g(i) \text{ for all } i \neq j\}.$$

If I is infinite and all the factors G_i have at least two vertices, then $\prod_{i\in I} G_i$ is necessarily disconnected. The weak Cartesian product is an induced subgraph of the Cartesian product where vertices can differ in only finitely many coordinates. For $f \in V(\prod_{i\in I} G_i)$, the *weak Cartesian product* of $\{G_i\}_{i\in I}$ with *base* f is the subgraph $\prod^f_{i\in I} G_i$ of $\prod_{i\in I} G_i$ induced by the functions $g \in V(\prod_{i\in I} G_i)$ such that $\{i \in I : g(i) \neq f(i)\}$ is finite.

If the graphs $G_i, i \in I$, are connected, then $\prod^f_{i\in I} G_i$ is the connected component of $\prod_{i\in I} G_i$ containing f. Hence, a weak Cartesian product of connected graphs is necessarily connected. If for some infinite cardinal κ we have $|V(G_i)| \leq \kappa$ for all $i \in I$ and $|I| \leq \kappa$, then $|V(\prod^f_{i\in I} G_i)| \leq \kappa$.

A graph G is called *prime* if it cannot be expressed as a weak Cartesian product of graphs in a non-trivial way. The unique factorization properties of connected graphs under the weak Cartesian product can be summarized as follows.

Theorem 2.1 (see Theorem B.5 of [6]). *Every connected graph admits a unique representation as a weak Cartesian product of prime graphs. Furthermore for any* *isomorphism* ϕ *between* $G = \prod_{i\in I}^{f} G_i, H = \prod_{j\in J}^{g} H_j$, *where all factors are connected and prime, there exists a bijection* $\psi : J \mapsto I$ *and isomorphisms* $\psi_j : G_{\psi(j)} \mapsto H_j$, *for* $j \in J$ *such that for all* $f \in V(G)$,

$$\phi(f)(j) = \psi_j(f(\psi(j))).$$

We will also use the following:

Remark 1. *The weak Cartesian product of connected vertex-transitive graphs is vertex-transitive.*

This result is not hard to deduce from the fact that automorphisms of the factors applied coordinatewise yield an automorphism of the product. Sabidussi [11], Miller [7], and Imrich [4] give more information on the structure of the automorphism group of the Cartesian product of graphs. Imrich [5] was the first to note that the converse does not hold: there exist vertex-transitive connected graphs whose prime factors are not vertex-transitive. The construction is adapted to a family of suitable graphs in the next section.

3. THE CONSTRUCTION

Throughout, κ will be a fixed infinite cardinal, identified with the set $\{\alpha < \kappa : \alpha$ is an ordinal$\}$. We first define by transfinite induction a family of rooted trees $\{T_\alpha\}_{\alpha\in\kappa}$, as follows:

(i) T_0 is the one-vertex tree with root u_0.
(ii) $T_{\alpha+1}$ is obtained from two disjoint copies of T_α by adding a new root $u_{\alpha+1}$ adjacent to the two copies of the root u_α of T_α.
(iii) If α is a limit ordinal, then T_α is obtained from the disjoint union of $\{T_\beta\}_{\beta<\alpha}$ by adding a new root u_α adjacent to the copies of the root u_β of T_β for all $\beta < \alpha$.

The following lemma has a simple proof, but it is fundamental to our approach.

Lemma 3.1. *For ordinals* $0 \le \alpha < \beta < \kappa$, $T_\alpha \le T_\beta$, *but* $T_\alpha \ncong T_\beta$.

Proof. The ordinal α is an isomorphism invariant of T_α: it is the (ordinal) number of times we need to repeat the operation of removing all vertices of degree 1 in order to reduce T_α to a single vertex. Hence, $T_\alpha \ncong T_\beta$. Suppose that $T_\alpha \nleq T_\beta$. Then there exists a least ordinal $\gamma \le \beta$ such that $T_\alpha \nleq T_\gamma$. However, by the definition of T_γ, we have $T_\alpha < T_\gamma$ if γ is a limit ordinal, and $T_\delta < T_\gamma$ if $\gamma = \delta + 1$. Hence $T_\alpha \nleq T_\gamma$ implies $T_\alpha \nleq T_\delta$, which contradicts the minimality of γ. ■

We now use the weak Cartesian product to construct a family $\{G_\alpha\}_{\alpha<\kappa}$ of vertex-transitive graphs with the properties of the T_α described in Lemma 3.1.

