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Metrically Universal Generic Structures in Free Amalgamation Classes

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Abstract. We prove that each $\forall_1$ free amalgamation class $\mathcal{K}$ over a finite relational language L admits a countable generic structure M isometrically embedding all countable structures in $\mathcal{K}$ relative to a fixed metric. We expand L by infinitely many binary predicates expressing distance, and prove that the resulting expansion of $\mathcal{K}$ has a model companion axiomatized by the first-order theory of M. The model companion is non-finitely axiomatizable, even over a strong form of the axiom scheme of infinity.

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1 Introduction and preliminaries

As is well-known, FRAÏSSÉ's theorem (see Theorem 7.1.2 of HODGES [7]) proves that each $\forall_1$ class $\mathcal{K}$ with the amalgamation property (AP) and the joint-embedding property (JEP) has a countable universal homogeneous structure F. In this situation, $\mathcal{K}$ has a model companion axiomatized by the first-order theory of F.

In an $\forall_1$ class $\mathcal{K}$ over a finite relational language L, each structure S is an integral metric space, relative to the *least-path metric* on the graph of S (see Definition 2). In the present article, when $\mathcal{K}$ has the free amalgamation property we prove a "metric" analogue of FRAÏSSÉ's theorem by constructing a countable structure M which isometrically embeds all countable structures in $\mathcal{K}$. The general method is to extend L to L^+ by adding infinitely many predicates expressing distance. We show that $\mathcal{K}^+$ (the class of L^+ expansions of $\mathcal{K}$-structures) has a model companion whose theory is the theory of the expanded structure M^+ (Theorem 3). We present an explicit axiomatization of the model companion in Theorem 4 and prove that if $\mathcal{K}$ has edges (see Section 5), then the model companion is non-finitely axiomatizable even modulo a strong form of the axiom scheme of infinity. We note that our results generalize the results of MOSS [9]. We wish to emphasize that the advantage of our methods is that our results apply simultaneously to many diverse classes of combinatorial structures,

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including the classes of all graphs, oriented graphs, k-uniform hypergraphs, for $k \geq 3$, and the class of all L-structures for a given L.

We assume basic model theory as found in Hodges [7], especially material on existentially closed structures (Ch. 8) and Fraïssé limits (Ch. 7). Unless otherwise stated, $\mathcal{K}$ is a class of structures over a finite relational language with equality; $\mathcal{K}_{\text{fin}}$ is the class of finite structures in $\mathcal{K}$. We abuse notation and use A, for example, to denote both a structure and its universe; for structures A, B, $A \leq B$ means that A is a substructure of B. If A is a subset of B, $B \restriction A$ denotes the substructure of B defined by restricting the relations of B to A. The cardinality of a structure is the cardinality of its domain and is written $|A|$. Given a structure A and $\bar{a} \in A^n$, for some $n \geq 1$, we sometimes blur the distinction between $\bar{a}$ and its underlying set; the length of $\bar{a}$ is written $|\bar{a}|$. The arity of a relation symbol R is denoted $\text{ar}(R)$. All graphs are simple (that is, no loops or multiple edges).

As is well known, given an $\forall_1$ class $\mathcal{K}$, the e. c. models of $\mathcal{K}$ are contained in $\mathcal{K}$; further, every $\mathcal{K}$-structure of infinite cardinality κ can be embedded in an e. c. structure of cardinality κ.

If $\mathcal{K}$ is an $\forall_2$ class over a countable relational language, with only countably many isomorphism types of finite models, and whose finite models have AP and JEP, then there is a countable structure in $\mathcal{K}$, called the *$\mathcal{K}_{\text{fin}}$-generic M.* M is the unique (up to isomorphism) countable structure in $\mathcal{K}$ satisfying the following properties:

(G1) Every finite structure in $\mathcal{K}$ embeds in M.

(G2) Each finite set $S \subseteq M$ can be extended to a finite structure $S' \leq M$ with $S' \in \mathcal{K}$.

(G3) (Amalgamating into). For each finite $A \leq M$ with $A \in \mathcal{K}$, and each finite extension $A \leq B \in \mathcal{K}$, there is a finite $C \leq M$ and an isomorphism $\beta : B \longrightarrow C$ that is the identity on A.

We restate the following result from [8] that we will use later.

Proposition 1. *Let $\mathcal{K}$ be a class of finite L-structures. Assume A is $\mathcal{K}$-generic and $T = \text{Th}(A)$. If A is a saturated model of T, then T is model complete.*

Throughout the rest of the article, $\mathcal{K}$ is an $\forall_1$ class over a finite relational language L.

The next definition is crucial to our discussion.

Definition 1. Let A be an L-structure. Define the *graph of A*, denoted by $G(A)$, to be the graph with vertices $x \in A$ and edges (x, y), where $x, y \in A$ with $x \neq y$ so that there exist $R \in L$ and $\bar{a} \subseteq A$ so that $x, y \in \bar{a}$ and $R^A\bar{a}$.

Using the notion of a graph of a relational structure, we may classify relational structures via properties of their graphs. For example, we say that a structure is *connected* if and only if its graph is connected.

Definition 2.

1. Let G be a graph. Define d_G to be the usual least path metric on G so that $d_G(x, y) \leq n$ if and only if there exists a path from x to y having $\leq n$ edges, with value ∞ on pairs of vertices in different connected components of G.

2. Let $A \in \mathcal{K}(L)$. Define $d_A : A \times A \longrightarrow \omega \cup \{\infty\}$ by $d_A(x, y) = d_{G(A)}(x, y)$.

Note that since $d_{G(A)}$ is a metric, so is d_A.

Definition 3.

1. For $A \in \mathcal{K}$, take d_A as defined in Definition 2. Define 2-ary predicates $\{d_n : n \in \omega\}$ with d_n^A interpreted as tuples (x, y) with $d_A(x, y) = n$ (we interpret d_0 as equality).

2. Let $L^+ = L \cup \{d_n : n \in \omega\}$. For $A \in \mathcal{K}$, define $A^+ = \langle A, (d_n^A)_{n\in\omega}\rangle$. Define $\mathcal{K}^+ = \{A^+ : A \in \mathcal{K}\}$. $\mathcal{K}^+$ is the class of *distanced $\mathcal{K}$-structures*. For $A \in \mathcal{K}^+$, A^- denotes the reduct of A to L.

3. Let A, $B \in \mathcal{K}$. Then A is an *isometric substructure of* B if $A^+ \leq B^+$; we write this as $A \leq^{\mathrm{i}} B$.

4. $\mathcal{K}$ admits a *metrically universal countable structure* if there is a countable $M \in \mathcal{K}$ so that for each countable $A \in \mathcal{K}$, there is an $A' \leq^{\mathrm{i}} M$ so that $A \cong A'$.

In general, $\mathcal{K}^+$ is not $\forall_1$, but it is at worst $\forall_2$. (Recall that $\mathcal{K}$ is $\forall_1$.) In particular, the predicates d_n are L-definable in $\mathcal{K}^+$ by $\forall_2$ sentences; this axiomatization may be found in [3, Chap. 1.5].

We note that, in general, $A \leq B$ does not imply $A \leq^{\mathrm{i}} B$.

2 Good classes

We supply a sufficient condition for a metrically universal structure to exist in a class.

Definition 4. $\mathcal{K}$ is *good* if $\mathcal{K}^+_{\mathrm{fin}}$ satisfies JEP and AP, and $\mathcal{K}^+_{\mathrm{fin}}$ is large in $\mathcal{K}^+$, that is, every countable $\mathcal{K}^+$-structure is contained in the union of a chain of $\mathcal{K}^+_{\mathrm{fin}}$-structures.

