GEOMETRIC RANDOM GRAPHS AND RADO SETS IN SEQUENCE SPACES

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Abstract. We consider a random geometric graph model, where pairs of vertices are points in a metric space and edges are formed independently with fixed probability p between pairs within threshold distance δ. A countable dense set in a metric space is *Rado* if this random model gives, with probability 1, a graph that is unique up to isomorphism. In earlier work, the first two authors proved that, in finite dimensional spaces $\mathbb{R}^n$ equipped with the ℓ_∞ norm, all countable dense set satisfying a mild non-integrality condition are Rado. In this paper we extend this result to infinite-dimensional spaces. If the underlying metric space is a separable Banach space, then we show in some cases that we can almost surely recover the Banach space from such a geometric random graph. More precisely, we show that in the sequence spaces c and c_0, for measures μ satisfying certain conditions, $\mu^{\mathbb{N}}$-almost all countable sets are Rado. Moreover, with probability 1, in c as in c_0, all graphs obtained from the random geometric model with a randomly chosen dense countable vertex set are isomorphic to each other. Finally, we show that representatives of the isomorphism classes obtained in this way from c and c_0 are non-isomorphic to each other, and also non-isomorphic to their counterparts obtained from finite dimensional spaces.

1. Introduction

The well known binomial random graph is a probability space $G(\mathbb{N}, p)$ that consists of graphs with vertices $\mathbb{N}$, so that each distinct pair of integers is adjacent independently with a fixed probability $p \in (0, 1)$. Over fifty years ago, Erdős and Rényi [12] discovered that with probability 1, a graph $G \in G(\mathbb{N}, p)$ is isomorphic to a particular graph, the Rado graph R. Further, the almost sure isomorphism class does not depend on p for any fixed $0 < p < 1$. A graph G is *existentially closed* (or *e.c.*) if for all finite disjoint sets of vertices A and B (one of which may be empty), there is a vertex $z \notin A \cup B$ adjacent to all vertices of A and to no vertex of B. We say that z is *correctly joined* to A and B. The unique isomorphism type of a countably infinite e.c. graph is named the *infinite random graph*, or the *Rado* graph, and is written R. See Chapter 6 of [4] and the surveys [7, 8] for additional background on R.

Isomorphism types distinct from R but arising from infinite random geometric graphs were first considered in [5]. Consider a normed space X with norm $\|\cdot\|$, a positive real number δ, a countable subset V of X, and $p \in (0, 1)$. The *Local Area Random Graph* $\mathrm{LARG}(V, \delta, p)$ has vertices V, and for each pair of vertices u and v with $\|u - v\| < \delta$, an edge is added independently with probability p. In other words, we consider a random, constant-radius disk model on a subset of a metric space. The LARG model generalizes some well-known classes of random graphs. For example, special cases of the LARG model include the random geometric graphs (where the space is Euclidean, the vertex set is chosen uniformly at random

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from a bounded subspace, and $p = 1$), and the finite binomial random graph $G(n, p)$ (where the base set is a metric space of finite diameter, D, and $\delta > D$). Since V is required to be countable and we will focus on the case where V is dense in X, we require X to be separable.

A general question is the classification of sets and metric spaces for which the LARG model, like the random graph model, leads to a unique isomorphism type. The following notation from [2] is helpful. A countable dense set V in a normed space X is *Rado* if for all $\delta > 0$ and $p \in (0, 1)$, with probability 1, $\mathrm{LARG}(V, \delta, p)$ generates a unique isomorphism type of graph. The set V is *strongly non-Rado* if any two such graphs are, with probability 1, not isomorphic. A fundamental question when studying graphs generated by the LARG model is to determine which sets are Rado or strongly non-Rado.

For a real number $1 \le p \le \infty$ and $d \ge 1$ an integer, the vector space $\mathbb{R}^d$ of dimension d equipped with the metric derived from the p-norm is denoted by ℓ_p^d. If $p = \infty$, then in [5] it was shown that almost all countable dense sets are Rado (here and for the rest of this section, "almost all" refers to a suitable measure constructed in the paper). The proof used a back-and-forth argument, coupled with a geometric analogue of the e.c. property (see the next section). The unique countable limits in the d-dimensional case were named GR_d; for a fixed d, these graphs are all isomorphic regardless of the choice of δ or p. In contrast, it was also shown in [5] that if $p = 2$, there are sets in ℓ_p^2 which are strongly non-Rado. The latter result was significantly generalized in [2], which proved, using tools from functional analysis, that if X is a finite-dimensional normed space not isometric to ℓ_∞^d, then almost every random dense set V is strongly non-Rado.

A question posed at the end of [2] was to classify the Rado sets in the infinite dimensional case. As discussed in [6], the existence of Rado sets in sequence spaces is open and this problem is settled in the current paper.

Let c be the space of all convergent, real sequences equipped with the ℓ_∞ norm. Let c_0 denote the subspace of c consisting of sequences converging to 0. It is well known that c and c_0 are separable normed spaces. We prove in the following two theorems that $\mu^{\mathbb{N}}$-almost all countable dense sets in the space of convergent sequences c, and $\mu_0^{\mathbb{N}}$-almost all countable subsets of the subspace c_0 (consisting of sequences in c that converge to 0) are Rado, where μ and μ_0 are measures on c and c_0, respectively, satisfying certain natural conditions.

The proofs of the following theorems are deferred to Sections 4 and 5, respectively. See Section 3 for the definitions of non-aligned, and fully supported measures, but we comment here that any reasonable fully supported measure μ on c and any reasonable fully supported product measure μ_0 on c_0 has the non-aligned property. Briefly, a measure is non-aligned if each affine coordinate hyperplane is a null set, and by a product measure, we mean one where the coordinates are mutually independent (but not identically distributed).

Theorem 1. *There exists a graph, $GR(c)$ such that, for every non-aligned fully supported measure μ on c, for $\mu^{\mathbb{N}}$-almost every countable subset V of c, for each $p \in (0, 1)$,* $\mathrm{LARG}(V, 1, p)$ *is almost surely isomorphic to $GR(c)$.*

Theorem 2. *There exists a graph, $GR(c_0)$, such that for every non-aligned fully supported measure μ of product type on c_0, for $\mu^{\mathbb{N}}$-almost every countable set V, for each $p \in (0, 1)$,* $\mathrm{LARG}(V, 1, p)$ *is almost surely isomorphic to $GR(c_0)$.*

The above theorems (as well as those of the previous paper [5]) show that if X in one of the Banach spaces ℓ_∞^d, c or c_0, there exists a local Rado graph $GR(X)$ such that for many countable dense subsets V of X, and any $p \in (0, 1)$, $\mathrm{LARG}(V, 1, p)$ is almost surely isomorphic to $GR(X)$. The following theorem, proved in section 6, shows that these local Rado graphs are mutually non-isomorphic: one can recover the underlying Banach space from the local Rado graph.

Theorem 3. *Suppose that X and Y are real Banach spaces with local Rado graphs $GR(X)$ and $GR(Y)$. If $GR(X)$ is isomorphic to $GR(Y)$, then X is isometrically equivalent to Y.*

Throughout, all graphs considered are simple, undirected, and countable unless otherwise stated. We will encounter two distinct notions of distance: metric distance (derived from a given norm) and graph distance. In a normed space, we write $\|u-v\|$ for the metric distance of the points. For a graph G, we write $d_G(u,v)$ for the graph distance. Given a normed space X, denote the (open) *ball of radius δ around x* (or *δ-ball*) by

$$B_\delta(x) = \{u \in X : \|u - x\| < \delta\}.$$

In large parts of the paper, we will take $\delta = 1$. In that case, we will use $B(x)$ instead of $B_1(x)$. A subset V is *dense* in X if for every point $x \in X$, every ball around x contains at least one point from V. We use the notation $\lfloor t \rfloor$ for the integer part of $t \in \mathbb{R}$, and let $\langle t \rangle = t - \lfloor t \rfloor$.