Note that all trees are prime, since a Cartesian product of non-trivial graphs necessarily contains 4-cycles; hence, all the graphs T_α are prime. Following [6], we construct vertex-transitive weak Cartesian products whose prime factors are all isomorphic to the same T_α.

For $\alpha \in \kappa$, let $I_\alpha = \kappa \times V(T_\alpha)$, and define $f_\alpha : I_\alpha \mapsto V(T_\alpha)$ by $f_\alpha(\beta, v) = v$. Let $G_\alpha = \prod_{i\in I_\alpha}^{f_\alpha} T_\alpha$. Note that $|V(T_\alpha)| \leq \kappa$, hence $|V(G_\alpha)| \leq \kappa$.

Lemma 3.2. *The graphs G_α, where $\alpha \in \kappa$, are vertex-transitive, and for ordinals $0 \leq \alpha < \beta < \kappa$, $G_\alpha \leq G_\beta$, but $G_\alpha \not\cong G_\beta$.*

Proof. For every $g \in V(G_\alpha)$, $v \in V(T_\alpha)$, we have that

$$|g^{-1}(v)| = |f_\alpha^{-1}(v)| = \kappa.$$

Therefore, there exists a bijection from $g^{-1}(v)$ to $f_\alpha^{-1}(v)$. Fixing such a bijection for each $v \in V(T_\alpha)$, we get a bijection ϕ from the set I_α to itself, such that for every $(\gamma, v) \in I_\alpha$, $g(\gamma, v) = f_\alpha(\phi(\gamma, v))$. Let $\psi : V(G_\alpha) \mapsto V(G_\alpha)$ be the function defined by $\psi(f) = f'$, where $f'(\gamma, v) = f(\phi(\gamma, v))$. Then ψ is an automorphism of G_α, with $\psi(f_\alpha) = g$. This shows that G_α is vertex-transitive.

By Lemma 3.1, for $\alpha < \beta$, there exists an embedding $g_{\alpha,\beta}$ of T_α into T_β. Define a function $F_{\alpha,\beta} : G_\alpha \to G_\beta$ by $F_{\alpha,\beta}(f) = \hat{f}$, where

$$\hat{f}(\gamma, v) = \begin{cases} g_{\alpha,\beta} \circ f(\gamma, g_{\alpha,\beta}^{-1}(v)) & \text{if } v \in g_{\alpha,\beta}(T_\alpha); \\ v & \text{otherwise.} \end{cases}$$

To see that $F_{\alpha,\beta}$ is an embedding, suppose that f and g are adjacent in G_α. Hence, there is a $(\gamma, v) \in I_\alpha$ so that $f(\gamma, v)g(\gamma, v) \in E(T_\alpha)$, and $f(j) = g(j)$ for all $j \in I_\alpha \backslash \{(\gamma, v)\}$. Then $\hat{f}(\gamma, g_{\alpha,\beta}(v)) = f(\gamma, v)$ is adjacent to $\hat{g}(\gamma, g_{\alpha,\beta}(v)) = g(\gamma, v)$. As $\hat{f}$ and $\hat{g}$ are equal on all $j \in I_\alpha \backslash \{(\gamma, g_{\alpha,\beta}(v)\}$, they are adjacent in G_β. The verification that $F_{\alpha,\beta}$ is injective and preserves non-edges is similar, using the facts that each $g_{\alpha,\beta}$ is an embedding.

The prime factors of G_α are isomorphic to T_α and all those of G_β to T_β. Hence, by Theorem 2.1 and Lemma 3.1, we have that $G_\alpha \not\cong G_\beta$. ∎

Note that $T_0 \cong G_0$ is the graph with one vertex, which is the unit for the Cartesian product. We will discard this graph and base our construction on the other graphs in the family $\{G_\alpha : \alpha < \kappa\}$.