Theorem 1. *If $\mathcal{K}$ is a good class, then $\mathcal{K}$ contains a countable metrically universal structure M.*

Proof. $\mathcal{K}^+_{\mathrm{fin}}$ has only countably many isomorphism types of finite structures. As $\mathcal{K}^+_{\mathrm{fin}}$ satisfies JEP and AP by hypothesis, there is a $\mathcal{K}^+_{\mathrm{fin}}$-generic $M^+ \in \mathcal{K}$. We claim that $(M^+)^-$ is the desired metrically universal structure.

Let $A \in \mathcal{K}^+$ countable. By largeness, there exists a $\leq$-chain of $\mathcal{K}^+_{\mathrm{fin}}$-structures $\{A_i : i \in \omega\}$ so that $A \leq \bigcup_{i\in\omega} A_i$. By an inductive use of the genericity of M^+ there is an $A' \leq M^+$ so that $\bigcup_{i\in\omega} A_i \cong A'$. More explicitly, by axiom (G1) of genericity, there is an embedding $f_0 : A_0 \longrightarrow M^+$. Assume that for $n \geq 0$ fixed, $f_n : A_n \longrightarrow M^+$ is an embedding. As $A_n \leq A_{n+1}$, A_n, $A_{n+1} \in \mathcal{K}^+_{\mathrm{fin}}$, by (G3) there is an embedding $f_{n+1} : A_{n+1} \longrightarrow M^+$ so that $f_{n+1} \restriction A_n = f_n$. Then $f = \bigcup_{i\in\omega} f_i : \bigcup_{i\in\omega} A_i \longrightarrow M^+$ is an embedding; choose $A' = f(\bigcup_{i\in\omega} A_i)$. □

2.1 Free amalgamation classes

Definition 5. Let A, B be L-structures so that $A \restriction (A \cap B) = B \restriction (A \cap B)$ if $A \cap B \neq \emptyset$. The *union of A and B*, $A \cup B$, is the L-structure with universe $A \cup B$ and with relations $R^{A\cup B} = R^A \cup R^B$, for $R \in L$.

We next introduce a form of the amalgamation property that will be the centerpiece of our discussions.

Definition 6. Let $\mathcal{K}$ be a class of L-structures.

1. $\mathcal{K}$ is *closed under disjoint union* if for A, $B \in \mathcal{K}$ with $A \cap B = \emptyset$, $A \cup B \in \mathcal{K}$, we write $A \uplus B$ for $A \cup B$ in this case.

2. $\mathcal{K}$ has the *free amalgamation property* if (a) $\mathcal{K}$ is closed under disjoint union, and (b) for A, $B \in \mathcal{K}$ with $A \cap B \neq \emptyset$, if $A \restriction (A \cap B) = B \restriction (A \cap B)$, then $A \cup B \in \mathcal{K}$; we say that $A \cup B$ is the *free amalgam of A and B.*

The aim of this subsection is to prove that a free amalgamation class is good, and therefore has a metrically universal structure. To accomplish this, we must extend several constructions of Moss [9] to relational structures.

Our first task is to isolate the notion of a *minimal path* in a relational structure. As we will see, the generalization of minimal paths from graphs to relational structures brings with it some new features.

We first need the notion of an *edge* in a relational structure.

Definition 7. For some $n \geq 2$, assume $\bar{a} \subseteq A \in \mathcal{K}$ is so that $\bar{a}$ contains at least two distinct elements and there is some $R \in L$ with $\bar{a} \in R^A$. Then $A \restriction \bar{a} \in \mathcal{K}$ is called an *edge.*

We may now define *minimal path.*

Definition 8. Let $A \in \mathcal{K}$ and a, $b \in A$.

1. A *path in A of length n from a to b* is any substructure of A of the form $P = A \restriction \bigcup_{1 \leq i \leq n} \overline{a_i}$, where $(\overline{a_i} : 1 \leq i \leq n)$ is a sequence of underlying sets of edges from A with the property that if $\overline{a_i}$ enumerates the subset S_i of A, then (a) $a \in S_1$ and $b \in S_n$, and (b) for $1 \leq i < j \leq n$, $S_i \cap S_j \neq \emptyset$ if and only if $j = i + 1$.

2. If a and b are in the same component of A, a *minimal path from a to b* is a path from a to b of length $d_A(a, b)$.

With notation as in Definition 8, $d_P(a, b) = d_A(a, b)$. Minimal paths from a to b may not be unique, and they need not be isomorphic. See Figure 1 for a pictorial representation of a path of length n from a to b.

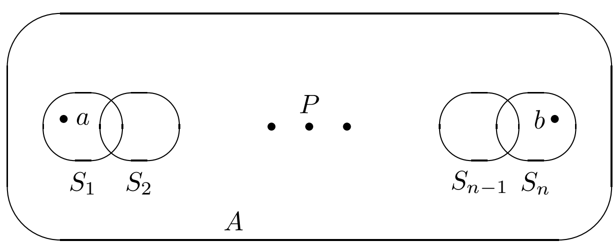


Figure 1. A minimal path P between a, b in A

With notation as in Definition 8 and a, b, $c \in P$, then it may happen that $d_A(a, b) < d_A(a, c) + d_A(c, b)$, even if P is a minimal path. For example, if $\mathcal{K}$ is the class of 3-uniform hypergraphs, A is the structure depicted in Figure 2 with circles representing hyperedges S_1, S_2, S_3 as shown, and our minimal path is $P = A$, then $d_A(a, b) = 3$ but $d_A(a, c) = d_A(c, b) = 2$.

Definition 9. Let $A \in \mathcal{K}$ and a, $b \in A$, and let P be a minimal path in A connecting a to b with domain $\{\overline{a_i} : 1 \leq i \leq n\}$. Assume that, for $1 \leq i \leq n$, $\overline{a_i}$ enumerates the subset S_i of A and that $\{\overline{a_i} : 1 \leq i \leq n\}$ is ordered successively. Then $c \in P$ is called *special* if either $c = a$ or $c = b$ or $c \in S_i \cap S_{i+1}$ for some $1 \leq i \leq n - 1$.

If A is a graph, then each vertex of a minimal path in A is special.

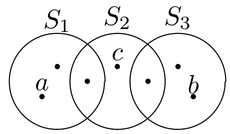


Figure 2. The 3-uniform hypergraph A

With notation as in Definition 9, choose a special $a_i \in S_i \cap S_{i+1}$ for $1 \le i \le n-1$. Then $\{a, a_1, \ldots, a_{n-1}, b\}$ is called a *set of special representatives of* P. The choice of a_i may not be unique. The path $aa_1 \ldots a_{n-1}b$ witnesses $d_A(a,b) = n$ in $G(A)$.

Lemma 1. *If A, a, b and P are as in Definition 9, and if $c \in P$ is special, then $d_A(a,b) = d_A(a,c) + d_A(c,b)$.*

Proof. If $c = a$ or $c = b$, the conclusion is immediate. Assume c is not equal to either a or b. Let $d_A(a,b) = n > 1$. Let $\{a, a_1, \ldots, a_{n-1}, b\}$ be a set of special representatives of P. Without loss of generality, we may assume that $c = a_i$ for some $1 \le i \le n-1$. As $aa_1 \ldots a_n c$ is a path in $G(A)$ from a to c, $d_A(a,c) \le i$. Similarly, $d_A(c,b) \le n - i$. If $d_A(a,c) < i$, let Q be a path between a and c witnessing this. Then Q adjoined with P restricted to the tuples of P connecting c to b is a path of length $< n$ from a to b in A, which is a contradiction. The same contradiction arises if $d_A(c,b) < n - i$. Hence, $d_A(a,c) = i$ and $d_A(c,b) = n - i$ as desired. □

The *canonical structure* of A over a finite subset B of A relative to a set of minimal paths of A is the structure built over $A \restriction B$ by freely adjoining the minimal paths representing the distances in A of tuples from B.

