GEOMETRIC RANDOM GRAPHS AND RADO SETS OF CONTINUOUS FUNCTIONS

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ABSTRACT. We prove the existence of Rado sets in the Banach space of continuous functions on $[0,1]$. A countable dense set S is Rado if with probability 1, the infinite geometric random graph on S, formed by probabilistically making adjacent elements of S that are within unit distance of each other, is unique up to isomorphism. We show that for a suitable measure which we construct, almost all countable dense sets in the subspaces of piecewise linear functions and of polynomials are Rado. Moreover, all graphs arising from such sets are of a unique isomorphism type. For the subspace of Brownian motion paths, almost all countable subsets are Rado (for a suitable measure) and the resulting graphs are of a unique isomorphism type. We show that the graph arising from piecewise linear functions and polynomials is not isomorphic to the graph arising from Brownian motion paths. Moreover, these graphs are non-isomorphic to graphs arising from Rado sets in $\mathbb{R}^n$, or the sequence spaces c and c_0.

1. INTRODUCTION

Erdős and Rényi [13] discovered that for $p \in (0,1)$, almost surely realizations of the infinite binomial random graph $G(\mathbb{N}, p)$ are isomorphic (and the isomorphism class is the same for all $p \in (0,1)$). The almost sure isomorphism class is named the *infinite random graph*, or the *Rado* graph, and is written R. The graph R is the unique countable graph satisfying the *existentially closed* (or *e.c.*) property: for all finite disjoint sets of vertices A and B (one of which may be empty), there is a vertex $z \notin A \cup B$ adjacent to all vertices of A and to no vertex of B. We say that z is *correctly joined* to A and B. See Chapter 6 of [4] and the surveys [8, 9] for additional background on R.

In [5], infinite random geometric graphs with analogous properties to R were introduced. Consider a normed space X with norm $\|\cdot\|$, a countable subset V of X, and $p \in (0,1)$. The *Local Area Random Graph* $\mathrm{LARG}(V,p)$ has vertices V, and for each pair of vertices u and v with $\|u-v\| < 1$, an edge is added joining u and v independently with probability p. Since V is required to be countable and we will focus on the case where V is dense in X, we require X to be separable. A countable dense set V in a normed space X is *Rado* if for all $p \in (0,1)$, with probability 1, $\mathrm{LARG}(V,p)$ generates a unique isomorphism type of graph. For simplicity, we call these *LARG graphs*. For a real number $1 \le p \le \infty$ and $d \ge 1$ an integer, the vector space $\mathbb{R}^d$ of dimension d equipped with the metric derived from the p-norm is denoted by ℓ_p^d. If $p = \infty$, then in [5] it was shown that almost all countable dense sets are Rado (here and for the rest of this section, "almost all" refers to a suitable measure constructed in the paper). The unique countable limits in the d-dimensional case were named GR_d; for a fixed d, these graphs are all isomorphic regardless of the choice of p. In contrast,

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it was also shown in [5] that if $p = 2$, there are sets in ℓ_p^2 which are strongly non-Rado. The latter result was generalized in [1], which proved that if X is a finite-dimensional normed space not isometric to ℓ_∞^d, then almost every random dense set V is not Rado.

A question posed at the end of [1] was to classify the Rado sets in the infinite dimensional case. As referenced in [6], even finding examples of Rado sets in infinite dimensional space was left as an open problem, and this was settled in [7]. Let c be the space of all convergent, real sequences equipped with the ℓ_∞ norm. Let c_0 denote the subspace of c consisting of sequences converging to 0. It was shown in [7] that $\mu^{\mathbb{N}}$-almost all countable dense sets in the space of convergent sequences c, and $\mu_0^{\mathbb{N}}$-almost all countable subsets of the subspace c_0 are Rado, where μ and μ_0 are measures on c and c_0, respectively, satisfying certain natural conditions.

We extend our results to the setting of $C[0,1]$, the Banach space of continuous functions on $[0,1]$. We show that Rado sets exist in $C[0,1]$, and they are abundant when restricted to certain subspaces including piecewise linear functions, polynomials, or Brownian motion paths. One of our main tools is the new notion of smoothly dense sets, which will be defined in Section 2. As we will prove in Theorems 7 and 8, respectively, there exist measures μ such that $\mu^{\mathbb{N}}$-almost all countable sets in the subspace of piecewise linear functions and polynomials are smoothly dense. In Theorem 4, we show that there is a graph $GR(\mathrm{SD})$ such that for any smoothly dense subset V of $C([0,1])$ and any $p \in (0,1)$, $\mathrm{LARG}(V,p)$ is almost surely isomorphic to $GR(\mathrm{SD})$.

In Section 3, we prove parallel results for so-called infinite crossing dense sets. We prove that countable families of Brownian motions are almost surely infinite crossing dense, and establish the existence of a graph $GR(\mathrm{ICD})$ such that for any infinite crossing dense subset V of $C([0,1])$ and any $p \in (0,1)$, $\mathrm{LARG}(V,p)$ is almost surely isomorphic to $GR(\mathrm{ICD})$.

We showed in [7] that if a countable dense subset of a Banach space is Rado, then the almost sure isomorphism type of the local area Rado graph determines the Banach space. In Section 4, we show that in $C([0,1])$, the Rado graph "sees" properties of the dense collection of functions and not just the underlying Banach space. In particular, we show that the graph isomorphism, if it exists, preserves crossings of pairs of functions, allowing us to prove that $GR(\mathrm{SD})$ and $GR(\mathrm{ICD})$ are non-isomorphic (and are non-isomorphic to the graphs $GR(X)$ studied in previous papers for other Banach spaces X).

Throughout, all graphs considered are simple, undirected, and countable unless otherwise stated. We will encounter two distinct notions of distance: metric distance (derived from a given norm) and graph distance. In a normed space, we write $\|u - v\|$ for the metric distance of the points. For a graph G, we write $d_G(u,v)$ for the graph distance. For a reference on graph theory the reader is directed to [10, 18], while [3] is a reference on normed spaces.

2. Smoothly dense sets

In this section we define a property called *smoothly dense*, and show that there is a graph $GR(\mathrm{SD})$ such that if V is a smoothly dense set, then for $p \in (0,1)$, a realization of $\mathrm{LARG}(V,p)$ is almost surely isomorphic to $GR(\mathrm{SD})$. We will show that for suitable measures which we define, almost all dense sets in the subspace of polynomials and in the subspace of piecewise linear functions are smoothly dense.

We first give a corollary of Theorem 2.4 in [5] which demonstrates that there is a close relationship between graph distance and metric distance in any graph that is a geometric 1-graph.

Corollary 1 ([5])**.** *Suppose V and W are countable dense subsets of a separable Banach space X such that graphs Γ sampled from* $\mathrm{LARG}(V,p)$ *and Γ' sampled from* $\mathrm{LARG}(W,p')$ *with $p,p' \in (0,1)$ are almost surely isomorphic. Then randomly sampled elements Γ from* $\mathrm{LARG}(V,p)$ *and Γ' from* $\mathrm{LARG}(W,p')$ *almost surely have the property that every graph isomorphism $f\colon \Gamma \to \Gamma'$ has the property that for every pair of vertices $u,v \in V$,*

$$\lfloor \|u-v\| \rfloor = \lfloor \|f(u)-f(v)\| \rfloor.$$

Corollary 1 suggests the following generalization of isometry. Given metric spaces (S,d_S) and (T,d_T), and subsets $V \subseteq S$ and $W \subseteq T$, a *step-isometry* from V to W is a bijective map $f: V \to W$ with the property that for every pair of vertices $u,v \in V$,

$$\lfloor d_S(u,v) \rfloor = \lfloor d_T(f(u),f(v)) \rfloor.$$

It is evident that every isometry is a step-isometry. As one would expect, the converse is false, but surprisingly, the paper [1] shows that for any finite-dimensional Banach space that is not isometric to ℓ_∞^d, a step-isometry between a pair of dense subsets is necessarily an isometry. Similar results probably hold in infinite-dimensional separable Banach spaces, and certainly hold in a separable Hilbert space.

We first introduce some terminology. For any $f \in C[0,1]$ the *remainder of f*, written $\langle f \rangle$, is defined point-wise by

$$\langle f \rangle(x) = \langle f(x) \rangle,$$

where $\langle x \rangle = x - \lfloor x \rfloor$. For functions f and g in $C[0,1]$, we say f and g *cross* at x if $\langle f(x) \rangle = \langle g(x) \rangle$ and we define $\mathsf{cr}(f,g) = \{x\colon \langle f(x) \rangle = \langle g(x) \rangle\}$. If A and B are finite subsets of $[0,1]$, then we write $A \sim_\epsilon B$ if they have the same cardinality, and when comparing elements of the sets in increasing order, for all i, the ith element of A is within ϵ of the ith element of B.

A subset $\mathcal{F}$ of $C[0,1]$ is *transverse* if it has the following properties:

(1) The sets $\mathsf{cr}(f,g)$ are disjoint for distinct pairs $f,g \in \mathcal{F}$;
(2) For any $f,g \in \mathcal{F}$, $\mathsf{cr}(f,g)$ is finite, does not contain 0 or 1, and if $t \in \mathsf{cr}(f,g)$, then t is not a local extremum of $f-g$.

One consequence of the set being transverse is that no two distinct elements are at integer distance from each other. In that sense, it is a generalization of the notion of *integer distance free (idf)* sets as defined in [5] for sets in $\mathbb{R}^n$ and in [7] for sets of sequences.

A countable subset V of $C[0,1]$ is said to be *smoothly dense* if every appropriate function in $C[0,1]$ can be approximated in such a way that the crossing behaviour of the approximation with a given finite set of functions mimics that of the original function. In addition, the functions in the set must possess a number of nice qualities. In particular, each member must be *Lipschitz*: a function is Lipschitz if there exists $K_f > 0$ such that $|f(x)-f(y)| \le K_f|x-y|$ for each $x,y \in [0,1]$).

Precisely, a set V is smoothly dense if it has the following properties:

(1) The set V is dense in $C[0,1]$;
(2) The set V is transverse;
(3) Each $f \in V$ is Lipschitz;
(4) For each finite subset $\mathcal{F} \subset V$, and each $f \in C[0,1]$ such that $\mathcal{F} \cup \{f\}$ is transverse, and for each $\epsilon > 0$, there exists $g \in V$ such that:
   (a) $\|f-g\| < \epsilon$.
   (b) $\mathsf{cr}(f,h) \sim_\epsilon \mathsf{cr}(g,h)$ for all $h \in \mathcal{F}$.