Let

$$X_\kappa = \{I \subseteq \kappa \backslash \{0\} : |I| = \kappa\}.$$

Then $|X_\kappa| = 2^\kappa$. We say that a set of ordinals S is *cofinal in* κ if for each ordinal $\gamma < \kappa$, there exists $\alpha \in S$, so that $\gamma < \alpha$. Note that each $I \in X_\kappa$ is cofinal in κ. If not, then there is an ordinal $\gamma < \kappa$ so that all $\alpha \in I$ satisfy $\alpha \leq \gamma$. But then $|I| \leq |\gamma| < \kappa$, which is a contradiction.

For $I \in X_\kappa$, we define $H_I = \prod_{\alpha \in I}^{f_I} G_\alpha$, where $f_I : I \mapsto \cup_{\alpha \in I} V(G_\alpha)$ is any function such that $f_I(\alpha) \in V(G_\alpha)$ for all $\alpha \in I$. (The vertex-transitivity of G_α makes the choice of $f_I(\alpha)$ irrelevant.) Since $|I| = \kappa$, we have that $|V(H_I)| = \kappa$.

Lemma 3.3. *The family $\{H_I\}_{I \in X_\kappa}$ is mutually embeddable and consists of vertex-transitive non-isomorphic graphs.*

Proof. The graphs H_I are all vertex-transitive by Remark 1. For $I, J \in X_\kappa$, we can define an injection $\phi : I \mapsto J$ such that $\phi(\alpha) > \alpha$ for all $\alpha \in I$. We define ϕ by transfinite induction. If S is a nonempty set of ordinals, then let $\min(S)$ be the least ordinal in S.

(i) If $\alpha_0 = \min(I)$, then let $\phi(\alpha_0) = \min(J \backslash (\{\alpha_0\}))$. Observe that $J \backslash (\{\alpha_0\}) \neq \emptyset$ since J is cofinal in κ.

(ii) If $\alpha + 1 \in I$, then define

$$\phi(\alpha + 1) = \min(J \backslash ((\alpha + 1) \cup \{\phi(\beta) : \beta \leq \alpha \text{ and } \beta \in I\})).$$

(iii) If α is a limit ordinal in I, then

$$\phi(\alpha) = \min(J \backslash (\alpha \cup \{\phi(\beta) : \beta < \alpha \text{ and } \beta \in I\})).$$

For all $\alpha \in I$, we can select, by Lemma 3.2 and since $G_{\phi(\alpha)}$ is vertex-transitive, an embedding $\psi_\alpha : G_\alpha \mapsto G_{\phi(\alpha)}$ such that $\psi_\alpha(f_I(\alpha)) = f_J(\phi(\alpha))$. The embedding $\psi : H_I \mapsto H_J$ is then defined by $\psi(f) = f'$, where

$$f'(j) = \begin{cases} \psi_{\phi^{-1}(j)}(f(\phi^{-1}(j))) & \text{if } j \in \phi(I); \\ f_J(j) & \text{otherwise.} \end{cases}$$

Observe that $\psi(f_I) = f_J$. To show that ψ is an embedding, consider an edge $fg \in E(H_I)$. Then there is a $j \in I$ so that $f(j)g(j) \in E(G_j)$ and $f(i) = g(i)$ for all $i \in I \backslash \{j\}$. Then $f'(\phi(j)) = \psi_j(f(j))$ is adjacent to $g'(\phi(j)) = \psi_j(g(j))$ in $G_{\psi(j)}$, since ψ_j is an embedding. As f' and g' are equal on all $i \in I \backslash \{\psi(j)\}$, they are adjacent in H_J. The verification that ψ is injective and preserves edges is similar, and so is omitted.

Thus, the graphs $\{H_I\}_{I \in X_\kappa}$ are mutually embeddable. However, for $I \neq J$, the prime Cartesian factors of H_I are those of $\{G_\alpha\}_{\alpha \in I}$, namely the trees $\{T_\alpha\}_{\alpha \in I}$, while those of H_J are the trees $\{T_\alpha\}_{\alpha \in J}$. Lemma 3.1 shows that H_I and H_J are not isomorphic. ∎

Our main theorem is the following.