Definition 10. Let $A \in \mathcal{K}$ and $B \subseteq A$ finite. Define an L-structure as follows. List the distinct pairs from B as $\{(a_i, b_i) : 1 \le i \le n\}$. Define $\mathrm{can}_A(B)_0 = A \restriction B$. Let $1 \le k \le n$. Assume $\mathrm{can}_A(B)_{k-1}$ has been defined, is finite, and contains B. Let P_k be some minimal path in A between a_k, b_k of length ≥ 1, if a_k, b_k are connected or $\{a_k, b_k\}$ otherwise. Let $B_k = A \restriction P_k$. We may assume (by taking isomorphic copies if necessary) the elements of $B_k - \{a_k, b_k\}$ are neither in A nor $\mathrm{can}_A(B)_{k-1}$. Define $\mathrm{can}_A(B)_k$ to be the free amalgam of B_k with $\mathrm{can}_A(B)_{k-1}$ over $\{a_k, b_k\}$. Define $\mathrm{can}_A(B)$ *relative to* $\{P_k : 1 \le k \le n\}$ to be $\mathrm{can}_A(B)_n$.

When the choice of paths is irrelevant, we may abuse notation and write $\mathrm{can}_A(B)$. If A, B are as in Definition 10 and $\mathcal{K}$ is closed under unions, then $\mathrm{can}_A(B)$ is in $\mathcal{K}$. With notation as in Definition 10, $B \le \mathrm{can}_A(B)$.

Definition 11. Let $A \in \mathcal{K}$. Let $B \subseteq C$ in A, and define $\mathrm{can}_A(B)$ relative to a set of paths $\mathcal{P}$ in A. Define $\mathrm{can}_A(C)$ relative to a set of minimal paths from a to b in $\mathcal{P}'$ so that for a, $b \in B$ the minimal path in $\mathcal{P}'$ connecting a to b is the same as the minimal path in $\mathcal{P}$ connecting a to b. If $\mathrm{can}_A(C)$ is defined in this way, we say $\mathrm{can}_A(C)$ is *built over* $\mathrm{can}_A(B)$.

Lemma 2.

1. *For $B \subseteq A$, if x, $y \in B$, then $d_{\mathrm{can}_A(B)}(x,y) = d_A(x,y)$.*
2. *If we assume $\mathrm{can}_A(C)$ is built over $\mathrm{can}_A(B)$, then $\mathrm{can}_A(B) \le^{\mathrm{i}} \mathrm{can}_A(C)$.*
3. *If $B \le^{\mathrm{i}} A$, then $B \le^{\mathrm{i}} \mathrm{can}_A(B)$.*
4. *If A is connected, then $\mathrm{can}_A(B)$ is connected.*

P r o o f.

1. In the inductive step of the definition of $\mathrm{can}_A(B)$, minimal paths of elements not in A are added, with no relations added except relations on the minimal paths themselves.

2. By our assumptions, we certainly have that $\mathrm{can}_A(B) \le \mathrm{can}_A(C)$. We must show that for all $x, y \in \mathrm{can}_A(B)$,

$$d_{\mathrm{can}_A(B)}(x,y) = d_{\mathrm{can}_A(C)}(x,y). \tag{2.1}$$

C a s e 1. $d_{\mathrm{can}_A(B)}(x,y) < \infty$.

C a s e i. $x, y \in B$. Immediate by 1.

C a s e ii. x, y are not both from B. Let $B' = \mathrm{can}_A(B)$, $C' = \mathrm{can}_A(C)$. Without loss of generality, assume $x \in B' - B$. Hence, x is a member of a minimal path P added to B in B' connecting two elements of B. If $d_{B'}(x,y) = 1$, then $d_{C'}(x,y) = 1$ as $B' \le C'$. We can assume therefore that $d_{B'}(x,y) > 1$. To obtain a contradiction, assume that $n = d_{C'}(x,y) < d_{B'}(x,y)$. Let Q be a minimal path in C' witnessing $d_{C'}(x,y)$ with special representatives $\{x, a_1, \ldots, a_{n-1}, y\}$ (as $B' \le C'$, $n > 1$).

C l a i m. $a_1 \in B' - \{y\}$. As $d_{B'}(x,y) > 1$, $a_1 \ne y$. As P is freely adjoined to B in B', and C' is built over B', we have the crucial fact that x cannot be adjacent to an element of $C' - P$. Hence, $a_1 \in P \subseteq B'$. The claim follows.

Now, $d_{C'}(x,y) = d_{C'}(x,a_1) + d_{C'}(a_1,y) = d_{B'}(x,a_1) + d_{B'}(a_1,y) \ge d_{B'}(x,y)$ (the first equality follows by Lemma 1, the second equality by inductive hypothesis and by Case i – depending on whether a_1 and y are in B or not – the inequality by the triangle inequality in B'). Contradiction.

C a s e 2. $d_{\mathrm{can}_A(B)}(x,y) = \infty$. Say $x \notin B$ and x connected in $\mathrm{can}_A(B)$ to $x' \in B$. Then $d_{\mathrm{can}_A(B)}(x',y) = \infty$. If $d_{\mathrm{can}_A(C)}(x,y) < \infty$, then $d_{\mathrm{can}_A(C)}(x',y) < \infty$, so that $d_{\mathrm{can}_A(B)}(x',y) < \infty$ by previous cases. Contradiction.

3. Let $x, y \in B$. Then $d_B(x,y) = d_A(x,y) = d_{\mathrm{can}_A(B)}(x,y)$, (the first equality as $B \le^{\mathrm{i}} A$, by hypothesis, the second equality by 1). But then $B \le^{\mathrm{i}} \mathrm{can}_A(B)$.

4. Fix $x, y \in \mathrm{can}_A(B) = C$. We show that $d_C(x,y)$ is finite. If $x, y \in B$, then this follows from 1. Suppose x, say, is not in B. Then x is an element of a copy of some path from A adjoined to B. Hence, x is of finite distance from some $z \in B$. If $y \in B$, then $d_C(y,z)$ is finite by 1, and the result follows. If y is not in B, then y is of finite distance from some $z' \in B$, and the result follows. □

T h e o r e m 2. *If $\mathcal{K}$ is an $\forall_1$ L-class that is closed under unions, then $\mathcal{K}$ is a good class.*

P r o o f. We first prove the following Claim.

C l a i m 1. *$\mathcal{K}^+_{\mathrm{fin}}$ satisfies JEP and AP.*

P r o o f. JEP follows as $\mathcal{K}^+_{\mathrm{fin}}$ is closed under disjoint union. For $A, B, C \in \mathcal{K}^+_{\mathrm{fin}}$ so that $A = B \cap C$, define $D = (B^- \cup C^-)^+$. The verification that $B, C \le D$ is similar to the proof of Strong Amalgamation Lemma in [9]. Assume that $B \not\le D$ (the case when $C \not\le D$ is handled similarly). Let m be the least positive integer so that there exist $x, y \in B$ with

$$m = d_D(x,y) < d_B(x,y). \tag{2.2}$$

Note that $d_D(x,y) \le d_B(x,y)$ for every $x, y \in B$. Let $P \subseteq D$ be a minimal path witnessing (2.2) in D.