The function g is called a *smooth ϵ-approximation* of f relative to $\mathcal{F}$.

We will show in the last two subsections that smoothly dense sets not only exist in $C[0,1]$, but they are abundant in each of the subspaces of piecewise linear functions and polynomials. More precisely, we will define a measure so that almost all countable, randomly chosen sets of piecewise linear functions or polynomials are smoothly dense.

2.1. **Smoothly dense sets are Rado.** We begin with a lemma providing a crucial step in the inductive proof of graph isomorphism. It states that, in smoothly dense sets, if we can find a smooth ϵ-approximation of a given function, then we can find infinitely many such approximations, and at least one of them will have the correct adjacency pattern with a given finite set of vertices.

Lemma 2. *Let V be smoothly dense and $p \in (0,1)$. If G is sampled from* $\mathrm{LARG}(V,p)$ *then the following property almost surely holds:*

For each finite subset $\mathcal{F}$ of V and $f \in C([0,1])$ such that $\mathcal{F} \cup \{f\}$ is transverse, for all $\mathcal{G} \subseteq \mathcal{F}$ so that $\|f-h\| < 1$ for all $h \in \mathcal{G}$, for all $\epsilon > 0$, there exists $g \in V \backslash \mathcal{F}$ so that g is a smooth ϵ-approximation of f with respect to $\mathcal{F}$, and g is adjacent to all vertices in $\mathcal{G}$ and no vertex in $\mathcal{F} \setminus \mathcal{G}$.

Proof. By the definition of smoothly dense, there exists a smooth ϵ-approximation of f with respect to $\mathcal{F}$. In fact, by decreasing ϵ we may find infinitely many such functions, and we may ensure that each such function g has the property that $\|g-h\| < 1$ for all $h \in \mathcal{G}$. Then for each such g, the probability of being correctly joined is $p^k(1-p)^\ell$, where $k = |\mathcal{G}|$ and $\ell = |\mathcal{F} - \mathcal{G}|$. We then observe that the probability of no such function g being correctly joined is 0. □

Given functions $f, g \in C[0,1]$ and an interval $I \subseteq [0,1]$, we say that $f < g$ *on* I if, for all $x \in I$, $f(x) < g(x)$. Let $f > g$ on I be similarly defined. Given finite sets $\mathcal{F}, \mathcal{G} \subseteq C[0,1]$, a bijection $\varphi : \mathcal{F} \to \mathcal{G}$, and intervals $I, J \subseteq [0,1]$, the map φ is *order-preserving* on (I, J) if for all $f, g \in \mathcal{F}$ we have that:

(1) For all $x \in I$ and $y \in J$, we have that $\lfloor f(x) \rfloor = \lfloor \varphi(f)(y) \rfloor$; and
(2) The remainders of f and g are similarly ordered, i.e. either $\langle f \rangle < \langle g \rangle$ on I and $\langle \varphi(f) \rangle < \langle \varphi(g) \rangle$ on J, or $\langle f \rangle > \langle g \rangle$ on I and $\langle \varphi(f) \rangle > \langle \varphi(g) \rangle$ on J.

If a bijection φ between sets of functions is given and there is no ambiguity, then we will use f^* to denote $\varphi(f)$.

Lemma 3. *Suppose $V \subseteq C[0,1]$ is smoothly dense, $\mathcal{F}, \mathcal{G} \subseteq V$ and we are given a bijection $\varphi : \mathcal{F} \to \mathcal{G}$. Let $X = \bigcup_{f,g \in \mathcal{F}} \mathsf{cr}(f,g) \cup \{0,1\}$ and $Y = \bigcup_{f,g \in \mathcal{F}} \mathsf{cr}(f^*,g^*) \cup \{0,1\}$. Suppose $|X| = |Y| = n+1$, and for $i = 1, \ldots n$, let $I_i = (x_{i-1}, x_i)$ and $J_i = (y_{i-1}, y_i)$, where x_i, y_i denote the i-th largest element of X, Y, respectively. If for all i, $0 < i \le n$, φ is order-preserving on (I_i, J_i), then φ is a step-isometry.*

Proof. Let $x_{\max}$ be chosen so that $(f-g)(x_{\max}) = \|f-g\|$. Note that $x_{\max}$ cannot be in X since V is transverse. Hence, $x_{\max}$ must lie in some interval I_i. Now take any point $y_{\max} \in J_i$.

For real numbers a and b, knowing $\lfloor a \rfloor$, $\lfloor b \rfloor$ and whether $\langle a \rangle < \langle b \rangle$ or $\langle a \rangle > \langle b \rangle$ is sufficient to determine $\lfloor |a-b| \rfloor$. Hence, from the order-preserving property,

$$\lfloor \|f-g\| \rfloor = \lfloor |f(x_{\max}) - g(x_{\max})| \rfloor = \lfloor |f^*(y_{\max}) - g^*(y_{\max})| \rfloor \le \lfloor \|f^* - g^*\| \rfloor.$$

A similar argument shows that $\lfloor \|f^* - g^*\| \rfloor \le \lfloor \|f-g\| \rfloor$. □

We now arrive at our main result of this section, that there is a unique isomorphism type for LARG graphs on smoothly dense sets.

Theorem 4. *There exists a graph $GR(\mathrm{SD})$ such that for every countable smoothly dense subset, V, of $C[0,1]$, and $p \in (0,1)$, $\mathrm{LARG}(V,p)$ is almost surely isomorphic to $GR(\mathrm{SD})$.*

Proof. Let V and W be smoothly dense sets and let $p, p' \in (0,1)$. Let G be sampled from $\mathrm{LARG}(V,p)$ and H be sampled from $\mathrm{LARG}(W,p')$. Using a back-and-forth argument, we will construct a sequence of bijections $\varphi_n : V_n \to W_n$ between finite subsets of V and W such that φ_n is an isomorphism from $G[V_n]$ to $H[W_n]$ and φ_{n+1} extends φ_n.

Let $V = \{g_i : i \geq 0\}$ and $W = \{h_i : i \geq 0\}$. Without loss of generality, assume g_0 is equal to the constant zero function (we may achieve this by subtracting one function in V from the remaining ones, which does not affect any of our conditions). We may similarly assume h_0 is the constant zero function.

We proceed by induction on $n \geq 0$. Set $V_0 = \{g_0\}$ and $W_0 = \{h_0\}$ and let $\varphi_0(g_0) = h_0$. Fix $n \geq 0$, and assume $\varphi_n : V_n \to W_n$ is defined. (We use the notation f^* to denote $\varphi_n(f)$.) Our induction hypothesis consists of the following items:

(1) $g_n \in V_n$ and $h_n \in W_n$;

(2) Let $X = \bigcup_{f,g \in V_n} \mathsf{cr}(f,g) \cup \{0,1\}$ and $Y = \bigcup_{f,g \in W_n} \mathsf{cr}(f^*,g^*) \cup \{0,1\}$. Then $|X| = |Y|$, and for $i = 1, \ldots m$, where $m+1 = |X|$, let $I_i = (x_{i-1}, x_i)$ and $J_i = (y_{i-1}, y_i)$, where x_i, y_i denote the ith largest elements of X and Y respectively. (Starting at $x_0 = y_0 = 0$.) Then for all i, $0 < i \leq m$, φ_n is order-preserving on (I_i, J_i),

For the induction step, we go forth. Going back is analogous and so is omitted. Let $g = g_{n+1}$ and assume $g \notin V_n$. Let the sets $X = \{x_0, x_1, \ldots, x_m\}$ and $Y = \{y_0, y_1, \ldots, y_m\}$ be as in item (2) of the induction hypothesis.

We build a continuous target function t, working separately on each J_i. Fix i such that $1 \leq i \leq m$. Let $q(y) = x_{i-1} + \frac{x_i - x_{i-1}}{y_i - y_{i-1}}(y - y_{i-1})$, that is the increasing linear function that maps J_i to I_i. Next, suppose that g crosses V_n inside I_i at points $x_{i,1}, \ldots, x_{i,k-1}$ with $x_{i-1} =: x_{i,0} < x_{i,1} < \ldots < x_{i,k-1} < x_{i,k} := x_i$. On each interval, $I_{i,l} = (x_{i,l-1}, x_{i,l})$, let $\gamma_l^- + n_l^- < g < \gamma_l^+ + n_l^+$, where $\gamma_l^+, \gamma_l^- \in V_n$, $n_l^+, n_l^- \in \mathbb{Z}$ and the left side is the maximal element of $V_n + \mathbb{Z}$ lying below g on $I_{i,l}$, while the right side is the minimal such function lying above g on $I_{i,l}$.

Define $\alpha(x) = (g(x) - (\gamma_l^-(x) + n_l^-))/((\gamma_l^+(x) + n_l^+) - (\gamma_l^-(x) + n_l^-))$, so that $0 < \alpha(x) < 1$ on $I_{i,l}$. By definition, g crosses either $\gamma_l^+ + n_l^+$ or $\gamma_l^- + n_l^-$ at $x_{i,l}$ and $x_{i,l+1}$. Thus, $\alpha(x_{i,l}) \in \{0,1\}$ and $\alpha(x_{i,l+1}) \in \{0,1\}$, and there are four different possibilities for the crossing behaviour of g with respect to $\gamma_l^+ + n_l^+$ and $\gamma_l^- + n_l^-$. However, in each case, $g(x) = \alpha(x)(\gamma_l^+ + n_l^+) + (1 - \alpha(x))(\gamma_l^- + n_l^-)$.

Let $J_{i,l} = q^{-1}(I_{i,l})$ We then define the target function on $J_{i,l}$ by

$$t(y) = \alpha(q(y))\big({\gamma_l^+}^*(q(y)) + n_l^+\big) + (1 - \alpha(q(y)))\big({\gamma_l^-}^*(q(y)) + n_l^-\big),$$

so that $t(y)$ sits between the images of integer translates of ${\gamma_l^-}^*$, ${\gamma_l^+}^*$ on $J_{i,l}$ as $g(y)$ sits between the corresponding integer translates of γ_l^+, γ_l^- on $I_{i,l}$. This is illustrated in Figure 1.