For $a \in \mathbb{R}$, we write c_a for the subset of c consisting of sequences converging to a. If x is a point of c, then we write $(x_n)_{n\in\mathbb{N}}$ for the terms of the sequence defining x. We use the notation x_∞ to denote $\lim_{n\to\infty} x_n$. We use the notation $x \sim_G y$ to denote that x and y are adjacent in G. If it is clear from the context which graph is meant, then we omit the subscript and write $x \sim y$.

For a reference on graph theory the reader is directed to [9, 18], while [3] is a reference on normed spaces.

2. Geometrically Existentially Closed graphs

Let $G = (V, E)$ be a graph whose vertices are points in the normed space X. The graph G is *geometrically existentially closed at level δ* (or *δ-g.e.c.*) if for all δ' so that $0 < \delta' < \delta$, for all $x \in V$, and for all disjoint finite sets A and B so that $A \cup B \subseteq B_\delta(x)$, there exists a vertex $z \notin A \cup B \cup \{x\}$ so that

(i) z is correctly joined to A and B,
(ii) $A \cup B \subseteq B_\delta(z)$,
(iii) $\|x - z\| < \delta'$.

This definition implies that V is dense in itself, since we may choose A and B to be empty. Further, if G is δ-g.e.c., then G is δ'-g.e.c. for any $\delta' < \delta$. If V is dense in a normed space, then by scaling, without loss of generality, we may assume that $\delta = 1$, and we refer to 1-g.e.c. simply as g.e.c.

The g.e.c. property bears clear similarities with the e.c. property defined in the introduction. The important differences are that a correctly joined vertex must exist only for sets A and B which are contained in an open ball with radius δ and centre x, and it must be possible to choose the vertex z correctly joined to A and B arbitrarily close to x; see Figure 1.

The following result demonstrates that graphs generated by the LARG model are almost surely g.e.c. Note that this result applies broadly to *all* normed spaces, including c and c_0.

Theorem 4 ([5])**.** *Let $(X, \|\cdot\|)$ be a normed space and V a countable dense subset of X. If $p \in (0,1)$, then with probability* 1, LARG$(V, 1, p)$ *is g.e.c.*

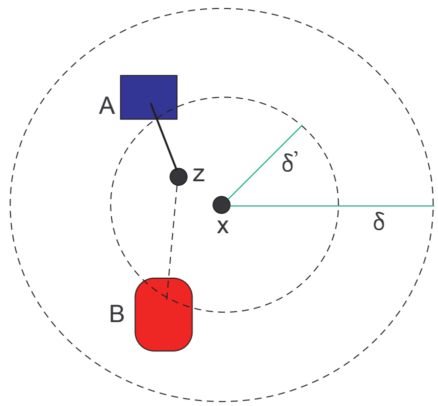

FIGURE 1. The δ-g.e.c. property.

Theorem 4 gives us a tool to prove that a set is Rado. In particular, we use a back-and-forth argument to show that g.e.c. graphs on the set are isomorphic.

The *length* of an edge uv is the distance between its endpoints, i.e. $\|u - v\|$. A countable graph that is g.e.c. and is such that all its edges have length less than δ, is called a *geometric δ-graph*. By definition, a graph G generated by LARG$(V, 1, p)$ has no edges of length more than 1, and, if V is countable and dense in X, then G is almost surely g.e.c. Thus, this random graph model generates geometric 1-graphs.

The following important theorem demonstrates that there exists a close relationship between graph distance and metric distance in any graph that is a geometric 1-graph.

Theorem 5 ([5])**.** *Let $(X, \|\cdot\|)$ be a normed space and $G = (V, E)$ be a geometric 1-graph with $V \subseteq X$. Suppose that $\overline{V}$ is convex. For any $u, v \in V$ so that $\|u - v\| \geq 1$, we have that*

$$d_G(u, v) = \lfloor \|u - v\| \rfloor + 1.$$

If $\|u - v\| < 1$, then

$$d_G(u, v) = \begin{cases} 0 & \text{if } u = v; \\ 1 & \text{if } u \sim v; \\ 2 & \text{otherwise.} \end{cases}$$

Theorem 5 directly leads to the following corollary.

Corollary 6 ([5])**.** *If $\overline{V}$ and $\overline{W}$ are convex, and there are geometric 1-graphs with vertex sets V and W, respectively, which are isomorphic via f, then for every pair of vertices $u, v \in V$,*

$$\lfloor \|u - v\| \rfloor = \lfloor \|f(u) - f(v)\| \rfloor.$$

Corollary 6 suggests the following generalization of isometry. Given metric spaces (S, d_S) and (T, d_T), sets $V \subseteq S$ and $W \subseteq T$, a *step-isometry* from V to W is a bijective map $f : V \to W$ with the property that for every pair of vertices $u, v \in V$,

$$\lfloor d_S(u, v) \rfloor = \lfloor d_T(f(u), f(v)) \rfloor.$$

Every isometry is a step-isometry, but the converse is false, in general.

3. Random infinite sets in c and c_0

We will show that sets that are dense in c or c_a are Rado if they satisfy certain mild properties, which we now define. A subset $A \subseteq c$ is *coordinate-wise integer distance free* or *i.d.f.* if for all $n \in \mathbb{N} \cup \{\infty\}$, $x_n \notin \mathbb{Z}$ for all $x \in A$ and $x_n - y_n \notin \mathbb{Z}$ for all distinct $x, y \in A$ (where x_∞ denotes $\lim_{n\to\infty} x_n$ as mentioned above). A subset $A \subseteq c_a$ is *i.d.f.* if for all $n \in \mathbb{N}$, $x_n \notin \mathbb{Z}$ for all $x \in A$, and $x_n - y_n \notin \mathbb{Z}$ for all distinct $x, y \in A$. Note that if a set is i.d.f. in c, it is i.d.f. in c_a, but the converse is false, since all sequences in c_a have the same limit.

When dealing with c_a, we need an additional second property (as will be explained in Section 5.) A countable subset of c_a $\{x^{(1)}, x^{(2)}, \dots\}$ satisfying the i.d.f. condition is said to have the *independent order property* (or i.o.p.) if for any finite sub-collection of the points, $x^{(i_1)}, \dots, x^{(i_k)}$, and any finite collection of linear orders $\prec_1, \dots, \prec_N$ of $\{1, \dots, k\}$, there exist distinct positions $j_1, \dots, j_N \in \mathbb{N}$ such that for all $\ell \in \{1, 2, \cdots N\}$,

$$m \prec_\ell n \text{ if and only if } \langle x^{(i_m)}_{j_\ell} \rangle < \langle x^{(i_n)}_{j_\ell} \rangle.$$

Let (μ_n) be a sequence of probability measures on $\mathbb{R}$, each continuous and fully supported. We construct a random element, x, of $\mathbb{R}^{\mathbb{N}}$ by independently sampling each coordinate x_n from the measure μ_n. Providing we impose suitable constraints on the μ_n (for example their distribution is symmetric around 0 and their variances are summable), the resulting point, x, is almost surely in c_0. To obtain a random point in c, we can just add the value of an independently sampled single continuous fully supported random variable taking values in $\mathbb{R}$ to each coordinate of x.

Theorem 7. *If μ is a measure on X (one of c or c_0) as described above, then $\mu^{\mathbb{N}}$-a.e. of $X^{\mathbb{N}}$ has the i.d.f. and i.o.p. properties.*

The proof, given for specific constructions of suitable μ, is contained in the appendix.

4. Proof of Theorem 1: almost all sets in c are Rado

We will show isomorphism of geometric 1-graphs in the spaces we consider by inductively constructing an isomorphism. By Corollary 6, we know that any isomorphism must be a step isometry. Thus, in the inductive construction, we need to make sure that the constructed map is a step-isometry. The following lemma gives sufficient conditions for a map on c to be a step-isometry.