Case i. $P \subseteq B$. Then $d_D(x,y) = d_B(x,y)$, as B acquires no new relations in D. Contradiction.

Case ii. $P \subseteq C$. Then $x, y \in A = B \cap C$ and $d_C(x,y) = d_B(x,y)$, which is a contradiction.

Case iii. $P \nsubseteq B$ and $P \nsubseteq C$. Let $a \in P \cap (C - A)$, $b \in P \cap (B - A)$. Let P_1 be the "subpath" of P connecting a to b. More precisely, assume $P = \bigcup\{\overline{a_i} : 1 \le i \le n\}$, and there are $1 \le j \le k \le n$ so that $a \in \overline{a_j}$ and $b \in \overline{a_k}$. Define P_1 to be the path with domain $\bigcup\{\overline{a_i} : j \le i \le k\}$. By the definition of $B \cup C$, $P_1 \cap A \ne \emptyset$. Let $w \in P_1 \cap A$, $w \ne x, y$. We can choose w to be special. Otherwise, each tuple $\bar{c}$ in P_1 containing some $w \in P_1 \cap A$ with $w \ne x, y$ intersects tuples in P_1 not in A. But then such a $\bar{c}$ would contain elements from $B - A$ and $C - A$ contradicting the definition of the relations of $B \cup C$. Then $d_D(x,y) = d_D(x,w) + d_D(w,y) = d_B(x,w) + d_B(w,y) \ge d_B(x,y)$, (the first equality holds by Lemma 1, the second equality follows by induction hypothesis, and the inequality follows by the triangle inequality in B). But this contradicts (2.2). □Claim 1

Claim 2. *For each $A \in \mathcal{K}^+$ countable, there is a $\le$-chain of $\mathcal{K}^+_{\text{fin}}$ structures $\{A_i : i \in \omega\}$ so that $A \le \bigcup_{i\in\omega} A_i$.*

Proof. Enumerate A as $\{a_i : i \in \omega\}$. Define A_i to be $\text{can}_{A^-}(\{a_0, \ldots, a_i\})^+$ built over $\text{can}_{A^-}(\{a_0, \ldots, a_{i-1}\})^+$. $\{A_i : i \in \omega\}$ is a $\le$-chain by Lemma 2.2. $A \le \bigcup_{i\in\omega} A_i$ by the remarks after Definition 10 and Lemma 2.1. □Claim 2

This finishes the proof of the theorem. □

Corollary 1. *Let $\mathcal{K}$ be as in Theorem 2. Then $\mathcal{K}$ admits a metrically universal countable structure.*

Proof. By Theorem 1 and Theorem 2. □

3 The model companion of $\mathcal{K}^+$

Let L be finite relational, and let $\mathcal{K}$ be an $\forall_1$ free amalgamation class over L. Let $M^+ \in \mathcal{K}^+$ be the associated generic structure. We now show that the model companion of $\mathcal{K}^+$ exists and $\text{Th}(M^+)$ axiomatizes $(\mathcal{K}^+)^{\text{mc}}$. The main complication that now arises is that L^+ is infinite. We need a modified version of *quantifier-free type*.

Definition 12. Let $A \in \mathcal{K}^+_{\text{fin}}$ and $\bar{a}$ list n distinct elements from A for some $n \in \omega^*$. Then $\text{qft}'_{A,\bar{a}}(\bar{x})$ denotes the conjunction of all atomic and negated atomic $L \cup \{d_1, \ldots, d_N\}$-formulas satisfied by $\bar{a}$ in A, where N is the maximum distance between two elements in a component of A.

To accomplish our objectives, we first introduce T', the *extension axioms* for the connected $\mathcal{K}^+_{\text{fin}}$-structures. Next, we introduce axioms T'' capturing: "each finite subset is contained in a finite isometric substructure". As we will see, M^+ models both T' and T'', and each pair of countable *connected* models of $T' \cup T''$ are isomorphic.

Definition 13. Let $n \ge 1$.

1. Let Θ consist of the $\exists_1$ L^+-sentences $\exists\bar{x}\,\text{qft}'_{A',\bar{a}}(\bar{x})$, where $|\bar{x}| = |\bar{a}| = n$, $A' \in \mathcal{K}^+_n$, A' is connected, and $\bar{a}$ enumerates A', for each $n \ge 1$.

2. Define φ'_n to be the L^+-sentence $\bigwedge \forall \bar{x} \exists \bar{y} \,(\mathrm{qft}'_{A,\bar{a}}(\bar{x}) \rightarrow \mathrm{qft}'_{B,\bar{a}\bar{b}}(\bar{x},\bar{y}))$, where the conjunction ranges over all sets of the form $(A, B, \bar{a}, \bar{b})$ so that (a) A, B are isomorphism types of connected structures in $\mathcal{K}^+_{\mathrm{fin}}$ with $A \leq B$, and with $|A| \leq n$ and $|B| \leq n+1$; (b) $\bar{a}$ is a set of distinct elements from A so that $A = \bar{a}$ and $|\bar{x}| = |\bar{a}|$; (c) $\bar{b}$ is a set of distinct elements from B so that $B = \bar{a}\bar{b}$ and $|\bar{x}| + |\bar{y}| \leq n+1$.

3. Let $T' = \Theta \cup \{\varphi'_n : n \in \omega^*\}$.

Remark 1. If $S \in \mathcal{K}^+$ satisfies T', then the following condition holds: if $A \leq S$ with $A \in \mathcal{K}^+_{\mathrm{fin}}$ and $A \leq B \in \mathcal{K}^+_{\mathrm{fin}}$ so that A and B are connected, then there is a $C \leq S$ with $C \in \mathcal{K}^+_{\mathrm{fin}}$ and an isomorphism $f : B \longrightarrow C$ so that $f \restriction A$ is the identity.

Given $A \in \mathcal{K}_{\mathrm{fin}}$ and $S \subseteq A$, $|\mathrm{can}_A(S)|$ is bounded above not by a function of $|S|$, but rather by a function of the maximum index of a symbol d_n appearing in $\mathrm{qft}'_{A,S}(\bar{x})$. We make this important observation precise in the following definition and lemma.

Definition 14. Let $S \subseteq A \in \mathcal{K}_{\mathrm{fin}}$. Let $m(L)$ be the maximum arity of a symbol from L. Define $\alpha(S)$ to be the maximum index of a symbol d_n appearing in $\mathrm{qft}'_{A,S}(\bar{x})$, and ∞ else. Define $\beta(S) = |S| + \binom{|S|}{2} \alpha(S) m(L)$.

The following Lemma is immediate from the definitions.

Lemma 3. *Let $S \subseteq A \in \mathcal{K}$ with A connected. Then $|\mathrm{can}_A(S)| \leq \beta(S)$.*

Definition 15. Let $n \geq 1$. Define φ''_n to be the L^+-sentence $\bigwedge \forall \bar{x} \,(\mathrm{qft}'_{A,\bar{a}}(\bar{x}) \rightarrow \bigvee \exists \bar{y} \,\mathrm{qft}'_{B,\bar{a}\bar{b}}(\bar{x},\bar{y}))$, where the conjunction ranges over all sets of the form $(A, \bar{a})$ such that (a) A is an isomorphism type of connected structure in $\mathcal{K}^+_{\mathrm{fin}}$ so that $|A| \leq n$ and (b) $\bar{a}$ is a set of distinct elements from A, so that $|\bar{a}| \leq n$ and $|\bar{x}| = |\bar{a}|$; and the disjunction ranges over all sets of the form $(B, \bar{b})$ such that (a) B is an isomorphism type of connected structure in $\mathcal{K}^+_{\mathrm{fin}}$ with $|B| \leq \beta(\bar{a})$ and (b) $\bar{b}$ is a set of distinct elements from B, so that $B = \bar{a}\bar{b}$, $A \restriction \bar{a} = B \restriction \bar{a}$, and $|\bar{y}| = |\bar{b}|$. Let T'' be $\{\varphi''_n : n \in \omega^*\}$.