Theorem 3.1. *For every infinite cardinal κ, there exist 2^κ many order κ, vertex-transitive, mutually embeddable, non-isomorphic connected graphs.*

Proof. Apply Lemma 3.3 along with the fact that $|X_\kappa| = 2^\kappa$. ∎

Several aspects of our construction can easily be modified. For instance, for every connected prime graph T and any infinite cardinal $\kappa \geq |V(T)|$, there exists a vertex-transitive graph $G(T, \kappa)$ which is a weak Cartesian product of copies of T and such that $|V(G(T, \kappa))| = \kappa$. The construction of $G(T, \kappa)$ parallels that of G_α given above. The chain $\{T_\alpha\}_{\alpha \leq \kappa}$ of graphs may also be replaced by certain other chains. For example, in step (ii) of the definition of the trees T_α, we may replace 2 by 3, or by n, if $n \geq 4$. If κ is a cardinal, then we say that a class $\mathcal{C}$ of graphs is a *κ-chain* if the following conditions hold.

1. If $G \in \mathcal{C}$, then $|V(G)| \leq \kappa$.
2. For each ordinal $\alpha \in \kappa \backslash \{0\}$, there is a unique graph $J_\alpha \in \mathcal{C}$.
3. If $0 \leq \alpha < \beta \in \kappa$, then $J_\alpha \leq J_\beta$, but $J_\alpha \not\cong J_\beta$.
4. The graphs in $\mathcal{C}$ are prime and connected.

By replacing the role of the trees T_α in Lemmas 3.2 and 3.3 by graphs in a κ-chain, we obtain the following strengthening of Theorem 3.1.

Theorem 3.2. *Let κ be a fixed infinite cardinal, and let $\mathcal{C}$ be a fixed κ-chain. There exists 2^κ many order κ, vertex-transitive, mutually embeddable, non-isomorphic connected graphs, each of whose prime factors are in $\mathcal{C}$.*

4. FAMILIES OF UNIVERSAL VERTEX-TRANSITIVE GRAPHS

A graph of order κ is *κ-universal* if it embeds all graphs of order at most κ as induced subgraphs. For example, the random graph R is $\aleph_0$-universal. This was first proved by Rado [8,9], who also proved that if the Generalized Continuum Hypothesis (GCH) holds, then κ-universal graphs exist for all $\kappa > \aleph_0$. (The (GCH) says that for every cardinal α, $\aleph_{\alpha+1} = 2^\alpha$.) Every family of κ-universal graphs is mutually embeddable. One question is whether there is a family of cardinality 2^κ consisting of non-isomorphic vertex-transitive graphs that are κ-universal. If so, then we say that such a family is *κ-good.*

The Cartesian product of *two* graphs G and H is written $G \square$ H.

Theorem 4.1. *Let $\kappa \geq \aleph_0$. If there is a κ-universal graph, then there is a κ-good family.*

Proof. Fix $\kappa \geq \aleph_0$ and let U be a κ-universal graph. Let T be the graph consisting of U and a vertex u adjacent to each vertex of U. Then T is a connected κ-universal graph. Furthermore, T is prime: the new vertex u is adjacent to every other vertex, therefore it is not contained in any induced 4-cycle. However every vertex of a non-trivial weak Cartesian product of graphs (with no isolated vertices) is contained in an induced 4-cycle.

As noted below the proof of Theorem 3.1 there exists a vertex-transitive graph $G(T, \kappa)$ of order κ which is a weak Cartesian product of copies of T; in particular, $G(T, \kappa)$ contains a copy of U. For a fixed $I \in X_\kappa$, define $U_I = G(T, \kappa) \square H_I$. By Remark 1, U_I is a κ-universal vertex-transitive graph whose prime factors consist of T and the trees T_α, where $\alpha \in I$. Since T contains cycles, $T \ncong T_\alpha$ for all α. The fact that there are 2^κ many non-isomorphic graphs of the form U_I now follows by Theorem 2.1. ■

Corollary 4.1. 1. *There is an $\aleph_0$-good family.*
2. *Assuming (GCH), there is a κ-good family for each $\kappa > \aleph_0$.*

Proof. For item (1), apply Theorem 4.1 using the infinite random graph R. For item (2), by [8,9] and by assuming (GCH), there is a κ-universal graph for each cardinal $\kappa > \aleph_0$. Now apply Theorem 4.1.

We close with the following question that we cannot answer.

Question. If G and H are mutually embeddable non-isomorphic graphs, then do G and H belong to an infinite family of mutually embeddable non-isomorphic graphs?

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