Lemma 8. *Let A, B be i.d.f. subsets of c, and let $f\colon A \to B$ be a bijection. If the following conditions are satisfied, then f is a step-isometry:*

1. *$\lfloor f(x)_i \rfloor = \lfloor x_i \rfloor$ for each $x \in A$ and each $i \in \mathbb{N}$;*
2. *For each $i \in \mathbb{N}$ and $x, y \in A$, we have that $\langle f(x)_i \rangle < \langle f(y)_i \rangle$ if and only if $\langle x_i \rangle < \langle y_i \rangle$.*

Proof. Let A be a countable i.d.f. subset of c. Notice that if $x > y$ and $\langle x \rangle \neq \langle y \rangle$, then

$$\lfloor |x - y| \rfloor = \begin{cases} \lfloor x \rfloor - \lfloor y \rfloor & \text{if } \langle x \rangle > \langle y \rangle; \\ \lfloor x \rfloor - \lfloor y \rfloor - 1 & \text{if } \langle x \rangle < \langle y \rangle. \end{cases}$$

Clearly, analogous statements hold if $y < x$ by swapping the roles of x and y.

Suppose that $f\colon A \to B$ is a bijection satisfying the hypotheses (1) and (2). If $x, y \in A$, then notice that $f(x)_i < f(y)_i$ if and only if $x_i < y_i$, and that $\langle f(x)_i \rangle < \langle f(y)_i \rangle$ if and only if $\langle x_i \rangle < \langle y_i \rangle$, so that $\lfloor |x_i - y_i| \rfloor = \lfloor |f(x)_i - f(y)_i| \rfloor$.

Note that, if we consider $x \in c$ such that $x_\infty \notin \mathbb{Z}$, then it is evident that there is an M such that for all $i \geq M$, $\lfloor x_i \rfloor = \lfloor x_\infty \rfloor$. Since A and B satisfy the i.d.f. property, the limits of x and y are not integers, and thus $\lfloor \|x - y\| \rfloor = \max_i \lfloor |x_i - y_i| \rfloor$ and $\lfloor \|f(x) - f(y)\| \rfloor = \max_i \lfloor |f(x)_i - f(y)_i| \rfloor$, so that $\lfloor \|f(x) - f(y)\| \rfloor = \lfloor \|x - y\| \rfloor$. □

The conditions can be loosely stated as follows. For each $i \in \mathbb{N}$, conditions (1) and (2) state that the i-th coordinate of the sequences have the same integer part, and their remainders are similarly ordered. This implies that the induced coordinate map which maps the i-th coordinate of every function in A to the i-th coordinate of the image of the function, is a step isometry. Since we are working with the ℓ^∞ norm, this immediately implies that f is a step isometry, but the condition is indeed stronger.

The proof of Theorem 1 follows directly from the following theorem, which states that all g.e.c. geometric 1-graphs obtained from i.d.f. sets in c are isomorphic. The unique isomorphism type is the graph $GR(c)$ mentioned in Theorem 1.

Theorem 9. *Let V and W be dense, countable i.d.f. sets in c. If G and H are g.e.c. geometric 1-graphs with vertex sets V and W, respectively, then $G \cong H$.*

Proof. We use the usual technique for proving that two countably infinite graphs are isomorphic: a back-and-forth argument. That is, to show that two countably infinite graphs G and H are equivalent, we inductively construct increasing sequences of finite subsets $V_t \subseteq V$ and $W_t \subseteq W$, together with a sequence of functions $\varphi_t \colon V_t \to W_t$ such that φ_t is an extension of φ_{t-1}, and has the property that for $x, y \in V_t$, x and y are adjacent in G if and only if $\varphi_t(x)$ and $\varphi_t(y)$ are adjacent in H. Throughout the proof, if $x \in V$, then we denote by x' its image in W (that is, if $x \in V_t$, then $x' = \phi_t(x) \in W_t$).

We just outline the forth part, as the argument going back is analogous. Let $V = \{x^{(1)}, x^{(2)}, \ldots\}$ and $W = \{\xi^{(1)}, \xi^{(2)}, \ldots\}$. The induction hypothesis requires not only that the partial maps constructed form an isomorphism between the relevant subgraphs, but also that the maps satisfy the coordinate-wise step isometry properties from Lemma 8. This will ensure that we can indeed invoke the g.e.c. condition to extend the map to a new vertex. Namely, the partial maps are step isometries, and thus the images under the partial map of the graph neighbours of a given vertex v are all contained in the 1-ball around the image of v. In addition, we maintain that the conditions from Lemma 8 also hold for the limits, x_∞ and x'_∞, so that the partial map also induces a step isometry on the limit elements.

Precisely, we maintain the following inductive hypotheses throughout the construction for a given $t \geq 1$.

(1) $x^{(t)} \in V_t$ and $\xi^{(t)} \in W_t$. Also, if $t > 0$ then $V_{t-1} \subseteq V_t$ and $W_{t-1} \subseteq W_t$.
(2) The map φ_t is a graph isomorphism from the subgraph $G[V_t]$ to $H[W_t]$. Moreover, φ_t extends φ_{t-1}.
(3) The following holds for all points x and y in V_t:

    For all $i \in \mathbb{N} \cup \{\infty\}$, we have that
    (a) $\lfloor x_i \rfloor = \lfloor x'_i \rfloor$
    (b) $\langle x_i \rangle < \langle y_i \rangle$ if and only if $\langle x'_i \rangle < \langle y'_i \rangle$.

If the induction hypothesis holds for all $t \in \mathbb{N}$, then by (1) and (2), the map $\varphi : V \to W$ defined as $\bigcup_{t \in \mathbb{N}} \varphi_t$ is an isomorphism, and the conclusion follows. Note that, by Lemma 8, condition (3) implies that φ_t is a step-isometry from V_t to W_t.

Set $V_0 = W_0 = \emptyset$.

For the induction step, let $x = x^{(t+1)}$. If $x \in V_t$, then there is nothing to be done, and we can proceed with the 'back'-step, which involves finding a pre-image for $\xi^{(t+1)}$. If not, then let $V_{t+1} = V_t \cup \{x\}$. Form the sequences $\ell = (\ell_1, \ell_2 \dots)$ and $u = (u_1, u_2, \dots)$ as follows. For all $i \in \mathbb{N}$,

$$\begin{aligned} u_i &= \min\big(\{1\} \cup \{\langle z_i' \rangle : z \in V_t \text{ and } \langle z_i \rangle > \langle x_i \rangle\}\big), \text{ and} \\ \ell_i &= \max\big(\{0\} \cup \{\langle z_i' \rangle : z \in V_t \text{ and } \langle z_i \rangle < \langle x_i \rangle\}\big). \end{aligned}$$

Also define

$$\begin{aligned} u_\infty &= \min\big(\{1\} \cup \{\langle z_\infty' \rangle : z \in V_t \text{ and } \langle z_\infty \rangle > \langle x_\infty \rangle\}\big), \text{ and} \\ \ell_\infty &= \max\big(\{0\} \cup \{\langle z_\infty' \rangle : z \in V_t \text{ and } \langle z_\infty \rangle < \langle x_\infty \rangle\}\big). \end{aligned}$$

That is, in each coordinate i, the remainder of x_i lies between ℓ_i and u_i, and no other coordinates from sequences in V_t have remainders (in the i-th coordinate) between ℓ_i and u_i. The elements ℓ_∞ and u_∞ are defined so that the remainder of x_∞ lies between ℓ_∞ and u_∞. The notation ℓ_∞ and u_∞ suggests that these values are the limits of the sequences (ℓ_i) and (u_i). We now claim and prove that this is indeed the case.