Remark 2. 1. For each $n \geq 1$, φ''_n is first-order. 2. Let C be a connected model of T''. Fix a finite (possibly disconnected) subset S of C contained in a component of C. Let $\bar{a}$ list the elements of S. Then, by Lemma 2.4., $A = \mathrm{can}_C(S)$ is a connected structure in $\mathcal{K}^+_{\mathrm{fin}}$, and it follows by Lemma 2.1 that $A \models \mathrm{qft}'_{C,\bar{a}}(\bar{a})$. Now, as $C \models T''$, S is contained in $S' \leq C$ with $S' \in \mathcal{K}^+_{\mathrm{fin}}$ connected and $|S'| \leq \beta(S)$.

Lemma 4. *Let A, $B \in \mathcal{K}^+$ be connected models of $T' \cup T''$. Then $A \equiv_{\infty\omega} B$; in particular, if A and B are countable, then $A \cong B$.*

Proof. We first note the following Claim, which follows from Remark 2, and which should help to clarify the role of T''.

Claim 1. *Let C be a connected model of T''. Then each finite $S \subseteq C$ is contained in $S' \leq C$ with $S' \in \mathcal{K}^+_{\mathrm{fin}}$ connected.*

We show that A and B are back-and-forth equivalent. To accomplish this, we first play a "modified" game whereby the two players, who we name $\forall$ and $\exists$, make moves by choosing certain *subsets* of A and B. More precisely, $\forall$ still chooses elements, while $\exists$ chooses finite isometric substructures. We then use this game to show that $\exists$ can always win the "usual" game where moves consist of choices of *elements* of A and B.

Claim 2. *If $\forall$ and $\exists$ have chosen $\{a_1, \ldots, a_n\} \subseteq A$, $\{b_1, \ldots, b_n\} \subseteq B$ and connected A_n, $B_n \in \mathcal{K}^+_{\mathrm{fin}}$ with $\{a_1, \ldots, a_n\} \subseteq A_n \leq A$, $\{b_1, \ldots, b_n\} \subseteq B_n \leq B$ and*

$A_n \cong B_n$, *then for each* $a_{n+1} \in A$ *there are* A_{n+1}, $B_{n+1} \in \mathcal{K}^+_{\text{fin}}$ *connected with* $A_n \cup \{a_{n+1}\} \subseteq A_{n+1} \leq A$, $B_n \subseteq B_{n+1} \leq B$ *and* $A_{n+1} \cong B_{n+1}$*; and for each* $b_{n+1} \in B$ *there are* $A_{n+1}, B_{n+1} \in \mathcal{K}^+_{\text{fin}}$ *connected with* $A_n \subseteq A_{n+1} \leq A$, $B_n \cup \{b_{n+1}\} \subseteq B_{n+1} \leq B$ *and* $A_{n+1} \cong B_{n+1}$.

Proof. Let $\{a_1, \ldots, a_n\}$, $\{b_1, \ldots, b_n\}$, A_n, B_n, a_{n+1} be as in the hypothesis of Claim 2. By Claim 1, there is $A_{n+1} \leq A$ with $A_{n+1} \in \mathcal{K}^+_{\text{fin}}$ connected so that $A_n \cup \{a_{n+1}\} \subseteq A_{n+1}$. As B satisfies T' and A_{n+1} is isomorphic to a connected finite extension of B_n, B realizes a copy of A_{n+1} extending B_n. Define B_{n+1} to be this copy of A_{n+1} in B. □Claim 2

Now, $\exists$ wins any usual game by using Claim 2. In the first step of play, no matter what element a $\forall$ chooses, as A and B satisfy Θ, $\exists$ can counter by choosing an element (in the other structure from where $\forall$ has chosen) with same isomorphism type as a. In general, if $\{a_1, \ldots, a_n\}$, $\{b_1, \ldots, b_n\}$, A_n, B_n have been chosen, and $\forall$ chooses an element a_{n+1} in A (say), then $\exists$ can counter by using A_{n+1} and B_{n+1} as in Claim 2, and then choosing an appropriate element of B_{n+1}.

The last conclusion of the lemma follows by the well-known fact that if A and B are countable, then $A \equiv_{\infty\omega} B$ implies $A \cong B$. □

Lemma 5. *Let* $B \in \mathcal{K}^+$ *and let* A *be a connected component of* B*. Then* $A \leq B$ *and* $A \in \mathcal{K}^+$.

Proof. As $B \in \mathcal{K}^+$, B contains paths between elements in a single component; hence, so does A. Thus, $A \in \mathcal{K}^+$ as there is a minimal path in A connecting each pair of distinct elements. Fix $a, b \in A$. If $n = d_B(a, b) < d_A(a, b)$, then a path witnessing n in B is already in A. Contradiction. □

Lemma 6. *Let* $\mathcal{K}$ *be as in Theorem* 2*. Then* $M^+ \models T' \cup T''$.

Proof. By genericity, $M^+ \models \Theta$. Now, let A, B, $\bar{a}$, $\bar{b}$ be as in the definition of φ'_n. Let $\bar{c}$ be a finite subset of M^+ so that $M^+ \models \text{qft}'_{A,\bar{a}}(\bar{c})$. Then $M^+ \restriction \bar{c} \cong A$. Now as A, $B \in \mathcal{K}^+_{\text{fin}}$, use (G3) to amalgamate B into M^+ over A. This proves that $M^+ \models T'$.

Further, $M^+ \models T''$. Given A, $\bar{a}$ be as in the definition of φ''_n, let $\bar{c}$ be a finite subset of M^+ so that $M^+ \models \text{qft}'_{A,\bar{a}}(\bar{c})$. Let $\bar{c} \subseteq C$, a finite isometric substructure of M^+ (we use (G2)).

Claim. *We can choose* C *connected.*

Proof. As A is connected, $\bar{c} \subseteq M_i^+$ for some component M_i^+ of M^+. Define $C_i = M_i^+ \restriction C \cap M_i^+$. But C_i is a connected component of C. By Lemma 5, $C_i \in \mathcal{K}^+$ and $C_i \leq C$. As $M_i^+ \leq M^+$, by Lemma 5, $C_i \leq M^+$. Thus we may let $C = C_i$. □Claim

Define $D = \text{can}_M(C)^+$ (relative to some set of minimal paths $\mathcal{P}$). Let $\mathcal{P}_0$ be the minimal paths from $\mathcal{P}$ connecting elements of $\bar{c}$, and let $B = \text{can}_M(\bar{c})^+$ relative to $\mathcal{P}_0$. Then B is connected by Lemma 2.4, $B \leq D$ by Lemma 2.2, clearly $B \in \mathcal{K}^+_{\text{fin}}$, and $|B| \leq \beta(\bar{c}) = \beta(\bar{a})$ by Lemma 3. By Lemma 3.3 we have $C \leq D$. So by (G3) we can amalgamate D into M^+ over C to get $D \leq M^+$. Thus $\bar{c} \subseteq B \leq M^+$, verifying that $M^+ \models T''$. □

Theorem 3. *Let* $\mathcal{K}$ *be as in Theorem* 2*. Then* $(\mathcal{K}^+)^{\text{mc}}$ *exists and is axiomatized by* $T = \text{Th}(M^+)$.

Proof. The proof of the following claim is immediate.