Define $\tilde{\varphi} \colon V_n \cup \{g\} \to W_n \cup \{t\}$. The construction above ensures that $\tilde{\varphi}$ is order-preserving. Also, $W_n \cup \{t\}$ is clearly transverse. Let $\mathcal{G} = \{g_j \in W_n \colon g_j \sim g\}$. Since W is smoothly dense, Lemma 2 ensures for any $\epsilon > 0$, the existence of an $h \in W$ that is a smooth ϵ-approximation

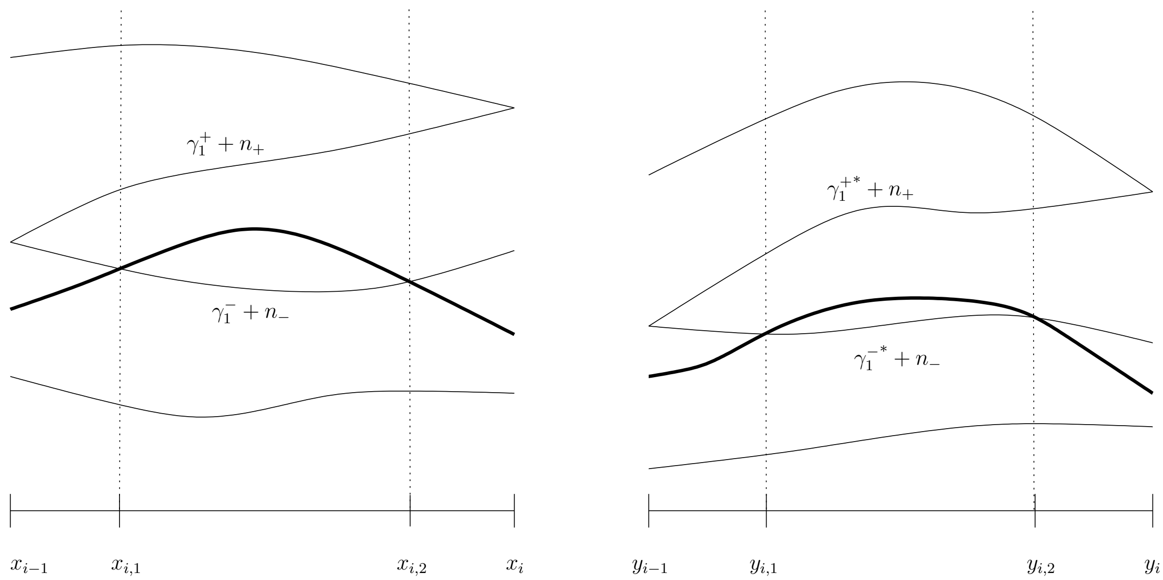

FIGURE 1. Schematic illustrating the new function to be added to $\mathcal{F}$, f (in bold) on an interval $[x_{i-1}, x_i]$; and the function t (also bold) on $[y_{i-1}, y_i]$ which is a "target" for f to be matched to.

of t with respect to W_n that is adjacent to $\mathcal{G}$ but not to $W_n \setminus \mathcal{G}$. By choosing ϵ sufficiently small, we are guaranteed that $\bar{\varphi}\colon V_n \cup \{g\} \to W_n \cup \{h\}$ is order-preserving. □

2.2. **Piecewise linear functions.** Define the set of *piecewise linear functions* in $[0,1]$, written $\mathsf{PL}[0,1]$, to be continuous functions with graphs consisting of finitely many line segments. Note that by a standard argument, the subspace $\mathsf{PL}[0,1]$ is dense in $C[0,1]$.

A function f in $\mathsf{PL}[0,1]$ is completely defined by a sequence of *change points* $(c_i^f)_{i=0}^{n+1}$, which is the collection of points in the graph of f where the slope of f changes, or meets the boundaries $x=0$ or $x=1$. Precisely, for all i, $i=0,\ldots,n+1$, $c_i^f \in [0,1]\times\mathbb{R}$, and the graph of f consists of the line segments connecting c_i^f to c_{i+1}^f for all i. Thus, we assume the change points to be ordered according to their x-coordinate. Precisely, $c_i^f = (x_i, y_i)$, where $0 = x_0 < x_1 < \cdots < x_{n+1} = 1$. If $f \in \mathsf{PL}[0,1]$, then we write $\mathsf{CP}(f)$ (for change points) for the sequence of x-coordinates of the change points.

We define a measure on $C[0,1]$ supported on $\mathsf{PL}[0,1]$. For $n \in \mathbb{N}_0$, and sequences $(x_i)_{i=0}^\infty$ of distinct points in $(0,1)$ and $(y_i)_{i=0}^\infty$ of points in $\mathbb{R}$, let $\varphi(n,(x_i),(y_i))$ be the function defined by change points $(c_i)_{i=0}^{n+1}$ where $c_0 = (0,y_0)$, $c_{n+1} = (1, y_{n+1})$, and, for all $1 \le i \le n$, $c_i = (\hat{x}_i, y_i)$, where $\hat{x}_i$ is the i-th largest element of $\{x_1,\ldots,x_n\}$.

We then equip $\mathbb{N}_0 \times [0,1]^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$ with the probability measure where the first coordinate is Poisson with mean 1; the middle coordinates are independent and uniform on $[0,1]$ and the remaining coordinates are independent standard normal. We write $\mathbb{P}$ for this measure on $\mathbb{N}_0 \times [0,1]^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$. Since Φ is measurable, this distribution pushes forward to a measure $\mathbb{P}_{\mathsf{P}L}$ on $\mathsf{PL}[0,1]$. That is, if S is a subset of $\mathsf{PL}[0,1]$, then $\mathbb{P}_{\mathsf{P}L}(S)$ is defined to be $\mathbb{P}\big(\{n,(x_i);(y_i)\}\colon \Phi(n,(x_i),(y_i)) \in S\big)$ (in particular, $\mathbb{P}_{\mathsf{P}L} = \mathbb{P}\circ\Phi^{-1}$).

We will prove that, a countable collection of piecewise linear functions, each sampled independently from $\mathbb{P}_{\mathsf{P}L}$ is almost surely smoothly dense. First we state and prove a simple lemma about real-valued random variables. A real-valued random variable X is said to be *continuous* if $\mathbb{P}(X = a) = 0$ for each $a \in \mathbb{R}$. It is said to be *absolutely continuous* if $\mathbb{P}(X \in A) = 0$ for each subset A of $\mathbb{R}$ of Lebesgue measure 0.

Lemma 5. *If X and Y are independent random variables with absolutely continuous distribution on $\mathbb{R}$, then $\mathbb{P}(X = Y) = 0$.*

In fact, we prove the stronger statement that even if just X has a continuous (not necessarily absolutely continuous) distribution, then $\mathbb{P}(X = Y) = 0$.

Proof. For any value y, conditioning on the event $\{Y = y\}$, since X has a continuous distribution, we have $\mathbb{P}(X = Y|Y = y) = 0$. It follows that the unconditional probability, $\mathbb{P}(X = Y)$ satisfies

$$\mathbb{P}(X = Y) = \int \mathbb{P}(X = y|Y = y) f_Y(y)\,dy = 0,$$

and the proof follows. □

We note that the argument above (that if some conditional probability vanishes whenever one conditions on any value of one of the random variables occurring in a condition, then that unconditional probability also vanishes) recurs throughout.

Lemma 6. *Let X_1, X_2, X_3, X_4 be independent absolutely continuously distributed random variables and a_1, a_2 be distinct real numbers. Then any of $\frac{X_4-X_3}{X_2-X_1}$, $\frac{X_4-X_3}{a_2-X_1}$, $\frac{X_4-X_3}{X_2-a_1}$, $\frac{X_4-X_3}{a_2-a_1}$ are absolutely continuously distributed on $\mathbb{R}$.*

Proof. It is straightforward to see that for $b \neq 0$ and $a \in \mathbb{R}$, $(X_4 - a)/b$ has an absolutely continuous distribution. Let G be the almost sure event where $X_2 \neq X_1$, $X_2 \neq a_1$ and $X_1 \neq a_2$. On the event G, conditioned on X_1, X_2, X_3, the slopes in the statement of the lemma all have absolutely continuous distributions. Since the conditional distributions are absolutely continuous, the unconditional distributions are also absolutely continuous. □

For $f \in \mathsf{PL}[0,1]$, let $\mathsf{Slope}(f)$ be the (finite) set of slopes that f assumes. Since the coordinates of the change points of a function chosen according to $\mathbb{P}_{\mathsf{P}L}$ are continuous random variables, by Lemma 6, for a given function $g \in \mathsf{PL}$ and randomly chosen f, the probability (under $\mathbb{P}_{\mathsf{P}L}$) that $\mathsf{Slope}(f)$ intersects $\mathsf{Slope}(g)$ is 0.

We now arrive at the main result of this subsection.

Theorem 7. *For $\mathbb{P}_{\mathsf{P}L}^{\mathbb{N}}$-almost every sequence $(f_n)_{n\in\mathbb{N}}$, the set $V = \{f_1, f_2, \ldots\}$ is smoothly dense.*

Proof. We first show that $\mathbb{P}_{\mathsf{P}L}^{\mathbb{N}}$-a.e. V is dense. Let $f \in C[0,1]$ and let $\epsilon > 0$. As mentioned at the beginning of this section, $\mathsf{PL}[0,1]$ is dense in $C[0,1]$. Let $\varphi \in \mathsf{PL}[0,1]$ be so that $\|\varphi - f\| < \frac{\epsilon}{2}$. Let $\mathsf{CP}(\varphi) = (c_i^\varphi)_{i=0}^{n+1}$, where $c_i^\varphi = (x_i, y_i)$ for $i = 0, \ldots, n+1$. We have that $\Phi(n, (\tilde{x}_i), (\tilde{y}_i)) = \varphi$ whenever $\tilde{y}_i = y_i$ for $i = 0, \ldots, n+1$ and x_i is the i-th largest element of $\{\tilde{x}_1, \ldots, \tilde{x}_n\}$, and in particular, if $\tilde{x}_i = x_i$ for $i = 1, \ldots, n$. Now there exists a $\delta > 0$ such that if $|\tilde{x}_i - x_i| < \delta$ for $i = 1, \ldots, n$ and $|\tilde{y}_i - y_i| < \delta$ for $i = 0, \ldots, n+1$, then $\|\Phi(n, (\tilde{x}_i), (\tilde{y}_i)) - \varphi\| < \frac{\epsilon}{2}$. Since the set $\{(n, (\tilde{x}_i), (\tilde{y}_i))\colon |\tilde{x}_i - x| < \delta \text{ for } i = 1, \ldots, n \text{ and } |\tilde{y}_i - y| < \delta \text{ for } i = 0, \ldots, n+1\}$ has positive measure with respect to $\mathsf{Pois}(1) \times \mathsf{Unif}[0,1]^{\mathbb{N}} \times \mathsf{Norm}(0,1)^{\mathbb{N}_0}$, V almost surely contains a g such that $\|g - f\| < \epsilon$ by the second Borel-Cantelli lemma.