Claim. *The sequences u and ℓ defined above are in c. Moreover, $\lim_{n\to\infty} u_n = u_\infty$ and $\lim_{n\to\infty} \ell_n = \ell_\infty$.*

Proof of Claim. By the i.d.f. property, for all distinct w and z in V_t, we have $\langle w_\infty \rangle \neq \langle z_\infty \rangle$. Also $\langle x_\infty \rangle$ is distinct from $\langle z_\infty \rangle$ for each $z \in V_t$ and $0 < \langle x_\infty \rangle < 1$. Hence, $F_\infty = \{\langle z_\infty \rangle : z \in V_t\} \cup \{0, 1, \langle x_\infty \rangle\}$ consists of $|V_t| + 3$ elements. Since F_∞ contains 0 and 1, $\langle x_\infty \rangle$ is neither the largest nor the smallest element of the set. For all sufficiently large i, the ordering of $F_i = \{\langle z_i \rangle : z \in V_t\} \cup \{0, 1, \langle x_i \rangle\}$ is identical to the ordering of F_∞ (that is if z_i is the kth largest element of F_i for some $z \in V_t$, then z_∞ is the kth largest element of F_∞). Since u_i is the next largest element of F_i after x_i and ℓ_i is the next smallest element of F_i before x_i, we deduce that $u_i \to u_\infty$ and $\ell_i \to \ell_\infty$ as required. □

To continue the proof of the theorem, we will first form a *target sequence* $y \in c$. The idea is to choose a sequence whose remainders are trapped above ℓ and below u coordinate-wise. If we would set $\varphi_{t+1}(x) = y$, then the extended function would satisfy condition (3) of the induction hypothesis, i.e. the map would be a coordinate-wise step isometry. The target sequence may not be in W, but we can use a density argument to find a sequence in W sufficiently close to our target sequence which satisfies condition (3) of the induction hypothesis, and also condition (2).

Let's define the target sequence. For all $i \in \mathbb{N}$, define

$$y_i = \lfloor x_i \rfloor + \frac{\ell_i + u_i}{2}.$$

Then by properties of limits, $y = (y_1, y_2, \dots)$ converges to $y_\infty = \lfloor x_\infty \rfloor + \frac{\ell_\infty + u_\infty}{2}$. As stated above, if we could choose $\varphi_{t+1}(x) = y$, then condition (3) of the induction hypothesis would be satisfied. However, we need to find an image that also fulfills the other conditions. Specifically, the image of x must be in W, and it must be correctly joined to the elements in W_t to maintain the isomorphism condition. Thus, we will define a ball around y with the property that each element in this ball will maintain condition (3). We can then invoke density and the g.e.c. condition (which holds by Theorem 4) to find another point in this ball that qualifies for the induction step.

Define $\alpha = \frac{u_\infty - \ell_\infty}{6}$. Note that $\alpha < 1$ by definition. Now we have that $|u_\infty - \langle y_\infty \rangle| = 3\alpha$ and $|\ell_\infty - \langle y_\infty \rangle| = 3\alpha$. As ℓ and u are convergent sequences, there is a positive integer M such that for all $i \geq M$, $|u_i - u_\infty| < \alpha$, $|\ell_i - \ell_\infty| < \alpha$, and $\lfloor y_i \rfloor = \lfloor x_i \rfloor = \lfloor x_\infty \rfloor$. Thus, we also have that $|\langle y_\infty \rangle - \langle y_i \rangle| = |y_\infty - y_i| < \alpha$. Then for all $i \geq M$ we have by the triangle equality that

$$\begin{aligned} |\langle y_i \rangle - u_i| &\geq |\langle y_\infty \rangle - u_\infty| - |u_i - u_\infty| - |\langle y_\infty \rangle - \langle y_i \rangle| \\ &\geq 3\alpha - \alpha - \alpha \\ &= \alpha. \end{aligned}$$

By a symmetric argument, for all $i \geq M$ we have that $|\ell_i - \langle y_i \rangle| \geq \alpha$. Therefore, for all $i \geq M$ and $t \in \mathbb{R}$ such that $|t - y_i| < \alpha$, we have that $\ell_i < t < u_i$.

Next, let $\alpha' < \alpha$ be such that

$$2\alpha' \leq \min\{u_i - \ell_i : 1 \leq i < M\}.$$

It follows that any point $v \in B_{\alpha'}(y)$, when taken as the image of x, would satisfy property (3) of the induction hypothesis.

We now invoke the g.e.c. condition to find a sequence in W that is correctly joined. We do this in two steps. First we identify the set A of all vertices in W_t that the image of x should be adjacent to. Then we find a point $v \in B_{\alpha'}(y) \cap W$ so that A is contained in a unit ball around v. Then we invoke the g.e.c. condition to find a sequence close to v which is adjacent to all vertices in A, and no other vertices in W_t.

Precisely, let $A = \{z' \in W_t : z \sim x\}$, the set of images of neighbours of x in W_t. Note that $A \subseteq B(y)$. Namely, since G is a geometric 1-graph, each neighbour z of x lies within $B(x)$. Our construction of y guaranteed that the step-isometry condition is satisfied if y is chosen as the image of x, and thus, the image z' lies in $B(y)$. Let $\beta > 0$ be so that $A \subseteq B_{1-\beta}(y)$, and $\beta \leq \alpha'/2$. We use the density of W to find a point $v \in W \setminus W_t$ so that $\|v - y\| < \beta$. By our choice of β, $A \subseteq B(v)$.

Next, we invoke the g.e.c. property to find a point $w \in W \cap B_\beta(v)$ which is adjacent to all vertices in A, and to no other vertices in W_t. In particular, for all $z \in V_t$, z is adjacent to x if and only if z' is adjacent to w. Now let $W_{t+1} = W_t \cup \{w\}$, and set $\varphi_{t+1}(x) = w$. Then condition (2) holds for φ_{t+1}. Since $\|w - y\| \leq \|w - v\| + \|v - y\| < 2\beta \leq \alpha'$, condition (3) holds as well.

Similarly, we can find a pre-image for $\xi^{(t+1)}$, and adjust V_{t+1}, W_{t+1} accordingly, to satisfy condition (1) and complete the reverse direction of the back-and-forth argument. □

Using Theorem 9 and the results of Section 3, Theorem 1 follows immediately.

5. Proof of Theorem 2: almost all sets in c_0 are Rado

In this section, our goal is to study the Rado property for countable subsets of c_0. It turns out that it is easier to first prove the Rado property for countable subsets of c_a for $a \in (0,1)$ (because $\lfloor x_i \rfloor$ is eventually 0 for all $x \in c_a$, while no such statement holds for c_0). It is then straightforward to deduce the result for c_0.

We prove the following theorem.

Theorem 10. *Let $a \in (0,1)$. Let V and W be dense, countable, i.d.f. sets in c_a satisfying the i.o.p. If G and H are g.e.c. geometric 1-graphs with vertex sets V and W respectively, then $G \cong H$.*

From Theorem 10, the conclusion of Theorem 2 follows as we now show.

Proof of Theorem 2. Let $\theta(x) = x + (\frac{1}{2}, \frac{1}{2}, \ldots)$, so that θ is a bijective isometry from c_0 to $c_{1/2}$. If μ is a non-aligned measure fully supported on c_0, then $\mu'(A) = \mu(\theta^{-1}(A))$ is a non-aligned measure fully supported on $c_{1/2}$. By Lemmas 15, 16 and 17, ${\mu'}^{\mathbb{N}}$-a.e. sequence of points, ${x'}^{(1)}, {x'}^{(2)}, \ldots$ in $c_{1/2}$ is dense, i.d.f., and satisfies the i.o.p., so that $W = \{{x'}^{(1)}, {x'}^{(2)}, \ldots\}$ is Rado by Theorems 4 and 10. Since the Rado property is preserved by isometries, we deduce that $V = \theta^{-1}(W)$ is Rado for ${\mu'}^{\mathbb{N}}$-almost every $({x'}^{(n)})$. Hence, for $\mu^{\mathbb{N}}$-almost all $x^{(1)}, x^{(2)}, \ldots$ in ${c_0}^{\mathbb{N}}$, $V = \{x^{(1)}, x^{(2)}, \ldots\}$ is Rado. □

Before proving Theorem 10, we comment on the argument, and the differences with the proof of Theorem 9. As before, the idea is to use a back-and-forth argument. Similarly, we use induction hypotheses to guarantee that at each step of the induction, we have a step isometry between the finite sets of vertices that are matched (otherwise Theorem 5 would show that we will not obtain a graph isomorphism in the limit). Lemma 8 still provides a sufficient condition for a matching to be a step isometry, but unfortunately, in the case of c_a, it is essentially impossible to satisfy the conditions of the lemma: if x is a randomly chosen element of c_a, write a sequence of $+$'s and $-$'s to denote if each coordinate is above or below a. One quickly sees that the probability that two random sets V and W contain elements with identical sign sequences is 0 (this is the union of countably many events of probability 0). This prevents us from directly applying Lemma 8. To circumvent this difficulty, we inductively build not only the bijection between the sets V and W, but also a bijection, g of the coordinates, so the partial bijection from V to W sends a point x with integer part $k \neq 0$ in position i to a point y with integer part k in position $g(i)$, and if $\langle x^{(i)} \rangle < \langle {x'}^{(i)} \rangle$, then $\langle y^{(g(i))} \rangle < \langle {y'}^{(g(i))} \rangle$. This is sufficient to maintain the step isometry property that we require.