Claim. *Let $A \in \mathcal{K}^+$ so that $A \vDash T' \cup T''$. Then $A_i \vDash T' \cup T''$ for every component A_i of A.*

By the Claim and Lemma 4, every pair of components of a $\mathcal{K}^+$-structure that is a model of $T' \cup T''$ are $L^+_{\infty\omega}$ equivalent. In particular, by Lemma 6, the components of M^+ are pairwise isomorphic. Let the isomorphism type of any of these components by labelled N.

Let $A \in \mathcal{K}^+$ be countable so that $A \vDash T$. Then by Lemma 6, $A \vDash T' \cup T''$, and so $A = \bigcup_{1 \leq i \leq n} N_i$, with $1 \leq n \leq \omega$, and $N_i \cong N$ for $1 \leq i \leq n$. In particular, T has only countably many countable models. Hence, T is small (that is, T realizes only countably many types), so that there is a countable saturated model S of T.

We claim that $M^+ \cong S$, and so M^+ is saturated. To see this, let γ be the maximum number of components of any countable model of T. (Note that γ is a countable cardinal.) As M^+ is metrically universal, it embeds every countable model of T, so the number of components of M^+ is γ. Since S is saturated, S must have γ components: this follows since S embeds every countable model of T (see [7, Theorem 10.1.6]). In particular, S embeds M^+. Hence, by previous discussion, both S and M^+ are isomorphic to γ disjoint copies of N. Thus, $M^+ \cong S$. But then by Proposition 1, T is model complete.

To show mutual model consistency, it is enough to show that every $B \in (\mathcal{K}^+)^{\mathrm{ec}}$ is a model of T. Using well-known results on e. c. structures, we may assume B is countable. But then $B \leq M^+$. As T is model complete, T is $\forall_2$ (see, for example, [7, Theorem 8.3.3]). But then $B \vDash T$ as B is e. c. □

4 Axiomatization of the model companion of $\mathcal{K}^+$

Recall that we have shown so far that for $\mathcal{K}$ an $\forall_1$ free amalgamation class over a finite language, and for the generic $M^+ \in \mathcal{K}^+$, the model companion of $\mathcal{K}$ exists and is axiomatized by $\mathrm{Th}(M^+)$. We are now interested in presenting an explicit axiomatization of $\mathrm{Th}(M^+)$. Two cases emerge, reflecting the nature of $\mathcal{K}$: either every e. c. structure in $\mathcal{K}^+$ has more than one component, or there are connected e. c. structures in $\mathcal{K}^+$. This appears to be a new phenomenon: the class of distanced graphs does have connected e. c. structures; we will see some examples where there are no connected e. c. structures in $\mathcal{K}^+$ (see Subsection 4.2).

Definition 16. Let $A, B \in \mathcal{K}^+$ and fix $m \geq 1$, $n \geq 0$. Let $\bar{a} = (a_1, \ldots, a_m)$ enumerate a subset of A and $\bar{b} = (b_1, \ldots, b_m)$ enumerate a subset of B.

1. We write $(A, \bar{a}) \approx (B, \bar{b})$ if $A^- \vDash \mathrm{qft}_{B^-, \bar{b}}(\bar{a})$ and $B^- \vDash \mathrm{qft}_{A^-, \bar{a}}(\bar{b})$.
2. We write $(A, \bar{a}) \equiv_n (B, \bar{b})$ if $(A, \bar{a}) \approx (B, \bar{b})$ and for $1 \leq i, j \leq m$, $0 \leq k \leq n$, $A \vDash d_k(a_i, a_j)$ iff $B \vDash d_k(b_i, b_j)$.
3. If $A, B \in \mathcal{K}^+_{\mathrm{fin}}$, $\bar{a} = A$ and $\bar{b} = B$, then we write $A \equiv_n B$ if $(A, \bar{a}) \equiv_n (B, \bar{b})$.

Remark 3. *If $A, B \in \mathcal{K}^+_{\mathrm{fin}}$ and $A \equiv_n B$ for each $n \geq 1$, then $A \cong B$. If $A, B \in \mathcal{K}^+_{\mathrm{fin}}$ are connected and $n = \max(\alpha(A), \alpha(B))$, then $A \equiv_n B$ implies $A \cong B$.*

Definition 17. Define φ^*_n for $n \in \omega^*$ fixed, to be the L^+-sentence

$$\bigwedge \forall \overline{x} \exists \bar{y} \, (\mathrm{qft}'_{B, \bar{b}}(\bar{y}) \wedge \bigwedge_{x \in \overline{x}, y \in \bar{y}, 1 \leq j \leq n} \neg d_j(x, y)),$$

where the conjunction ranges over all sets of the form $(B,\bar b)$ so that B is an isomorphism type of connected structure in $\mathcal{K}^+_{\text{fin}}$ with B connected, $B=\bar b$, $|\bar y|=|\bar b|$, $1\le|\bar x|\le n$, $|\bar x|+|\bar y|\le n+1$. Let T^* be $\{\varphi^*_n : n\in\omega^*\}$.

Lemma 7. 1. $M^+\vDash T^*$. 2. $(\mathcal{K}^+)^{\text{ec}}\vDash T^*$.

Proof. For 1 let $A\subseteq M^+$ be finite, and let $B\in\mathcal{K}^+_{\text{fin}}$ be connected (disjoint from A). By genericity, there is $A'\in\mathcal{K}^+_{\text{fin}}$ with $A\subseteq A'\le M^+$. By taking isomorphic copies if necessary, we may assume M^+ and B are disjoint. Form $C=M^+\uplus B$. Then $A'\le A'\uplus B$ in C, and by genericity there is a $D\le M^+$ and an isomorphism $f: A'\uplus B\longrightarrow D$ so that f is the identity on A'. But then $B\cong f(B)\le M^+$ is not in the same component of M^+ with A', and hence, A. The assertion 2 follows from 1 and by Theorem 3. □

4.1 An axiomatization

Theorem 4. $\text{Th}(\mathcal{K}^+)\cup T^*\cup T'\cup T''$ *axiomatizes* $(\mathcal{K}^+)^{\text{mc}}$. *If there is a connected e. c. structure in* $\mathcal{K}^+$*, then* $\text{Th}(\mathcal{K}^+)\cup T'\cup T''$ *axiomatizes* $(\mathcal{K}^+)^{\text{mc}}$.

Proof. For the first assertion of the theorem, because of Theorem 3 and Lemma 7 it is enough to show that for $A\in\mathcal{K}^+$, $A\vDash T^*\cup T'\cup T''$ implies that A is e. c. in $\mathcal{K}^+$.