Next, we show that V is transverse. Notice that if f and g are piecewise linear functions, then provided no piece of f has the same slope as a piece of g, then $\langle f\rangle$ and $\langle g\rangle$ have only finitely many intersections. Now fix a function $f \in \mathsf{PL}[0,1]$. By Lemma 6, conditional on

$\{f_i = f\}$, the probability that $\mathsf{Slope}(f_j)$ intersects $\mathsf{Slope}(f_i)$ is 0. Hence, the unconditional probability is also zero, so that with probability 1, all pairs of functions in V have disjoint slope sets. Hence, almost surely $\mathsf{cr}(f_i, f_j)$ is finite for all $i \neq j$. Since $f_i(0)$ and $f_j(0)$ are independently normally distributed, the probability of a crossing at 0 is zero and similarly at 1. For a fixed $f \in \mathsf{PL}[0,1]$ and for any other $g \in \mathsf{PL}[0,1]$ with distinct slopes, any extremum of $f-g$ occurs at a change point of f or g. By a similar argument to the above, the probability that g differs from f by an integer at an element of $\mathsf{CP}(f)$ is 0. Likewise, the probability that f differs from g by an integer at an element of $\mathsf{CP}(g)$ is 0. Hence, condition (2) of the definition of transversality is almost surely satisfied.

Similarly, for any finite subsets $A = \{t_1, \dots, t_l\}$ of $[0,1]$ and $B = \{s_1, \dots, s_l\}$ of $[0,1)$, then $\mathbb{P}_{\mathsf{P}L}(\{\langle f(t_i)\rangle = s_i \text{ for some } i\})$ is zero. Hence, conditional on $\{f_i = f\}$ and $\{f_j = g\}$ where $f, g \in \mathsf{PL}[0,1]$, $\mathbb{P}_{\mathsf{P}L}(\{\langle f_k(t)\rangle = \langle f(t)\rangle \text{ for some } t \in \mathsf{cr}(f,g)\}) = 0$. This shows that conditional on f_i and f_j, the probability that f_i, f_j and f_k have a common crossing is 0. The unconditional probability that three functions have a common crossing is therefore 0. Similarly conditioned on f_i and f_j, the probability that f_k and f_l have a crossing at an element of $\mathsf{cr}(f_i, f_j)$ is 0, so that condition (1) of the definition of transversality is almost surely satisfied and V is transverse with probability 1.

Since every piecewise linear function is Lipschitz, smoothly dense condition (3) is automatically satisfied. We finish the proof by showing that V satisfies smoothly dense condition (4). Let $f \in C[0,1]$ and $\mathcal{F} = \{f_1, \dots, f_r\} \subset V$ and suppose that $\mathcal{F} \cup \{f\}$ is transverse. Let K be such that each element of $\mathcal{F}$ is Lipschitz with Lipschitz constant at most K. Let $\epsilon > 0$. Let $C_j = \mathsf{cr}(f_j, f)$. By assumption, C_j does not contain any extrema of $f - f_j$. Let $C_j = \{t_{j,1}, \dots, t_{j,n_j}\}$ and let $f(t_{j,m}) - f_{i_j}(t_{j,m}) = a_{j,m} \in \mathbb{Z}$. For any m, by transversality of $\mathcal{F} \cup \{f\}$, we have that $t_{j,m}$ is not an extremum of $f - f_j$, and thus $f - f_j - a_{j_m}$ is negative on one side of $t_{j,m}$ and positive on the other. Let $\delta_{j,m} < \epsilon$ be such that:

(1) $\bigcup_{i=1}^r C_i \cap [t_{j,m} - 2\delta_{j,m}, t_{j,m} + 2\delta_{j,m}] = \{t_{j,m}\}$;
(2) $f - f_j - a_{j,m}$ takes one sign on the entire interval $[t_{j,m} - \delta_{j,m}, t_{j,m}]$ and the opposite sign on the interval $[t_{j,m}, t_{j,m} + \delta_{j,m}]$.

Let $\eta = \frac{1}{2}\min\{d(f(x) - f_i(x), \mathbb{Z}) : i \neq j \text{ and } x \in [t_{j,m} - \delta_{j,m}, t_{j,m} + \delta_{j,m}]\}$. That is, η is chosen so that any function $\tilde{f}$ within distance η of f on the intervals around the $t_{j,m}$ will be so that $\lfloor \tilde{f} - f_i \rfloor = \lfloor f - f_i \rfloor$ for all $f_i \in \mathcal{F} \setminus \{f_j\}$ on these intervals.

We define a target function, $\tilde{f}$ on $I_{j,m} = [t_{j,m} - \delta_{j,m}, t_{j,m} + \delta_{j,m}]$ such that:

(1) $\tilde{f}$ agrees with f at $t_{j,m} \pm \delta_{j,m}$;
(2) $\tilde{f}$ is linear with slope $\pm(K+2)$ on a neighbourhood $N_{j,m}$ of $t_{j,m}$ (so that $\tilde{f}$ crosses f_j in the same direction that f does at $t_{j,m}$);
(3) $\|(\tilde{f} - f)|_{I_{j,m}}\| < \eta$;
(4) $\tilde{f} - f_j - a_{j,m}$ has the same sign as $f - f_j - a_{j,m}$ on $I_{j,m}$.

Finally, let $\tilde{f}$ be defined to be equal to f on $[0,1] \setminus \bigcup_{j,m} I_{j,m}$. Let $\zeta < 1$ be such that $d(\tilde{f}(t) - f_j(t), \mathbb{Z}) \geq 2\zeta$ for each $j = 1, \dots, r$ and all t outside $\bigcup_{j,m} N_{j,m}$.

Let g be any function satisfying:

(1) $\|g - \tilde{f}\| < \zeta$;
(2) $|g' - \tilde{f}'| < 1$ on $N_{j,m}$ for each j, m.

Then we claim that $\mathsf{cr}(g, f_j) \sim_\epsilon \mathsf{cr}(f, f_j)$ for $j = 1, \dots, r$. To see this, note that the first condition ensures that no crossings are created outside $\bigcup N_{j,m}$. Since $|g' - \tilde{f}'| < 1$ on $\bigcup N_{j,m}$

and $\tilde{f} - f_j$ is monotone on $N_{j,m}$, we find that g crosses f_j at most once on each $N_{j,m}$. Finally, we observe that since g starts on one side of f_j and ends on the other, the Intermediate Value Theorem ensures that there is at least one (and thus, exactly one) crossing of g with f_j in each $N_{j,m}$, ensuring that $\mathsf{cr}(g, f_j) \sim_\epsilon \mathsf{cr}(f, f_j)$.

Finally, we observe that the above conditions are satisfied for a collection of piecewise linear functions with change points in a set of positive measure. Hence, by the second Borel-Cantelli lemma, there are infinitely many $g \in V$ satisfying $\|f - g\| < \epsilon$ and $\mathsf{cr}(f, h) \sim_\epsilon \mathsf{cr}(g, h)$ for all $h \in \mathcal{F}$. □

2.3. **Polynomials.** We equip $\mathbb{N}_0 \times \mathbb{R}^{\mathbb{N}_0}$ with the probability measure $\mathsf{Pois}(1) \times \mathsf{Norm}(0, 1)^{\mathbb{N}_0}$ and, for any sequence $(a_i)_{i\in\mathbb{N}_0}$, let $\Phi(n, (a_i))$ be the polynomial $\varphi(t) = a_0 + a_1 t + \ldots + a_n t^n$. The map Φ pushes forward the measure on parameters to a measure $\mathbb{P}_{\text{poly}}$ on $C[0, 1]$.

Theorem 8. *For $\mathbb{P}^{\mathbb{N}}_{poly}$-almost every sequence $(f_n)_{n\in\mathbb{N}}$, the set $V = \{f_1, f_2, \ldots\}$ is smoothly dense.*

Proof. We first show that V is dense. By Weierstrass's theorem, the collection of polynomials is uniformly dense in $C[0, 1]$. Given $f \in C[0, 1]$ and $\epsilon > 0$, let function φ given by $\varphi(t) = a_0 + a_1 t + \ldots + a_n t^n$ satisfy $\|f - \varphi\| < \frac{\epsilon}{2}$. Now $\{(n; (b_i)_{i\in\mathbb{N}_0}) : |b_i - a_i| \le \epsilon/(2(n+1)) \text{ for } i = 0, \ldots, n\}$ has positive measure and any polynomial with these parameters is $\frac{\epsilon}{2}$-close to φ, and so ϵ-close to f. Hence, as before, V is almost surely dense in $C[0, 1]$.

Next, we show that V is transverse. Recall that for any two polynomials f_1 and f_2 that differ in at least one non-constant coefficient, $\mathsf{cr}(f_1, f_2)$ is finite (if f_1 and f_2 are both constant polynomials, then they almost surely have no crossings). Conditional on the non-constant coefficients of a third polynomial f_3, the probability that its fractional part agrees with that of f_1 and f_2 at an element of $\mathsf{cr}(f_1, f_2)$ is 0. Similarly, conditioned on f_1 and f_2, the probability that two further polynomials f_3 and f_4 cross at an element of $\mathsf{cr}(f_1, f_2)$ is 0. Hence, condition (1) of transversality holds.

Given a polynomial f, and the non-constant coefficients of a polynomial g, the conditional probability that f and g cross at 0 or 1 is zero. Hence, the unconditional probability is also 0, so that almost surely $\mathsf{cr}(f, g)$ is finite and does not contain 0 or 1. Given the non-constant coefficients of g, provided f and g are not constant polynomials, $f - g$ has finitely many critical points. Now the conditional probability that when the constant term is added to g, that f and g cross at one of these critical points is 0. Hence, almost surely $\mathsf{cr}(f, g)$ does not contain any critical points of $f - g$ (in case f and g are constant, $\mathsf{cr}(f, g)$ is almost surely empty). Hence, condition (2) of transversality holds, so that V is almost surely a transverse subset of $C[0, 1]$.