Proof of Theorem 10. We show, using a back-and-forth type argument, that G and H are almost surely isomorphic. Enumerate V and W as $V = \{x^{(1)}, x^{(2)}, \ldots\}$ and $W = \{y^{(1)}, y^{(2)}, \ldots\}$ respectively. By assumption, these sets are i.d.f. and have the i.o.p.

We build bijections $f\colon \mathbb{N} \to \mathbb{N}$ and $g\colon \mathbb{N} \to \mathbb{N}$ so that:

(*i*) $\lfloor y^{(f(i))}_{g(j)} \rfloor = \lfloor x^{(i)}_j \rfloor$ for each i and j;

(*ii*) $\langle y^{(f(i))}_{g(j)} \rangle < \langle y^{(f(i'))}_{g(j)} \rangle$ if and only if $\langle x^{(i)}_j \rangle < \langle x^{(i')}_j \rangle$ for each i, i' and j;

(*iii*) $x^{(i)}$ is adjacent to $x^{(i')}$ if and only if $y^{(f(i))}$ is adjacent to $y^{(f(i'))}$.

The function f can be transformed into an isomorphism from G to H by defining $\phi : V \to W$ by

$$\phi(x^{(i)}) = y^{(f(i))}, \text{ for all } i \in \mathbb{N}.$$

Property (*iii*) guarantees that f is indeed an isomorphism; properties (*i*) and (*ii*) guarantee that f is a coordinate-wise step isometry under the map $j \to g(j)$.

We construct the bijections f and g by induction. That is, we inductively construct sets S_n and T_n, for $n \in \mathbb{N}$, and partially define f and g on these sets.

The sets S_n will represent the (super-)indices of the points $(x^{(i)})$ that have so far been matched. The point $x^{(i)}$ is matched to the point $y^{(f(i))}$. The sets T_n will represent the sets of coordinate positions (in the $(x^{(i)})$ sequences) that have been matched with coordinate positions in the $(y^{(f(i))})$ sequences: the jth coordinate of all sequences $x \in V$ that have been matched are matched with the $g(j)$-th coordinate of the image $y \in W$ of the sequence.

The induction hypotheses that we maintain are:

(1) $S_n \supseteq \{1, \ldots, n\}$ and $T_n \supseteq \{1, \ldots, n\}$;
(2) $f(S_n) \supseteq \{1, \ldots, n\}$ and $g(T_n) \supseteq \{1, \ldots, n\}$;
(3) $T_n \supseteq \{j \colon \text{there exists } i \in S_n \colon x_j^{(i)} \notin (0,1)\}$;
(4) $g(T_n) \supseteq \{j \colon \text{there exists } i \in f(S_n) \colon y_j^{(i)} \notin (0,1)\}$;
(5) $\langle x_k^{(i)} \rangle < \langle x_k^{(j)} \rangle$ if and only if $\langle y_{g(k)}^{(f(i))} \rangle < \langle y_{g(k)}^{(f(j))} \rangle$ for $i, j \in S_n$ and $k \in T_n$.
(6) $\lfloor y_{g(k)}^{(f(i))} \rfloor = \lfloor x_k^{(i)} \rfloor$ for each $i \in S_n$ and $k \in \mathbb{N}$;
(7) $x^{(i)} \sim x^{(j)}$ if and only if $y^{(f(i))} \sim y^{(f(j))}$ for $i, j \in S_n$.

Conditions (1) and (2) ensure that $\bigcup_{n \in N} S_n = \mathbb{N}$ and $\bigcup_{n \in N} T_n = \mathbb{N}$, and thus, that f and g are bijections of $\mathbb{N}$, as required. Conditions (5) and (6) ensure that f and g satisfy Properties (i) and (ii) above, and Condition (7) ensures that f and g satisfy Property (iii). Conditions (3) and (4) are necessary to ensure that the induction can be propagated.

We now proceed with the induction. Let $S_0 = T_0 = \emptyset$. Suppose that for a given $n \geq 1$, we have constructed S_{n-1} and T_{n-1}, and f and g are defined on those sets, and that the inductive properties hold. We now show how to extend the sets S_{n-1} and T_{n-1} in such a way that these properties are preserved. We focus on making an extension satisfying properties (1), (3), (5), (6) and (7) (that is, we go forth). A symmetric (and so omitted) "going back" argument applies for the extension properties (2), (4), (5), (6) and (7).

Here we match $x^{(n)}$ (if it is not already matched) with a sequence $y^{(i)}$, and we match some coordinate positions (including the n-th if it is not already matched, to ensure that the final map g is a bijection) with some coordinate positions in the y's.

First, set $S_n = S_{n-1} \cup \{n\}$ and $T_n = T_{n-1} \cup \{n\} \cup \{j \colon x_j^{(n)} \notin (0,1)\}$; this satisfies property (3). Let $T_n \setminus T_{n-1} = \{j_1, \ldots, j_L\}$. Notice that T_n is a finite set since $x^{(n)} \in c_a$ and $0 < a < 1$. For each k, $1 \leq k \leq L$, define an order $\prec_k$ on S_{n-1} by $i \prec_k i'$ if $\langle x_{j_k}^{(i)} \rangle < \langle x_{j_k}^{(i')} \rangle$. Let $r_1, \ldots, r_L$ be coordinate positions not in $g(T_{n-1})$ such that, for all $i, i' \in S_{n-1}$ and each $1 \leq k \leq L$, $\langle y_{r_k}^{(f(i))} \rangle < \langle y_{r_k}^{(f(i'))} \rangle$ if and only if $i \prec_k i'$. Such coordinate positions exist by the i.o.p.

Now define $g(j_k) = r_k$ for $k = 1, \ldots, L$. Notice that condition (6) is automatically satisfied for $i \in S_{n-1}$ and $k \in T_n \setminus T_{n-1}$ by conditions (3) and (4) of the induction hypothesis since one has $\lfloor x_k^{(i)} \rfloor = 0 = \lfloor y_{g(k)}^{(f(i))} \rfloor$. This has extended the collection of dimensions. Notice that property (5) is automatically satisfied (for $j \in T_n$) by the choice of $r_1, \ldots, r_L$ provided that i and i' belong to S_{n-1}.

To finish the argument, we need to choose $f(n)$ (assuming that $n \notin S_{n-1}$). For $j \in T_n$, let $k = g(j)$, let $a_k = \lfloor x_j^{(n)} \rfloor$, let $b_k = \max\big(\{0\} \cup \{\langle y_k^{(f(i))} \rangle \colon i \in S_{n-1}, \langle x_j^{(i)} \rangle < \langle x_j^{(n)} \rangle\}\big)$ and $c_k = \min\big(\{1\} \cup \{\langle y_k^{(f(i))} \rangle \colon i \in S_{n-1}, \langle x_j^{(i)} \rangle > \langle x_j^{(n)} \rangle\}\big)$. Let J_k be the open interval $(a_k + b_k, a_k + c_k)$. For $k \notin g(T_n)$, let $J_k = (0,1)$ and let $U = c_a \cap \prod_{k \in \mathbb{N}} J_k$. Notice that U contains an open set of radius $\frac{1}{2} \min_{k \in g(T_n)} (c_k - b_k)$.