Let $A\le B\vDash\exists\bar x\theta(\bar x,\bar a)$, for $\bar a$ in A, $B\in\mathcal{K}^+$, $\theta(\bar x,\bar a)$ a quantifier-free L^+-formula; let $\bar b$ witness $\exists\bar x\,\theta(\bar x,\bar a)$ in B. Without loss of generality, we may assume that B is e. c., and hence, by Lemma 7, that B is a model of $T^*\cup T'\cup T''$. Decompose $\bar a$ as $\overline{a_1}\uplus\cdots\uplus\overline{a_n}$ with $\overline{a_i}$ in a component $A_i\le A$ for $1\le i\le n$. As $A_i\vDash T'\cup T''$, by Claim 1 in the proof of Lemma 4, there is a finite connected $C_i\le A_i$ containing $\overline{a_i}$. Let $\bar b=\overline{b_1}\cup\overline{b_2}$, where each element of $\overline{b_1}$ is of finite distance to some element of $\bar a$ and no element of $\overline{b_2}$ is of finite distance to any element of $\bar a$. Let $\overline{b_1}=\overline{e_1}\uplus\cdots\uplus\overline{e_n}$ be a component decomposition of $\overline{b_1}$ in B, where each element $\overline{e_i}$ is of finite distance to some element of $\overline{a_i}$ and $\overline{e_i}$ in a component $B_i\le B$ for $1\le i\le n$ (in particular, $A_i\le B_i$ in B). As $B_i\vDash T'\cup T''$ there is a finite connected $D_i\le B_i$ containing $C_i\cup\overline{e_i}$. As $A_i\vDash T'$ using the fact that $C_i\le D_i$, realize D_i inside A_i as D_i'. Let $\overline{e_i}'$ name the image of $\overline{e_i}$ inside D_i', and define $\overline{b_1}'=\overline{e_1}'\uplus\cdots\uplus\overline{e_n}'$. Note that $(A,\overline{b_1}')\equiv_m(B,\overline{b_1})$ for each $m\ge0$. Let s be the maximum index of the d_s's in $\theta(\bar x,\bar a)$. Let $\overline{b_2}=\overline{f_1}\uplus\cdots\uplus\overline{f_r}$ be a component decomposition of $\overline{b_2}$ in B, where each $\overline{f_i}$ in a component $S_i\le B$ for $1\le i\le r$. As before, each $\overline{f_i}$ is contained in some finite connected $T_i\le S_i$. Using T^* with $C=A\restriction D_1'\uplus\cdots\uplus D_n'\le A$ playing the role of $\bar x$, and T_1 playing the role of B, realize T_1 inside A as T_1' of distance $>s$ from elements of C. Proceeding inductively, assume $T_1,\dots,T_k$ have been realized in A as $T_1',\dots,T_k'$ of distance $>s$ from C and each other. Define $C_k=C\uplus\{T_1'\uplus\cdots\uplus T_k'\}$. Then there is some $C_k'\in\mathcal{K}^+_{\text{fin}}$ so that $C_k\subseteq C_k'\le A$ (let $X_1\uplus\cdots\uplus X_t$ be a component decomposition of C_k, with X_i in a component $E_i\le A$; as before, we can find $Y_i\in\mathcal{K}^+_{\text{fin}}$ so that $X_i\subseteq Y_i\le E_i$; then $A\restriction Y_1\uplus\cdots\uplus Y_t\le A$). Using T^* with T_{k+1} and C_k' realize T_{k+1} in A as T_{k+1}' of distance $>s$ from C_k'. Hence, we can realize the elements of $\{\overline{f_1},\dots,\overline{f_r}\}$ in A as elements of distance $>s$ to each element of $\bar a$ in A, and so that the realizations of $\overline{f_i}$ and $\overline{f_j}$ in A for $1\le i<j\le r$ are of distance $>s$ to each other and to $\overline{b_1}'$ in A. Label the realizations of $\{\overline{f_1},\dots,\overline{f_r}\}$ in A as $\overline{b_2}'$. Then $(A,\overline{b_1}'\overline{b_2}'\bar a)\equiv_s(B,\overline{b_1}\,\overline{b_2}\,\bar a)$. Hence, $A\vDash\theta(\overline{b_1}'\overline{b_2}',\bar a)$ so that $A\vDash\exists\bar x\,\theta(\bar x,\bar a)$. This proves that A is e. c. in $\mathcal{K}^+$.

For the final statement of the theorem, let A be a connected e. c. structure in $\mathcal{K}^+$.

Let $B \vDash \mathrm{Th}(\mathcal{K}^+) \cup T' \cup T''$. We show B is e. c. in $\mathcal{K}^+$. Let $B = \bigcup_{i\in I} B_i$ be a component decomposition. Then $B_i \vDash \mathrm{Th}(\mathcal{K}^+) \cup T' \cup T''$, for each $i \in I$. By Lemma 4, $B_i \equiv_{\infty\omega} A$. But then B_i is e. c., since it is well-known that the property of being e. c. is expressible as an $L_{\infty\omega}$ sentence. Thus, $B_i \vDash T^* \cup T' \cup T''$. The argument that B is e. c. is now similar to the above argument. □

Problem. Find a necessary and sufficient condition for a free amalgamation class to have connected distanced e. c. structures.

4.2 Examples

Our results now apply to the following classes. Each of the classes listed in 1) and 2) below satisfy the conclusions of Corollary 1 and Theorem 3. Closure under union may be checked directly.

1) Let $L = \{E\}$ with E 2-ary. Let $\mathcal{K}$ be one of the L-classes of graphs (recovering Moss [9, Corollary 5.2]), digraphs, oriented graphs, K_n-free graphs ($n > 2$), Henson digraphs ($\forall_1$ classes of oriented digraphs defined by excluding a set of finite mutually non-embeddable tournaments). Each of the above classes has a connected e. c. structure and so the model companion of $\mathcal{K}^+$ is axiomatized by $\mathrm{Th}(\mathcal{K}^+) \cup T' \cup T''$.

2) Let $\mathcal{K}$ be the class of graphs. Let $G \in (\mathcal{K})_{\mathrm{fin}}$. Define $C(G)$ to be the graphs that admit a homomorphism into G; $C(G)$ is called the class of *G-colourable graphs.* If G is K_n, the *complete graph of order n* (each pair of distinct vertices is adjacent), $C(G)$ is the class of n-colourable graphs. For more on these classes, see Hell and Nešetřil [5].

Let $|G| = n \geq 2$ with $G = \{1, \ldots, n\}$. Let $L = \{E, P_1, \ldots, P_n\}$ with $\mathrm{ar}(E) = 2$ and $\mathrm{ar}(P_i) = 1$ for $1 \leq i \leq n$. Define the L-class $C(G)$ to be the $\forall_1$ class defined by the axioms for simple graphs and $\forall x\,(\bigvee_{i=1}^{n} P_i(x))$, $\forall x\,(\bigwedge_{1\leq i<j\leq n} \neg(P_i(x) \wedge P_j(x)))$ and $\forall xy\,(P_i(x) \wedge P_j(y) \rightarrow \neg Exy)$ with $(i,j) \notin E^G$. $C(G)$ is the class of $\{E\}$-reducts of $C(G)'$; $C(G)'$ is the class of *G-coloured graphs.* $C(G)'$ has the free amalgamation property. Hence, Theorem 2 applies to $C(G)'$, in turn yielding a metrically universal countable generic $C(G)$-structure. $((C(G)')^+)^{\mathrm{mc}}$ may not have any connected e. c. structures. For example, if $G = A \uplus B$ with $A = K_3$ and B a 4-chromatic K_3-free graph (for example, choose B to be the Grötzsch graph; see [2, p. 231]), the reader can check that the members of $((C(G)')^+)^{\mathrm{mc}}$ are disconnected.

Another (and more trivial) example is to let L consist of a fixed finite number of unary predicates and let $\mathcal{K}$ be the class of all L-structures. In $\mathcal{K}$, every model has only singleton components.

5 The model companion of $\mathcal{K}^+$ is strongly nqfa

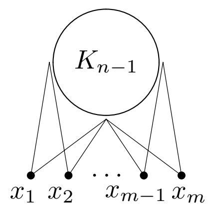
We say that a class $\mathcal{K}$ *has edges* if there is some $A \in \mathcal{K}$ so that $G(A)$ has edges. It follows from results we proved in [1], that if $\mathcal{K}$ is an $\forall_1$ free amalgamation class that has edges, then $\mathcal{K}$ has a model companion that is non-finitely axiomatizable modulo sentences asserting “I embed each finite structure in $\mathcal{K}$”. In the present section, we prove an analogous result for $\mathcal{K}^+$.