It is evident all polynomials are Lipschitz on $[0, 1]$, and so the smoothly dense condition (3) is satisfied. The proof that V satisfies smoothly dense condition (4) is almost exactly the same as that given for piecewise linear functions above. Namely, we identify neighbourhoods $N_{j,m}$ around the crossing points $t_{j,m}$ of f with all functions in the given set $\mathcal{F}$. We define a smooth target function $\tilde{f}$ which has the same crossing behaviour as f, but crosses steeply, with slope exceeding that of any function in $\mathcal{F}$. We need to show that the conditions:

(1) $\|g - \tilde{f}\| < \zeta$;
(2) $|g'(x) - \tilde{f}'(x)| < 1$ for all $x \in \bigcup N_{j,m}$ (with $N_{j,m}$ as in the previous subsection)

are satisfied by a collection of polynomials in a set of parameters of positive measure (since all of these functions have the required property that their crossings with $\mathcal{F}$ are close to those of f). To see this, let h_1 be a continuously differentiable function agreeing with $\tilde{f}$ on $\bigcup N_{j,m}$, and differing from $\tilde{f}$ by at most $\frac{\zeta}{4}$ elsewhere (such functions exist by density of the continuously differentiable functions in the continuous functions). Now, let $h_2(x) = a_1 + a_2 x + \dots a_n x^{n-1}$ be a polynomial differing from h_1' by at most $\frac{\zeta}{4}$. Finally, we claim that any polynomial $g(x) = b_0 + b_1 x + \dots b_n x^n$ satisfies the required conditions if $|b_0 - \tilde{f}(0)| < \frac{\zeta}{4}$ and $|kb_k - a_k| < \frac{\zeta}{4n}$ for $k = 1, \dots n$. To see this, notice that if these conditions are satisfied, then $|g'(x) - h_2(x)| \leq \frac{\zeta}{4}$ for each $x \in [0,1]$, so that $|g'(x) - h_1'(x)| \leq \frac{\zeta}{2}$ for each x. Since $|g(0) - h_1(0)| \leq \frac{\zeta}{4}$, we derive that $|g(x) - h_1(x)| \leq \frac{3\zeta}{4}$ for each x and so $|g(x) - \tilde{f}(x)| \leq \zeta$ for each x.

On $\bigcup N_{j,m}$, we have $h_1' = \pm(K+2)$ and $\|h_1' - g'\| < \frac{\zeta}{2}$ so that $|h_1'| > K+1$ on $\bigcup N_{j,m}$. This is sufficient to ensure that g crosses f_{i_j} at most once and the intermediate value theorem ensures that g crosses f_{i_j} on $N_{j,m}$.

As before, these conditions are satisfied by a collection of polynomials with parameters lying in a set of positive measure. It follows by the second Borel-Cantelli lemma that V contains infinitely many polynomials with the correct crossings. Hence, V is almost surely smoothly dense. □

Corollary 9. *If V is a countable set of independently sampled functions from $\mathbb{P}_{poly}$ or $\mathbb{P}_{\mathsf{PL}}$, then for all $p \in (0,1)$, a graph sampled form* LARG(V,p) *is almost surely isomorphic to* $GR(\mathrm{SD})$.

3. IC-dense sets and Brownian motion

By definition, smoothly dense sets must contain functions that are relatively smooth, have finitely many extrema, and cross finitely often with other functions in the set. In this section, we extend our results to functions in $C[0,1]$ that exhibit the opposite type of behaviour: pairs of functions cross infinitely often. We will call sets of such functions *infinite crossing dense* (IC-dense). An example of such functions is formed by Brownian motions. We will first show that any two LARG graphs with IC-dense vertex sets are almost surely isomorphic. Then, we will define a natural probability measure on the set of Brownian motions, and show that under this measure almost all countable sets of Brownian motions are IC-dense.

A countable subset V of $C[0,1]$ will be called *IC-dense* if it has the following properties:

(1) The set V is dense in $C[0,1]$;
(2) The sets $\mathsf{cr}(f,g)$ are disjoint for distinct pairs $f, g \in V$;
(3) For any $f, g \in V$, $\mathsf{cr}(f,g)$ does not contain 0 or 1;
(4) For each distinct pair $f, g \in V$, if $\langle f(x) \rangle = \langle g(x) \rangle$ for $x \in (0,1)$, then there exists a sequence of points x_n converging to x monotonically such that $\langle f(x_{2n}) \rangle < \langle g(x_{2n}) \rangle$ and $\langle f(x_{2n+1}) \rangle > \langle g(x_{2n+1}) \rangle$ for each n.

Note that the fact that the functions are continuous implies that $\mathsf{cr}(f,g)$ is closed for each $f, g \in V$.

3.1. IC-dense sets are Rado.

We will now show that IC-dense sets are Rado. As in the case for smoothly dense sets, the proof is based on the inductive construction of a graph isomorphism. But whereas in the smoothly dense case we required that the isomorphism exactly preserves the crossing behaviour of the functions, here we wish to preserve the decomposition of the interval $[0,1]$ into subintervals where pairs of functions cross infinitely often. Before we state the main theorem, we give definitions and useful lemmas.

Given finite sequences of real numbers $x_1, \dots, x_k$; and $y_1, \dots, y_k$, we say they have the same *circular order* if for any a, b, c in $\{1, \dots, k\}$, the orientation of $\langle x_a \rangle$, $\langle x_b \rangle$ and $\langle x_c \rangle$ on the circle is the same as that of $\langle y_a \rangle$, $\langle y_b \rangle$ and $\langle y_c \rangle$. We write $x_a \prec x_b \prec x_c \prec x_a$ if $\langle x_a \rangle$, $\langle x_b \rangle$ and $\langle x_c \rangle$ occur in a positive orientation on the circle.

It is easy to see that a function f defined by $f(x_i) = y_i$ for $i = 1, \dots, k$ is a step-isometry if $\lfloor x_i \rfloor = \lfloor y_i \rfloor$ and $x_1, \dots, x_k$ and $y_1, \dots, y_k$ have the same circular order. In the induction step of the isomorphism proof we will preserve the circular orders of the functions that have already been mapped.

Crossing partitions. If $\mathcal{F}$ is a finite IC-dense subset of $C[0, 1]$, then we build a finite partition of $[0, 1]$ into open and closed sub-intervals by the following procedure. Let

$$a_1 = \min\{x \in [0, 1] \colon \langle g(x) \rangle = \langle h(x) \rangle \text{ for some distinct } g, h \in \mathcal{F}\}. \tag{1}$$

Suppose $a_1, \dots, a_n$ have been found, and let $(g_i, h_i)_{i=1}^n$ be pairs of distinct elements of $\mathcal{F}$ so that $\langle g_i(a_i) \rangle = \langle h_i(a_i) \rangle$ for each $1 \le i \le n$. Then define

$$a_{n+1} = \min \big\{ x \in (a_n, 1] \colon \langle h(x) \rangle = \langle g(x) \rangle \text{ for some } g, h \in \mathcal{F} \tag{2}$$

$$\text{such that } \{g, h\} \neq \{g_n, h_n\} \big\}. \tag{3}$$

If a_{n+1} is defined (that is, the set is not empty), then set $g_{n+1} = g$ and $h_{n+1} = h$. If not, then the procedure terminates with a_n.

Lemma 10. *The procedure to generate the sequence (a_n) as defined in (1) and (2) terminates in a finite number of steps.*

Proof. Suppose for a contradiction that there are infinitely many a_n's. Since (a_n) is an increasing sequence lying in $[0, 1]$, let its limit be a. By the Pigeonhole Principle, there are functions f^1, f^2, f^3 and f^4 in $\mathcal{F}$ (not necessarily distinct, but such that $f^1 \neq f^2$, $f^3 \neq f^4$ and $\{f_1, f_2\} \neq \{f_3, f_4\}$) and a subsequence (n_j) such that $\{g_{n_j}, h_{n_j}\} = \{f^1, f^2\}$ and $\{g_{n_j+1}, h_{n_j+1}\} = \{f^3, f^4\}$ for all j. We have $\langle f^1(a_{n_j}) \rangle = \langle f^2(a_{n_j}) \rangle$ and $\langle f^3(a_{n_j+1}) \rangle = \langle f^4(a_{n_j+1}) \rangle$ for each j. Taking a limit, and using continuity of the functions involved, we see that $\langle f^1(a) \rangle = \langle f^2(a) \rangle$ and $\langle f^3(a) \rangle = \langle f^4(a) \rangle$. That is, $a \in \mathsf{cr}(f^1, f^2) \cap \mathsf{cr}(f^3, f^4)$. This contradicts the IC-dense condition (2), establishing the above procedure terminates as required. $\square$

Now for each i such that a_i is defined, let $b_i = \max\{x \in [a_i, a_{i+1}] \colon \langle g_i(x) \rangle = \langle h_i(x) \rangle\}$. Note that, since $\mathsf{cr}(g_i, h_i)$ is closed, this maximum is defined, and since h_i and g_i cross at b_i and a different pair h_{i+1}, g_{i+1} crosses at a_{i+1}, we have that $b_i < a_{i+1}$. The *crossing partition* $\mathcal{P}(\mathcal{F})$ is then given by

$$\mathcal{P}(\mathcal{F}) = \{[0, a_1), [a_1, b_1], (b_1, a_2), [a_2, b_2], \dots, [a_n, b_n], (b_n, 1]\},$$

so that on the (relatively) open sets $[0, a_1)$, (b_1, a_2), $\dots$, (b_{n-1}, a_n) and $(b_n, 1]$, there are no crossings and the functions in $\mathcal{F}$ maintain a fixed circular order, while on $[a_i, b_i]$, the order between g_i and h_i switches infinitely many times (on any neighbourhood of a_i or b_i), while the circular order of $\mathcal{F} \setminus \{g_i\}$ and $\mathcal{F} \setminus \{h_i\}$ remains fixed. The closed intervals $[a_i, b_i]$ are called the *crossing intervals*.