Claim. *If $z \in U$, then $\lfloor \|z - y^{(f(i))}\| \rfloor = \lfloor \|x^{(n)} - x^{(i)}\| \rfloor$ for each $i \in S_{n-1}$.*

Proof. Let $z \in U$. Since all points belong to c_a, we have $\|z - y^{(f(i))}\| = \max_k |z_k - y_k^{(f(i))}|$ and $\|x^{(n)} - x^{(i)}\| = \max_j |x_j^{(n)} - x_j^{(i)}|$. In particular, it suffices to show that $\max_k \lfloor |z_k - y_k^{(f(i))}| \rfloor = \max_j \lfloor |x_j^{(n)} - x_j^{(i)}| \rfloor$ for each $i \in S_{n-1}$.

Notice that if $k \notin g(T_n)$, then $z_k, y_k^{(f(i))} \in (0,1)$ for each $i \in S_{n-1}$ by Condition (4), so that $\lfloor |z_k - y_k^{(f(i))}| \rfloor = 0$. Similarly if $j \notin T_n$, then $\lfloor |x_j^{(n)} - x_j^{(i)}| \rfloor = 0$ for each $i \in S_{n-1}$ by Condition (3).

Hence, it suffices to show that $\lfloor |z_{g(j)} - y_{g(j)}^{(f(i))}| \rfloor = \lfloor |x_j^{(n)} - x_j^{(i)}| \rfloor$ for each $i \in S_{n-1}$ and each $j \in T_n$. First notice that $\lfloor z_{g(j)} \rfloor = \lfloor x_j^{(n)} \rfloor$ for each $z \in U$ and $\lfloor y_{g(j)}^{(f(i))} \rfloor = \lfloor x_j^{(i)} \rfloor$ for each $i \in S_{n-1}$ and $j \in T_n$ by Conditions (3), (4) and (6). Also $\langle z_{g(j)} \rangle < \langle y_{g(j)}^{(f(i))} \rangle$ if and only if $\langle x_j^{(n)} \rangle < \langle x_j^{(i)} \rangle$ by the choice of $(b_k)_{k \in g(T_n)}$ and $(c_k)_{k \in g(T_n)}$. An argument exactly analogous to the proof of Lemma 8 establishes that $\lfloor |z_{g(j)} - y_{g(j)}^{(f(i))}| \rfloor = \lfloor |x_j^{(n)} - x_j^{(i)}| \rfloor$ for each $i \in S_{n-1}$ and $j \in T_n$, completing the proof. □

Now let $A = \{i \in S_{n-1} \colon \|x^{(n)} - x^{(i)}\| < 1\}$. By the above claim, if $i \in A$, then $\|z - y^{(f(i))}\| < 1$ for each $z \in U$. Since H satisfies the g.e.c. property and U contains an open set, there exists $y^{(t)} \in U \cap H$ with $t \notin f(S_{n-1})$ such that $y^{(t)} \sim_H y^{(f(i))}$ if and only if $x^{(n)} \sim_G x^{(i)}$. Defining $f(n) = t$ ensures that the induction hypotheses are satisfied at the nth stage as required. □

6. NON-ISOMORPHISM OF RADO GRAPHS

Let $GR(c)$ and $GR(c_0)$ be the unique isomorphism type of the geometric Rado graphs provided by Theorems 9 and 10, respectively. Similarly, let $GR(\ell_\infty^d)$ be the unique isomorphism type of the geometric Rado graph on ℓ_∞^d provided by [6]. We show in this section that these graphs are mutually non-isomorphic. In fact, we show more generally that a geometric graph on a countable dense subset of a Banach space determines the space up to isometric isomorphism.

To show this result, we need to make use of a concept introduced in the 1940's by Hyers and Ulam [15] and developed over the next five decades by a number of authors. An ϵ-*isometry* from a Banach space X to a Banach space Y is a map $T \colon X \to Y$ such that $\Big| \|T(x) - T(x')\| - \|x - x'\| \Big| \le \epsilon$ for all $x, x' \in X$. A map T from a Banach space X to a Banach space Y is δ-*surjective* if for all $y \in Y$, there exists $x \in X$ with $\|T(x) - y\| < \delta$.

The following theorem states that δ-surjective ϵ-isometries of Banach spaces are uniformly approximated by genuine isometries.

Theorem 11 (Dilworth [10], Proposition 2)**.** *Let X and Y be real Banach spaces and let T be a δ-surjective ϵ-isometry from X to Y. Then there exists a $K < \infty$ and a surjective linear isometry $U \colon X \to Y$ such that $\|T(x) - U(x)\| \le K$ for all $x \in X$.*

Theorem 12. *Let X and Y be real Banach spaces and suppose V and W are countable dense subsets of X and Y respectively. Further, let G and H be geometric 1-graphs with vertex sets V and W. If G and H are isomorphic, then there is an surjective isometry between X and Y.*

Proof. Let θ be the isomorphism from G to H. Let V be enumerated as $\{x^{(1)}, x^{(2)}, \ldots\}$ and let $y^{(n)} = \theta(x^{(n)})$. By Theorem 5, we have that for $i \neq j$, $d_G(x^{(i)}, x^{(j)}) < \|x^{(i)} - x^{(j)}\| + 2$, so that $\|y^{(i)} - y^{(j)}\| < d_H(y^{(i)}, y^{(j)}) = d_G(x^{(i)}, x^{(j)}) < \|x^{(i)} - x^{(j)}\| + 2$; and similarly $\|x^{(i)} - x^{(j)}\| < \|y^{(i)} - y^{(j)}\| + 2$, so that $\Big| \|\theta(x^{(i)}) - \theta(x^{(j)})\|_Y - \|x^{(j)} - x^{(i)}\|_X \Big| < 2$. We show how to extend θ to a 3-surjective 4-isometry from X to Y. Namely, we define $n \colon X \to \mathbb{N}$

by $n(x) = \min\{i \colon \|x^{(i)} - x\| < 1\}$. We now define $T(x) = y^{(n(x))}$. We claim that T is a 3-surjective 4-isometry.

To see this, first let $x, x' \in X$. Let $i = n(x)$ and $i' = n(x')$. We then have that $\|T(x) - T(x')\| = \|\theta(x^{(i)}) - \theta(x^{(i')})\|$, so that

$$\Big| \|T(x) - T(x')\| - \|x^{(i)} - x^{(i')}\| \Big| < 2.$$

We also have that

$$\Big| \|x^{(i)} - x^{(i')}\| - \|x - x'\| \Big| < 2.$$

We deduce that

$$\Big| \|T(x) - T(x')\| - \|x - x'\| \Big| < 4,$$

so that T is a 4-isometry.

Next, let $y \in Y$. Then there exists a j such that $\|y - y^{(j)}\| < 1$. We consider $T(x^{(j)})$: let $i = n(x^{(j)})$. Notice that $\|x^{(i)} - x^{(j)}\| < 1$ by definition of $n(\cdot)$. By Theorem 5, we have that $\|y^{(i)} - y^{(j)}\| < 2$. Hence, $T(x^{(j)}) = y^{(i)}$ and $\|y^{(i)} - y\| < 3$, so that T is 3-surjective as required. Theorem 11 then implies that X and Y are linearly isometric as Banach spaces. □

The spaces ℓ_∞^d for $d \in \mathbb{N}$ (that is, $\mathbb{R}^d$ equipped with the ℓ_∞ norm) are evidently not isometrically equivalent to each other or to c_0 or c since surjective isometries preserve dimension. Further, c and c_0 are not isometrically equivalent. One way to see this is as follows; see [3] for additional background. The closed unit ball of c has *extreme points* (that is, points that are not a convex combination of other points in the set; for example, $(1, 1, \ldots)$). There are no extreme points of the closed unit ball of c_0: for any x in the closed unit ball of c_0, there is an n such that $|x_n| < \frac{1}{2}$. Then x is equal to $\frac{1}{2}(y + z)$, where y and z agree with x in every position except the nth; $y_n = x_n + \frac{1}{2}$ and $z_n = x_n - \frac{1}{2}$. However, it is evident that isometric equivalences preserve unit balls and extreme points.