Theorem 5. *Let $\mathcal{K}$ be $\forall_1$ free amalgamation class that has edges. Then $(\mathcal{K}^+)^{\mathrm{mc}}$ is non-finitely axiomatizable modulo $\Gamma = \mathrm{Th}(\mathcal{K}^+) \cup T^* \cup T'' \cup \Theta$, where Θ is as in Definition* 13.1 (*in which case we say that $\mathcal{K}^{\mathrm{mc}}$ is strongly nqfa*).

If $B \in \mathcal{K}_{\text{fin}}$ is connected, $\text{diam}(B)$ is the *diameter of* $G(B)$ (the maximum value attained by the distance function). Recall that we are assuming that $\mathcal{K}$ is closed under unions.

L e m m a 8. *Fix an edge* $A \in \mathcal{K}$ *with* $|A| = n \geq 2$.

1. *For* $m \geq 1$, *there is* $S_m(A) \in \mathcal{K}$ *with graph as in Figure* 3.
2. *For all* $m \geq 2$, *there is* $S_{m,m}(A) \in \mathcal{K}$ *with graph as in Figure* 4.
3. *For all* $m \geq 2$, $S_m(A)$ *and* $S_{m,m}(A)$ *in items* 1 *and* 2 *may be constructed so that* $S_m(A) \leq^{\text{i}} S_{m,m}(A)$.



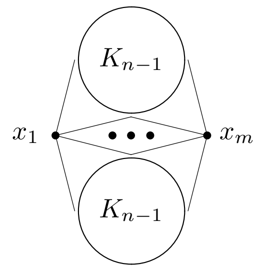


Figure 3. The graph of $S_m(A)$ Figure 4. The graph of $S_{m,m}(A)$

P r o o f . 1. Let $S_1 = A$. Fix $x \in A$. Assume S_m is defined. Let A_{m+1} be a copy of A with $x \in A_{m+1}$ represented by x_{m+1} disjoint from S_m. Let S_{m+1} be the free amalgam of S_m and A_{m+1} over $A \restriction A - \{x\}$. 2. Let $S_{m,m}$ be the free amalgam of S_m and S_m over $A \restriction \{x_1, \ldots, x_m\}$. 3. Immediate, as $\text{diam}(S_m) = 2$. □

R e m a r k 4. When A is clear from context, we write S_m and $S_{m,m}$ for $S_m(A)$ and $S_{m,m}(A)$, respectively.

5.1 Proof of Theorem 5

We define a set of countable structures $\{M_r : r \geq 1\}$ so that for each $r \geq 1$, $M_r \vDash \{\varphi'_r \wedge \neg\varphi'_{r+2n-1}\} \cup \Gamma$. If $(\mathcal{K}^+)^{\text{mc}}$ were not strongly nqfa, then by the Compactness Theorem and Theorem 4, $(\mathcal{K}^+)^{\text{mc}}$ would be axiomatized by $\Gamma \cup \{\varphi'_r\}$ for some $r \geq 1$. But then $M_r \in (\mathcal{K}^+)^{\text{mc}}$, contradicting the fact that $\text{Th}((\mathcal{K}^+)^{\text{mc}}) = \text{Th}(M^+)$ and $M^+ \vDash \varphi'_{r+2n+1}$.

For $r \geq 1$ define $M_r \in \mathcal{K}^+$ as follows. 1. Let $M_r^0 = S_{r+2}^+ \in \mathcal{K}^+$. 2. For $s \geq 0$ assume $M_r^s \in \mathcal{K}_{\text{fin}}^+$ and $M_r^0 \leq M_r^s$. 3. Form $M_r^{s+1''} \in \mathcal{K}_{\text{fin}}^+$ by forming all extensions of at most r-element substructures of M_r^s to at most $(r+1)$-element $\mathcal{K}^+$-structures (accomplished by iterated amalgamation); this makes sense as the number of isomorphism types of n-element $\mathcal{K}^+$-structures is finite for all $n \geq 1$. Let $M_r^{s+1'}$ be the disjoint union of $M_r^{s+1''}$ along with one isomorphic copy of every $\mathcal{K}^+$-structure of cardinality $\leq s+1$. Then $M_r^{s+1'} \in \mathcal{K}_{\text{fin}}^+$. Let $M_r^{s+1} = \text{can}_{(M_r^{s+1'})^-}(M_r^{s+1'})^+ \in \mathcal{K}_{\text{fin}}^+$ (relative to some set of paths). Note that $M_r^s \leq M_r^{s+1''} \leq M_r^{s+1'} \leq M_r^{s+1}$ (the last inequality following by Lemma 2.3). Let $M_r = \bigcup_{s \in \omega} M_r^s$. Then $M_r \in \mathcal{K}^+$ since $\mathcal{K}^+$ is $\forall_2$. By construction, $M_r \vDash \text{Th}(\mathcal{K}^+) \cup \{\varphi'_r\} \cup \Theta$, as the reader can check.

To see that $M_r \vDash T''$, let A, $\bar{a}$ be as in the definition of φ''_n, and let $\bar{c}$ be a finite subset of M_r so that $M_r \vDash \text{qft}'_{A,\bar{a}}(\bar{c})$. Let $s \geq 1$ be chosen so that $\bar{c} \subseteq M_r^s$. The restriction of M_r^{s+1} to a set of minimal paths connecting elements of $\bar{c}$ is then a connected substructure of M_r containing $\bar{c}$ with order $\leq \beta(\bar{a})$ (by Lemma 3).

To see that $M_r \models T^*$, let A be a finite subset of M_r and let $B \in \mathcal{K}^+_{\mathrm{fin}}$ be connected and disjoint from A with $|B| = s$. Then $A \subseteq M_r^t$ for some $t \geq 0$. Let $u = \max(s,t)$. Then B is realized in $M_r^{u+1'}$ in a different component of M_r than A.

Claim. *For $r \geq 2$, $M_r \not\models \varphi'_{r+2n-1}$.*

Consider the $\leq$-extension of M_r^0 by $C = S^+_{r+2,r+2}$. If $M_r \models \varphi'_{r+2n-1}$, then, as $M_r^0 \leq M_r$ and $M_r^0 \leq C$ (as $\mathrm{diam}(M_r^0) = 2$), M_r would realize the extension of M_r^0 by C. We find a contradiction.

The idea is to show inductively that C is not realized in the chain of structures defining M_r.

1. C is not realized in M_r^0 as $|C| > |M_r^0|$.

2. Assume C is not realized in M_r^s. (a) C is not realized in $M_r^{s+1''}$, since no element in $M_r^{s+1''} - M_r^s$ is adjacent in $G((M_r^{s+1''})^-)$ to more than r elements in M_r^s (amalgamation in $\mathcal{K}^+$ is not free, but is free in $\mathcal{K}$). (b) C is not realized in $M_r^{s+1'} - M_r^{s+1''}$, as C is connected. (c) C is not realized in $M_r^{s+1} - M_r^{s+1'}$, since elements in $M_r^{s+1} - M_r^{s+1'}$ are adjacent in $G((M_r^{s+1})^-)$ to at most two elements of M_r^0; but $r \geq 2$ so that there are elements in $C - M_r^0$ adjacent in $G(C^-)$ to at least 3 elements of M_r^0.

This completes the proof of Theorem 5. □

$A \in \mathcal{K}$ has the *finite model property* if every L-sentence satisfied by A has a finite model in $\mathcal{K}$. We do not know the answer to the following problem: Does M have the finite model property, even if $\mathcal{K}$ is the class of all graphs?

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