Given infinite graphs G and H on vertex sets V and W, respectively, and sets $\mathcal{F} = \{F_1, \dots, F_n\} \subseteq V$ and $\mathcal{G} = \{G_1, \dots, G_n\} \subseteq W$ are *suitably matched* if:

(1) For each i and j, F_i and F_j are joined by an edge if and only if G_i and G_j are joined by an edge;

(2) For each i and j, $\lfloor |F_i(0) - F_j(0)| \rfloor = \lfloor |G_i(0) - G_j(0)| \rfloor$ and $F_i(0) < F_j(0)$ if and only if $G_i(0) < G_j(0)$;

(3) (a) If the crossing partition $\mathcal{P}(\mathcal{F})$ consists of $2n+1$ intervals of the form $\{[0, a_1), [a_1, b_1], (b_1, a_2), [a_2, b_2], \ldots, [a_n, b_n], (b_n, 1]\}$, then $\mathcal{P}(\mathcal{G})$ also consists of $2n+1$ intervals of the same form. Call them

$$\{[0, a_1'), [a_1', b_1'], (b_1', a_2'), [a_2', b_2'], \ldots, [a_n', b_n'], (b_n', 1]\};$$

(b) F_i and F_j cross on interval $[a_k, b_k]$ if and only if G_i and G_j cross on $[a_k', b_k']$;

(c) the circular order of F_i, F_j and F_k on any interval of $\mathcal{P}(\mathcal{F})$ agrees with that of G_i, G_j and G_k on the corresponding interval of $\mathcal{P}(\mathcal{G})$ (provided that the interval is not a crossing interval of two of F_i, F_j and F_k).

If $\varphi : \mathcal{F} \to \mathcal{G}$ is so that $\varphi(F_i) = G_i$ for $i = 1, \ldots n$, then we say that $\mathcal{F}$ and $\mathcal{G}$ are suitably matched *via* φ. From condition (1), it follows immediately that φ is an isomorphism from $G[\mathcal{F}]$ to $H[\mathcal{G}]$.

Lemma 11. *, Let $\mathcal{F}$ and $\mathcal{G}$ be finite subsets of IC-dense sets. If $\mathcal{F}$ and $\mathcal{G}$ are suitably matched via φ, then φ is a step-isometry.*

Proof. We claim that if $\mathcal{F}$ and $\mathcal{G}$ are suitably matched, and $\mathcal{F} = \{F_1, \ldots, F_n\}$ and $\varphi(F_i) = G_i$ for $i = 1, \ldots, n$, then they have the additional property that $\lfloor \|F_i - F_j\| \rfloor = \lfloor \|G_i - G_j\| \rfloor$ for each i, j. To see this, notice that by condition (2) $\lfloor |F_i(0) - F_j(0)| \rfloor = \lfloor |G_i(0) - G_j(0)| \rfloor$. Since F_i and F_j then have the same crossings and the same circular order as G_i and G_j (replacing crossing intervals of F_i and F_j and of G_i and G_j by points), we deduce that $\lfloor \|F_i - F_j\| \rfloor = \lfloor \|G_i - G_j\| \rfloor$ as required. □

Theorem 12. *There exists a graph $GR(\mathrm{ICD})$ such that if V is IC-dense, and $p \in (0, 1)$, then $\mathrm{LARG}(V, p)$ is almost surely isomorphic to $GR(\mathrm{ICD})$.*

Proof. Let V and W be infinite crossing dense subsets of $C([0, 1])$ and let $p, p' \in (0, 1)$. We show that a realization of $\mathrm{LARG}(V, p)$ is almost surely isomorphic to a realization of $\mathrm{LARG}(W, p')$. As in the proof of Theorem 4 for smoothly dense sets, we construct an isomorphism via a back-and-forth argument. The key step of the proof is to show that if finite sets $\mathcal{F}$ and $\mathcal{G}$ are suitably matched finite subsets of V and W, then given any additional $f \in V$, there is almost surely a $g \in W$ such that $\mathcal{F} \cup \{f\}$ and $\mathcal{G} \cup \{g\}$ are suitably matched.

We focus on a single step in one direction. Let $\mathcal{F} = \{F_1, \ldots, F_n\} \subseteq V$ and $\mathcal{G} = \{G_1, \ldots, G_n\} \subseteq W$ be suitably matched via an isomorphism φ, so that F_l and G_l correspond in the suitable matching. We show that we can extend the isomorphism by adding the next element in V to $\mathcal{F}$, finding a suitable image for it, and adding this image to $\mathcal{G}$.

Let $f \in V \setminus \mathcal{F}$ and let the crossing partitions of $[0, 1]$ of $\mathcal{F}$ and $\mathcal{G}$ be

$$\mathcal{P}(\mathcal{F}) = \{[0, a_1), [a_1, b_1], (b_1, a_2), [a_2, b_2], \ldots, [a_N, b_N], (b_N, 1]\} \text{ and}$$
$$\mathcal{P}(\mathcal{G}) = \{[0, a_1'), [a_1', b_1'], (b_1', a_2'), [a_2', b_2'], \ldots, [a_N', b_N'], (b_N', 1]\}.$$

Consider $\mathcal{P}(\mathcal{F} \cup \{f\})$. This partition is a refinement of $\mathcal{P}(\mathcal{F})$: the open sub-intervals of $\mathcal{P}(\mathcal{F})$ may be sub-divided in $\mathcal{P}(\mathcal{F} \cup \{f\})$ due to intersections of f with elements of $\mathcal{F}$. On the closed intervals $[a_j, b_j]$, two functions F_l and F_m in $\mathcal{F}$ cross infinitely often, and, by construction, no other functions of $\mathcal{F}$ cross. When the intersections with f are used to refine the partition, one obtains a sub-partition of $[a_j, b_j]$ consisting of closed intervals in which F_l and F_m cross infinitely often; open intervals (with no crossings); and closed intervals in which f crosses an element of $\mathcal{F}$ (possibly F_l or F_m), again infinitely often. In each closed

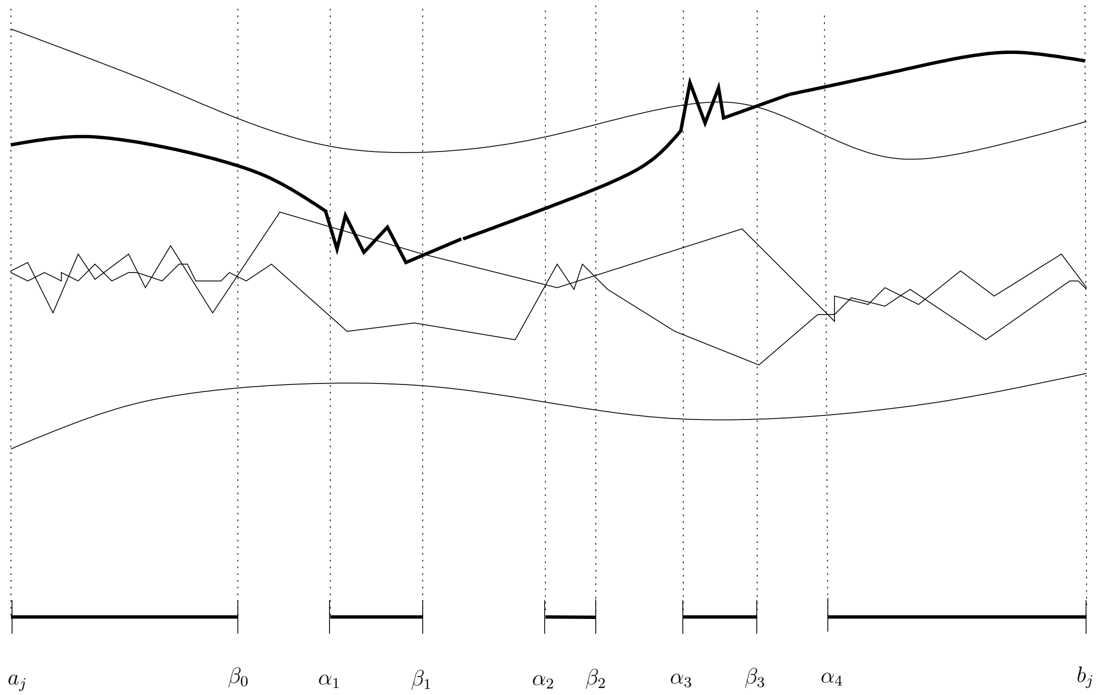

FIGURE 2. Schematic indicating the refinement of an interval $[a_j, b_j]$ in which the two central functions, F_l and F_m are crossing when a new function f (in bold) is added to the collection. The closed sub-intervals in the refinement are indicated below.

interval, there are crossings between exactly one pair of functions. If there are no crossings of f and $\mathcal{F}$, the sub-partition is trivial; otherwise there are sub-intervals of all three types. The situation is illustrated schematically in Figure 2.

We shall build a target function g such that all functions h in a suitable neighbourhood of g satisfy suitable matching conditions (2) and (3). This function is built sub-interval by sub-interval, interpolating various convex combinations of the (G_i).

We first deal with the (harder) case of the sub-partition of one of the closed intervals $[a_j, b_j] \in \mathcal{P}(\mathcal{F})$. Suppose that the functions crossing in the interval are F_l and F_m. In $\mathcal{P}(\mathcal{F} \cup \{f\})$, let $[a_j, b_j]$ be refined as

$$\mathcal{P}' = \{[a_j, \beta_0], (\beta_0, \alpha_1), [\alpha_1, \beta_1], (\beta_1, \alpha_2), [\alpha_2, \beta_2], \ldots, [\alpha_{M-1}, \beta_{M-1}], (\beta_{M-1}, \alpha_M), [\alpha_M, b_j]\}.$$

Since the refinement is obtained by adding f to $\mathcal{F}$, at each of $\alpha_1, \ldots, \alpha_M$, there is a crossing of f and an element of $\mathcal{F}$, or a crossing of F_l and F_m.

Let $[a'_j, b'_j]$ be the corresponding interval to $[a_j, b_j]$ in $\mathcal{P}(\mathcal{G})$. We say a sub-interval $[\alpha', \beta']$ of $[a'_j, b'_j]$ is a *crossing sub-interval* for G_l and G_m if G_l and G_m cross at α' and β', but not on small open intervals $(\alpha' - \eta, \alpha')$ and $(\beta', \beta' + \eta)$, for some $\eta > 0$. Such a crossing sub-interval is of *type* $<<$, $<>$, $><$ or $>>$ if respectively, $\langle G_l \rangle$ is below $\langle G_m \rangle$ to the left and the right of the interval; $\langle G_l \rangle$ is below $\langle G_m \rangle$ to the left and above to the right; $\langle G_l \rangle$ is above $\langle G_m \rangle$ to the left and below to the right; or $\langle G_l \rangle$ is above $\langle G_m \rangle$ to the left and right of the interval. Condition (3) ensures that there are infinitely many crossing sub-intervals of each type in any sub-interval $[a'_j, a'_j + \delta)$ or $(b'_j - \delta, b'_j]$.