Corollary 13. *The graphs $GR(\ell_\infty^d)$ for $d \in \mathbb{N}$, $GR(c)$ and $GR(c_0)$ are mutually non-isomorphic.*

We finish with some open problems. We would like to consider other infinite dimensional separable spaces, such as the space of continuous functions $C([0, 1])$ on $[0, 1]$. Are almost all countable sets in $C([0, 1])$ Rado with respect to some natural measure? The abundance (or non-abundance) of Rado sets for other normed spaces such as ℓ_p and L^p (where $1 < p < \infty$) remains open.

7. Appendix

In this appendix, we show that dense countable i.d.f. sets are abundant, in the sense that we give a probabilistic construction that almost surely yields a dense countable i.d.f. subset of c. We also probabilistically construct dense countable i.d.f. subsets of c_a with the i.o.p. We remark that we make extensive use of the Borel-Cantelli lemmas. We recall that the first Borel-Cantelli lemma states that if (A_n) is a countable collection of events in a probability space satisfying $\sum_n \mathbb{P}(A_n) < \infty$, then almost surely the number of those events that occur is finite. The second Borel-Cantelli lemma is a partial converse, stating that if (A_n) is a countable collection of independent events such that $\sum_n \mathbb{P}(A_n) = \infty$, then almost surely the number of events that occur is infinite.

An *affine coordinate hyperplane* is a subset of c (or c_a) of the form $\{x \in c\colon x_j = \alpha\}$ for some $j \in \mathbb{N}$ and some $\alpha \in \mathbb{R}$. A *limit coordinate hyperplane* is a subset of c of the form $\{x \in c\colon \lim_{j\to\infty} x_j = \alpha\}$ for some $\alpha \in \mathbb{R}$.

A measure μ on c is said to be *non-aligned* if $\mu(H) = 0$ for any affine coordinate hyperplane or limit coordinate hyperplane. A measure μ on c_a is said to be *non-aligned* if $\mu(H) = 0$ for any affine coordinate hyperplane. A measure μ on the space X (c or c_a) has the *integer-distance free* (or i.d.f.) property if, for all $n \in \mathbb{N}$, when $X = c_a$, or for all $n \in \mathbb{N} \cup \{\infty\}$, when $X = c$, it holds that $\mu\{x \in X\colon x_n \in \mathbb{Z}\} = 0$ and $\mu \times \mu\{(x, y) \in X \times X\colon x_n - y_n \in \mathbb{Z}\} = 0$. Intuitively, a countable subset of X chosen according to an i.d.f. measure μ should give μ-almost surely an i.d.f. set. We will show below that this is indeed the case in Lemma 14.

Given a measure μ on X (one of c and c_a), we can build a measure $\mu^{\mathbb{N}}$, so that sampling a sequence from this measure $(Z_1, Z_2, \ldots)$ one obtains a sequence of elements of X, where each coordinate is independently chosen with distribution μ.

A measure μ on a topological space X is said to be *fully supported* if $\mu(U) > 0$ for each non-empty open set U (or equivalently if $\mu(B) > 0$ for each non-empty open ball).

Lemma 14. *If μ is a measure on the space X (one of c or c_a) with the i.d.f. property, then $\mu^{\mathbb{N}}$-a.e. element of $X^{\mathbb{N}}$ is a set with the i.d.f. property.*

Proof. The set of elements $(x^{(1)}, x^{(2)}, \ldots)$ of $\Omega^{\mathbb{N}}$ where there is some pair $x^{(i)}$ and $x^{(j)}$ such that, for some $n \in \mathbb{N} \cup \{\infty\}$, (or $n \in \mathbb{N}$ in case where $X = c_a$), satisfying $x_n^{(i)} \in \mathbb{Z}$ or $x_n^{(i)} - x_n^{(j)} \in \mathbb{Z}$, is the countable union of sets of measure 0. □

In turn, we argue that it is sufficient to have a non-aligned measure in each of c and c_0.

Lemma 15. *If μ is a non-aligned measure on c or c_a, then μ has the i.d.f. property.*

Proof. We deal with the two spaces in turn, although the proofs are nearly identical.

First if $X = c$, we observe that μ has the i.d.f. property if $\mu \times \mu(L) = 0$, $\mu(B_j) = 0$ for each j and $\mu \times \mu(C_j) = 0$ for each j, where $L = \{(x, y) \in c \times c\colon \lim_j (x_j - y_j) \in \mathbb{Z}\}$, $B_j = \{x \in c\colon x_j \in \mathbb{Z}\}$, and $C_j = \{(x, y) \in c \times c\colon x_j - y_j \in \mathbb{Z}\}$.

$$\begin{aligned}
\mu \times \mu(L) &= \sum_{n\in\mathbb{Z}} \int \mathbf{1}_{\{\lim_j (x_j - y_j) = n\}}\, d(\mu \times \mu)(x, y) \\
&= \sum_{n\in\mathbb{Z}} \int \left(\int \mathbf{1}_{\{\lim_j (x_j - y_j) = n\}}\, d\mu(x) \right) d\mu(y) \\
&= \sum_{n\in\mathbb{Z}} \int \left(\int \mathbf{1}_{\{\lim_j x_j = \lim_j y_j + n\}}\, d\mu(x) \right) d\mu(y),
\end{aligned}$$

where we used Fubini's theorem. By the non-aligned property, the inner integral is 0 for each value of $\lim_j y_j + n$, and so $\mu \times \mu(L) = 0$. An exactly similar argument shows that $\mu(B_J) = 0$ and $\mu \times \mu(C_j) = 0$ for each j.

In the case $X = c_a$, there is no L to consider. Define C_j as above, but applied to elements of c_a. Redefine $B_j = \{x \in c_a\colon x_j \in \mathbb{Z}\}$. Now μ has the i.d.f. property if $\bigcup_j B_j \cup \bigcup_j C_j$ has measure zero. The countable union of the C_j's and B_j's has measure 0 by a similar argument. □

For c_0, our vertex sets will need to satisfy the i.o.p. This can be achieved by imposing an additional condition on the measure, which we define here. A measure μ on subsets of $\mathbb{R}^N$

is of *product type* if it is of the form $\mu = \prod_{n=1}^{\infty} \nu_n$ where $(\nu_n)_{n\in\mathbb{N}}$ is a collection of measures on $\mathbb{R}$ (that is, the law of μ is a sequence of independent random variables, where the jth coordinate has law ν_j).

A measure ν on $\mathbb{R}$ is said to be *non-atomic* if $\nu(\{a\}) = 0$ for each $a \in \mathbb{R}$. Note that a measure of product type is non-aligned if and only if each ν_n is non-atomic.

Lemma 16. *If μ is a non-aligned measure of product type, then $\mu^{\mathbb{N}}$-almost every sequence of points in $\mathbb{R}^{\mathbb{N}}$ has the i.o.p.*

Proof. Fix positive integers k and $i_1 < i_2 < \ldots < i_k$, and let $\prec$ be an ordering on $\{1, \ldots, k\}$. Let $m_1, \ldots, m_k$ be the enumeration of $\{1, \ldots, k\}$ in increasing $\prec$ order. We consider realizations of $\mu^{\mathbb{N}}$, which we denote as $(x^{(i)})_{i\in\mathbb{N}}$, where each $x^{(i)}$ is an element of $\mathbb{R}^{\mathbb{N}}$.

Let $E_j = \{(x^{(i)}) \in (\mathbb{R}^{\mathbb{N}})^{\mathbb{N}} \colon \langle x_j^{(i_{m_1})} \rangle < \ldots < \langle x_j^{(i_{m_k})} \rangle\}$, that is, the collection of those realizations such that in the jth coordinate, the order among the fractional parts of $x^{(i_1)}, \ldots, x^{(i_k)}$ is $\prec$. Since $x_j^{(i_1)}, \ldots, x_j^{(i_k)}$ are chosen independently, we see that $\mu^{\mathbb{N}}(E_j) = 1/k!$. Further, since μ is a product measure, we see that the events $(E_j)_{j=1}^{\infty}$ are mutually independent.

By the second Borel-Cantelli lemma, $\mu^{\mathbb{N}}$-almost every element of $(\mathbb{R}^{\mathbb{N}})^{\mathbb{N}}$ belongs to infinitely many E_j's. Since this holds for each of the $n!$ choices of $\prec$, it follows that for $\mu^{\mathbb{N}}$-a.e. choice of sequence $x^{(1)}, x^{(2)}, \ldots$ of points in $\mathbb{R}^{\mathbb{N}}$, one can find a finite sequence of coordinates on which the ordering of the fractional parts matches any chosen finite sequence of orderings. Hence, $\mu^{\mathbb{N}}$-almost every element of $(\mathbb{R}^{\mathbb{N}})^{\mathbb{N}}$ satisfies the i.o.p., as required. □

Lemma 17. *Let X be a separable Banach space and let μ be a fully supported measure on X. Then for $\mu^{\mathbb{N}}$-a.e. sequence $(x^{(i)})_{i\in\mathbb{N}}$ of points in X, the set $V = \{x^{(i)} \colon i \in \mathbb{N}\}$ is a dense subset of X.*

Proof. Let (z_n) be a fixed countable dense subset of X, so that $\{B_{1/m}(z_n) \colon m, n \in \mathbb{N}\}$ is a countable neighbourhood basis of X. For fixed m and n, by the second Borel-Cantelli lemma, with probability 1, the random set, V, intersects $B_{1/m}(z_n)$ infinitely often since $\mu(B_{1/m}(z_n)) > 0$. Let $E_{m,n}$ be the event that V intersects $B_{1/m}(z_n)$. Intersecting this countable collection of events of measure 1 over all choices for n, one deduces that with probability 1, the random set, V intersects every $B_{1/m}(z_n)$. That is, V is a dense subset of X. □

We now construct some non-aligned fully supported measures on c and c_a. We base the construction on normal random variables, but we point out that there is nothing delicate about the construction: all that is required is that the distributions of all each coordinate is non-atomic and fully supported on $\mathbb{R}$, but that the distributions of the successive coordinates become more and more concentrated near a to ensure that with probability 1, the sequence of coordinates approaches a.

First, we construct a measure on c_a. Let $(Y_n)_{n\in\mathbb{N}}$ be a family of independent normal random variables with mean a and variance $(\log n)^{-2}$ and let μ_a be the law of $(Y_1, Y_2, \ldots)$. (That is, $\mu_a(A) = \mathbb{P}\Big((Y_1, Y_2, \ldots) \in A\Big)$, for each Borel subset, A, of X). Note that μ_a is of product type. Since the random variables $(Y_j)_{j\in\mathbb{N}}$ are independent, μ_a is exactly as required by the hypothesis of Lemma 16, so that $\mu_a^{\mathbb{N}}$-almost every sequence in $(c_a)^{\mathbb{N}}$ has the i.o.p.

Lemma 18. *The measure μ_a as constructed above is non-aligned and fully supported on c_a.*

Proof. To prove that μ_a is fully supported on c_a, we must prove two things: firstly, that $\mu_a(c_a) = 1$, and secondly, that $\mu_a(U) > 0$ for any non-empty open ball in c_a. We work in the case $a = 0$. For any $a \neq 0$, the law of $(Y_1 - a, Y_2 - a, \ldots)$ is just μ_0, so that if μ_0 is fully supported on c_0, then μ_a is fully supported on $c_0 + (a, a, \ldots) = c_a$.

To prove that $\mu_0(c_0) = 1$, we show for μ_0-almost every sequence $(x_1, x_2, \ldots)$ that $x_n \to 0$. To see this, write $N_j = (\log j)Y_j$. Having done this, the random variables (N_j) are independent standard normal random variables.

Fix $\epsilon > 0$. Then $\mu_0(\{x\colon \limsup_j |x_j| > \epsilon\}) \leq \mathbb{P}(\{|N_j| > \epsilon \log j \text{ infinitely often}\})$.

We have by a standard estimate on the tail of the normal distribution that $\mathbb{P}(N_j > \epsilon \log j) < (2\pi)^{-1/2} \exp(-\epsilon^2 (\log j)^2/2)/(\epsilon \log j)$ (see [11], Theorem 1.2.3). Thus this probability goes to zero as j increases, which tells us that $\mu_0(\{x\colon \limsup_j |x_j| > \epsilon\}) = 0$ for each $\epsilon > 0$. Now taking a countable sequence of ϵ's converging to 0 and taking the union, we deduce that $\mu_0(\{x\colon \limsup_j |x_j| = 0\}) = 1$, so that $\mu_0(c_0) = 1$.

Now let $U = B(x, r)$ be an open ball in c_0. Since $x \in c_0$, we have $|x_j| < \frac{r}{2}$ for all j greater than some N. Now $U \supseteq \left(\prod_{j=1}^{N}(x_j - r, x_j + r) \times \prod_{j>N}(-\frac{r}{2}, \frac{r}{2})\right) \cap c_0$. The above estimate on the tail of a normal distribution shows that $\prod_{j>N} \mathbb{P}(|Y_j| < \frac{r}{2}) > 0$. Since normal random variables are fully supported on $\mathbb{R}$, we also have $\prod_{j \leq N} \mathbb{P}(Y_j \in (x_j - r, x_j + r))$ is the product of a finite number of positive reals, and hence is positive. Hence, we deduce $\mu_0(U) > 0$ as required.

Finally, if H is a coordinate hyperplane, $\{x\colon x_j = \alpha\}$, then $\mu_0(H) = \mathbb{P}(Y_j = \alpha) = 0$, so that μ is non-aligned. □

To build a fully supported measure on c, we start from the previous construction. Let (Y_n) be as above and let Z be a standard normal random variable (or any fully supported non-atomic random variable). Then by the above, the sequence $(Y_1 + Z, Y_2 + Z, \ldots)$ almost surely takes values in c and the limiting value of such a random sequence is just the random variable Z. Let μ be the distribution of this random element of c.

Lemma 19. *The measure, μ, as constructed above is non-aligned and fully supported on c.*

Proof. As mentioned above, sequences sampled from μ almost surely lie in c, so that the support of μ is a subset of c. If $U = B(x, r)$ is an open ball in c, let $x_\infty = \lim_j x_j$ and let (y_j) be defined by $y_j = x_j - x_\infty$, so that $y \in c_0$. Now if $(Y_1, Y_2, \ldots)$ lies in $B_{c_0}(y, \frac{r}{2})$ and Z lies in $(x_\infty - \frac{r}{2}, x_\infty + \frac{r}{2})$, then $(Y_n + Z)_n$ lies in U. Since the events that $(Y_n)_n$ lies in $B_{c_0}(y, \frac{r}{2})$ and Z lies in $(x_\infty - \frac{r}{2}, x_\infty + \frac{r}{2})$ are independent and both of positive measure, we see that $\mu(U)$ is bounded below by the product of the measures, and hence is positive, proving that μ is fully supported.

We now verify that μ is non-aligned. If $H = \{x \in c\colon x_j = \alpha\}$, then $\mu(H) = \mathbb{P}(Y_j + Z = \alpha)$. Since Y_j and Z are independent normal random variables, their sum is another normal random variable. Since normal random variables are non-atomic, we see $\mu(H) = 0$.

Similarly, if $L = \{x \in c\colon \lim_j x_j = \alpha\}$, then the event L is equal to the event $\{Z = \alpha\}$ up to a set of measure 0. Since Z is non-atomic, we see $\mu(L) = 0$, so that L is non-aligned. □

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