We now proceed to define a "target sub-partition" of $\mathcal{P}(\mathcal{G})$ which corresponds to the restriction of $\mathcal{P}(\mathcal{F}\cup\{f\})$ to $[a'_j, b'_j]$. We will subsequently construct a target function g such that any sufficiently nearby function $\tilde{g}$ in W has the property that $\mathcal{P}(\mathcal{G}\cup\{\tilde{g}\})$ is close to $\mathcal{P}(\mathcal{G})$.

First, let $[a_j, \beta_0]$, $[\alpha_{k_i}, \beta_{k_i}]$ for $i = 1, \ldots, L$ and $[\alpha_M, b_j]$ be the sub-intervals in $\mathcal{P}(\mathcal{F}\cup\{f\})$ where F_l and F_m cross (or just $[a_j, b_j]$ if f does not intersect $\mathcal{F}$ on $[a_j, b_j]$). For each, pick a crossing sub-interval $[\alpha'_{k_i}, \beta'_{k_i}]$ of $[a'_j, b'_j]$ for G_l and G_m of the same type, arranged in the same order on the interval. This is possible by the above observation that there are infinitely many crossing sub-intervals of each type near a'_j and b'_j. Between each pair, $[\alpha'_{k_i}, \beta'_{k_i}]$ and $[\alpha'_{k_{i+1}}, \beta'_{k_{i+1}}]$, pick a sub-interval O_i on which G_l and G_m do not cross and have the same ordering as F_l and F_m have on $(\beta_{k_i}, \alpha_{k_{i+1}})$.

Next, for the remaining sub-intervals $[\alpha_k, \beta_k]$ between $[\alpha_{k_i}, \beta_{k_i}]$ and $[\alpha_{k_{i+1}}, \beta_{k_{i+1}}]$ where f crosses some F_j, arbitrarily pick disjoint sub-intervals $[\alpha'_k, \beta'_k]$ of O_i that respect the left-to-right ordering.

Finally, write

$$\mathcal{Q}' = \{[a'_j, \beta'_0], (\beta'_0, \alpha'_1), [\alpha'_1, \beta'_1], (\beta'_1, \alpha'_2), [\alpha'_2, \beta'_2], \ldots, [\alpha'_{M-1}, \beta'_{M-1}], (\beta'_{M-1}, \alpha_M), [\alpha_M, b'_j]\},$$

This process gives a partition of the closed sub-intervals that has similar behaviour, with respect to the functions in $\mathcal{G}$, as the functions in $\mathcal{F}$ have on the sub-intervals induced by introducing f to the crossing partition of $\mathcal{F}$. We will now do the same for the open sub-intervals.

In the case that one is sub-partitioning one of the open sub-intervals (b_j, a_{j+1}), things are easier as the only crossings that occur in the sub-interval are those between f and the F_j's. In this case, as before write

$$\mathcal{P}' = \{(b_j, \alpha_1), [\alpha_1, \beta_1], (\beta_1, \alpha_2), [\alpha_2, \beta_2], \ldots, [\alpha_M, \beta_M], (\beta_M, a_{j+1})\}.$$

Recall that, by construction, in each of the open intervals no functions cross, while in each closed intervals, f crosses infinitely often with exactly one of the functions in $\mathcal{F}$. Define an arbitrary set of division points α'_i and β'_i such that $b'_j < \alpha'_1 < \beta'_1 < \ldots < \beta'_M < a'_{j+1}$, and let

$$\mathcal{Q}' = \{(b'_j, \alpha'_1), [\alpha'_1, \beta'_1], (\beta'_1, \alpha'_2), [\alpha'_2, \beta'_2], \ldots, [\alpha'_M, \beta'_M], (\beta'_M, a'_{j+1}]\}.$$

Given an interval I of $\mathcal{P}'$ in which f does not cross any of the F_j's, let F_p be the function immediately below f in the circular ordering on I and F_q be the function immediately above. Let the integers r and s be chosen so that $F_p + r < f < F_q + s$ on I and r and s are respectively, the largest and smallest integers with that property. Let J be the interval in $\mathcal{Q}'$ corresponding to I. We then define g on J to be $\frac{1}{2}((G_p + r) + (G_q + s))$. If $I \in \mathcal{P}'$ is such that f crosses the function F_j, then let the circular predecessor and successor of F_j be F_p and F_q respectively, and as above, let $F_p + r < f < F_q + s$. Also, suppose that the integer t is chosen so that $F_j + t$ intersects f (in $\mathbb{R}$; not just in the circle), so that $F_j + t$ satisfies the same inequality at f on I: $F_p + r < F_j + t < F_q + s$. We then define g by interpolating between midpoints of pairs of $F_p + r$, $F_j + t$ and $F_q + s$. In particular, if $J = [\alpha', \beta']$ is the corresponding interval to I and the intersection between f and F_j on I is of type <> for example, then we define g restricted to J by

$$g(x) = \frac{\beta' - x}{2(\beta' - \alpha')}(G_p + r + G_j + t) + \frac{x - \alpha'}{2(\beta' - \alpha')}(G_j + t + G_q + s),$$

so that g crosses from the midpoint below G_j to the midpoint above G_j. For a second example, if f is of type $<<$, we set $\gamma' = \frac{\alpha'+\beta'}{2}$ and define

$$g(x) = \begin{cases} \frac{\gamma'-x}{2(\gamma'-\alpha')}(G_p + r + G_j + t) + \frac{x-\alpha'}{2(\gamma'-\alpha')}(G_j + t + G_q + s) & x \in [\alpha', \gamma'], \\ \frac{\beta'-x}{2(\beta'-\gamma')}(G_j + t + G_q + s) + \frac{x-\gamma'}{2(\beta'-\gamma')}(G_p + r + G_j + t) & x \in [\gamma', \beta'] \end{cases},$$

so that g crosses from the midpoint below G_j to the midpoint above and back down again. The piecewise interpolated function, g, defined on all of [0,1] interval by interval in this way is continuous, and has the same pattern of crossings with $\mathcal{G}$ that f has with $\mathcal{F}$.

Note that for each crossing interval of f with one of the functions F_k, g likely crosses G_k infinitely often, but this is not guaranteed from the definition. Nonetheless, when g is approximated by a $\tilde{g}$ belonging to W, crossings between $\tilde{g}$ and the G_k do occur with infinite multiplicity.

Let C be the union of all of the sub-intervals $[\alpha', \beta']$ in [0,1] on which g crosses $\mathcal{G}$, and let ϵ be smaller than half the distance between two consecutive sub-intervals in C, and also smaller than a'_1 and b'_N. Let $\mathcal{N}$ be the ϵ-neighbourhood of C, so that it consists of disjoint open intervals, one for each of the closed sub-intervals forming C. There exists a $\delta_1 > 0$ such that $|g(x) - \big(G_j(x) + m\big)| \geq \delta_1$ for all $x \in [0,1] \setminus \mathcal{N}$, all $j = 1, \ldots, n$ and all $m \in \mathbb{Z}$. Also, there exists $\delta_2 > 0$ such that in each of the sub-intervals in $\mathcal{P}'$ where g crosses any $G_j + m$, it attains values less than $G_j + m - \delta_2$ and values greater than $G_j + m + \delta_2$. Finally, there exists a $\delta_3 > 0$ such that if $\|\tilde{g} - g\| < \delta_3$, $\lfloor |f(0) - F_j(0)| \rfloor = \lfloor |\tilde{g}(0) - G_j(0)| \rfloor$. Now if $\|\tilde{g} - g\| < \delta = \min(\delta_1, \delta_2, \delta_3)$, we see that $\mathcal{F} \cup \{f\}$ and $\mathcal{G} \cup \{\tilde{g}\}$ satisfy conditions (2) and (3) of suitable matching. By Lemma 11, for any $\tilde{g}$ such that $\|\tilde{g} - g\| < \delta$, we have $\lfloor \|\tilde{g} - G_j\| \rfloor = \lfloor \|f - F_j\| \rfloor$ for each $1 \leq j \leq n$. In particular, for each j such that f is adjacent to F_j, we have $\|\tilde{g} - G_j\| < 1$. Hence, since $\{g_n\}$ is dense, there are infinitely many k's such that $\|g_k - g\| < \delta$ and g_k has the correct adjacencies to $\mathcal{G}$. Thus, $\mathcal{F} \cup \{f\}$ is suitably matched to $\mathcal{G} \cup \{g_k\}$. This completes the inductive step, and hence, the proof. □

3.2. **Brownian motion.** We consider the following procedure to select random functions: a starting point (the value at 0) is chosen from a standard normal distribution and from there the function follows a standard Brownian path. Formally, let $(N_n)_{n\in\mathbb{N}}$ be a family of countably many independent standard normal random variables and let $(X_n(t))_{n\in\mathbb{N}}$ be a family of countably many independent standard Brownian motions (with $X_n(0) = 0$, $\mathbb{E}X_n(t) = 0$ and $\operatorname{Var} X_n(t) = t$ for each $n \in \mathbb{N}$ and $t \in [0,1]$). Let $f_n(t) = N_n + X_n(t)$ be a random function obtained in this manner, which we refer to as a *shifted Brownian motion*.

Theorem 13. *If $(f_n)_{n\in\mathbb{N}}$ is a family of countably many independent shifted Brownian motions sampled from the above distribution, then the collection is almost surely IC-dense.*

Proof. Let $\epsilon > 0$ be given. Let f be a continuous function. There exists a piecewise linear function g with change points at $(0, y_0), (x_1, y_1), \ldots, (x_{n-1}, y_{n-1}), (1, y_n)$ such that $\|g - f\| < \frac{\epsilon}{2}$. By standard properties of Brownian motion, it is known that $\mathbb{P}(|N - y_0| < \frac{\epsilon}{2n}) > 0$ and that $\mathbb{P}(|X(x_j) - X(x_{j-1}) - (y_j - y_{j-1})| < \frac{\epsilon}{2n} \,|\, X(x_1), \ldots, X(x_{j-1})) > 0$. Finally, it is known that conditioned on the values of $X(x_1), \ldots, X(x_n)$, the probability that X never strays by more than $\frac{\epsilon}{2}$ from the linear function joining $X(x_i)$ and $X(x_{i+1})$ is positive (the distribution of $(X(t))_{x_i \leq t \leq x_{i+1}}$ conditioned on $X(x_i)$ and $X(x_{i+1})$ is a so-called *Brownian bridge*). Multiplying these probabilities together, we note that the probability that $\|f_j - f\| < \epsilon$ is positive, and is the same for each j. Hence, by the second Borel-Cantelli lemma, there almost surely exists a j such that $\|f_j - f\| < \epsilon$. Since $C([0,1])$ is separable, the above argument shows that (f_n) is almost surely dense.

IC-dense Condition (2) follows from a corollary of Lévy's theorem (Corollary 2.23 in Mörters and Peres [15]) which states that for any non-zero point p in $\mathbb{R}^2$, the standard 2-dimensional Brownian motion almost surely does not pass through p. To see this, we deal with two cases: we first show that there are no three functions all crossing at a single x value; and then show that there are not two distinct pairs of functions with both pairs crossing at a single x value. For the first of these, it suffices (since there are countably many triples) to show that with probability 1, f_1, f_2 and f_3 do not all cross at any x. That is, it suffices to show that for any integers m and n, with probability 1, there does not exist an x such that $f_2(x) - f_1(x) = m$ and $f_3(x) - f_1(x) = n$. Conditional on $N_1 = y_1$, $N_2 = y_2$ and $N_3 = y_3$, the values at $x = 0$, we therefore have to show that

$$\mathbb{P}(\exists x\colon X_2(x) - X_1(x) = m + y_2 - y_1 \text{ and } X_3(x) - X_1(x) = n + y_3 - y_1) = 0.$$

This does not immediately follow from the theorem mentioned above about 2-dimensional Brownian motion since $((X_2 - X_1)(x), (X_3 - X_1)(x))$ is not a standard Brownian motion: the coordinates are correlated. It is, however, expressible as a fixed linear transformation of a standard Brownian motion (see, for example, [15] page 14) and hence, the previous theorem applies.

Similarly, to show that f_1, f_2, f_3 and f_4 do not have the property that f_1 and f_2 cross at some x and f_3 and f_4 cross at the same x, we need to show that

$$\mathbb{P}(\exists x\colon X_2(x) - X_1(x) = m + y_1 - y_2 \text{ and } X_4(x) - X_3(x) = n + y_3 - y_4) = 0.$$

This follows directly from the quoted theorem since $((X_2 - X_1)(x), (X_4 - X_3)(x))$ is a scaled copy of the standard Brownian motion (by a scale factor of $\sqrt{2}$).

To see that f_1 and f_2 do not cross at 0, we observe that $\mathbb{P}(N_2 = N_1 + n | N_1 = x) = 0$ and so integrating over x, the probability of a crossing at 0 is 0, similar to the argument in Lemma 5. A similar argument involving N_1, N_2, $X_1(1)$ and $X_2(1)$ shows that the probability of a crossing at 1 is 0, and this proves (3).

To verify the last condition, notice that $f_2(x) - f_1(x) = X_2(x) - X_1(x) + N_2 - N_1$. By Theorem 2.11 of [15], a Brownian motion has countably many local extrema and by Theorem 5.14 of [15], no local points of increase or decrease (a point of local increase for a Brownian motion $Z(x)$ is an x_0 such that for some $\delta > 0$, on $(x_0, x_0 + \delta)$, $Z(x) > Z(x_0)$ and on $(x_0 - \delta, x_0)$, $Z(x) < Z(x_0)$, with a point of local decrease being defined similarly). Given X_1 and X_2, the probability that $N_2 - N_1$ is such that value taken at one of the countably many local extrema of $f_2 - f_1$ is an integer is 0 (since this only occurs for countably many values of $N_2 - N_1$ and N_1 and N_2 have continuous distributions). Hence, we have that almost surely, each crossing point of $f_2 - f_1$ is not a local extremum or a point of local increase or decrease. Since these possibilities are ruled out, the remaining possibility is IC-density condition (4). □

Corollary 14. *If V is a countable collection of independent samples of shifted Brownian motion, then for all $p \in (0,1)$, $\mathrm{LARG}(V,p)$ is almost surely isomorphic to $GR(\mathrm{ICD})$.*

Proof. By Theorem 13, a countable collection of realizations of Brownian motions with random starting points is almost surely IC-dense. By Theorem 12, if one builds a LARG graph from such a realization, it is almost surely isomorphic to $GR(\mathrm{ICD})$. □

4. Non-isomorphism

In the previous sections we showed that LARG graphs on smoothly dense sets are almost surely isomorphic to $GR(\mathrm{SD})$, while LARG graphs on IC-dense sets are almost surely isomorphic to $GR(\mathrm{ICD})$. In this section, we will show that these two graphs are not isomorphic, and that they are not isomorphic to the infinite graphs obtained from LARG graphs obtained from some other metric spaces.

A central tool in this discussion is a corollary of a theorem of Dilworth (Theorem 16 of [7]), which states that for every step-isometry θ between dense subsets V and W of a normed space X, there is a true isometry $\tilde{\theta}$ such that $\|\tilde{\theta}(v) - \theta(v)\| < K$ for all $v \in V$. Recall that any isomorphism between LARG graphs must be a step-isometry.

We first focus on $GR(\mathrm{SD})$. In [7] it was shown that for suitable random subsets of c, the LARG graphs are Rado with isomorphism type $GR(c)$ and similarly for random subsets of c_0, the isomorphism type is $GR(c_0)$. Using the fact that c and c_0 are not isometric, we We showed in [7] that the graphs $GR(c)$ and $GR(c_0)$ are non-isomorphic. Our first result is derived similarly to the results in [7].

Corollary 15. *The graphs $GR(\mathrm{SD})$ and $GR(\mathrm{ICD})$ are not isomorphic to either $GR(c)$ or $GR(c_0)$.*

This follows from the fact that $C[0,1]$ is not isometric to either c_0 or c (for example, its dual is not separable).

We next consider $GR(\mathrm{ICD})$, the almost-sure isomorphism type of shifted Brownian motions (as shown in Corollary 14.) We show that $GR(\mathrm{SD})$ and $GR(\mathrm{ICD})$ are not isomorphic.

Before we give this theorem, we present some classical results which will be used in the proof. A theorem of Mazur and Ulam [14] states that a surjective isometry of real normed spaces is necessarily affine. Dilworth [11] extended this by showing that an approximate surjective isometry of Banach spaces may be uniformly approximated by a linear isometry. Recall that any isomorphism between LARG graphs must almost surely be a step-isometry (see Corollary 5 of [5]). The following corollary of Dilworth's result states that any step-isometry gives rise to a surjective linear isometry that bounds the distance between pre-image and image.

Theorem 16. *([7]) Suppose (f_n) and (g_n) are dense sequences in separable Banach spaces X and Y and $\lfloor \|f_i - f_j\| \rfloor = \lfloor \|g_i - g_j\| \rfloor$ for each i, j. Then there exists a surjective linear isometry, L, from X to Y and a constant $\kappa > 0$ such that $\|g_i - L(f_i)\| < \kappa$ for all i.*

A classical result then shows that the existence of a surjective linear isometry implies that there exists a homeomorphism with certain nice properties.

Theorem 17 (Banach–Stone)**.** *Let X and Y be compact Hausdorff spaces and L be a surjective linear isometry from $C(X,\mathbb{R})$ to $C(Y,\mathbb{R})$. Then there exists a homeomorphism $\varphi\colon Y \to X$ and a function $g \in C(Y,\mathbb{R})$ with $|g(y)| = 1$ for all y such that*

$$L[f](y) = g(y) f(\varphi(y)) \text{ for all } f \in C(X,\mathbb{R}) \text{ and } y \in Y.$$

We apply this in the special case that X and Y are both the closed unit interval, so that g is a constant function, either $g \equiv 1$ or $g \equiv -1$. This will lead to necessary conditions for two sets to give rise to isomorphic graphs, as stated in the following theorem.

Theorem 18. *Let (f_n) and (g_n) be countable dense sets in $C([0,1])$ and suppose that the LARG graphs on (f_n) and on (g_n) are almost surely isomorphic. Then there exists a homeomorphism ψ from $[0,1]$ to itself and a permutation of $\mathbb{N}$, $i \mapsto n_i$, such that one of the following holds:*

(1) *(Order-preserving case) For each $x \in [0,1]$, if $f_i(x) > f_j(x)$ then $g_{n_i}(\psi(x)) \geq g_{n_j}(\psi(x))$;*
(2) *(Order-reversing case) For each $x \in [0,1]$, if $f_i(x) > f_j(x)$ then $g_{n_i}(\psi(x)) \leq g_{n_j}(\psi(x))$.*

Proof. Let (f_n) and (g_n) be as described. Fix LARG graphs G_1 and G_2 on (f_n) and (g_n), respectively, and suppose that they are isomorphic. We may re-label (g_n) so that for each n, f_n is identified with g_n in the isomorphism. By Corollary 1, such an isomorphism must almost surely be a step-isometry: $\lfloor \|f_i - f_j\| \rfloor = \lfloor \|g_i - g_j\| \rfloor$ for each i, j. Then by Theorems 16 and 17, there exist a self-homeomorphism φ of $[0,1]$, a constant $\rho = \pm 1$ and a $\kappa > 0$ such that

$$\|g_i - \rho f_i \circ \varphi\| < \kappa \text{ for all } i. \tag{4}$$

We deal with the order-preserving case, $\rho = 1$, although the case $\rho = -1$ is analogous. Let $\psi = \varphi^{-1}$.

Let $x_0 \in [0,1]$ and suppose that $f_i(x_0) > f_j(x_0)$. We let $\Delta = f_i(x_0) - f_j(x_0)$. For any positive numbers, $l \in \mathbb{N}$ and $M \in \mathbb{R}$, define $h_{l,M}(x) = \max(f_i, f_j + \Delta) - M + l|x - x_0|$.

Now given $l \in \mathbb{N}$, Let M_l be of the form $N + \frac{\Delta}{2}$, where $N \in \mathbb{N}$ is chosen sufficiently large that

$$h_{l,M_l} < \min(f_i, f_j) - 2\kappa - \Delta. \tag{5}$$

We write h for h_{l,M_l}. We now have that, for each $x \in [0,1]$,

$$\begin{aligned} f_i - h(x) &\leq M - l|x - x_0| \\ f_j - h(x) &\leq M - \Delta - l|x - x_0|, \end{aligned}$$

where both inequalities are equalities at $x = x_0$. In particular, we see $\|f_i - h\| = N + \frac{\Delta}{2}$ and $\|f_j - h\| = N - \frac{\Delta}{2}$ and $\lfloor \|f_i - h\| \rfloor \geq N$ while $\| \lfloor f_j - h\| \rfloor < N$.

Let f_k be an element such that $\|f_k - h\| < \frac{\Delta}{2}$. This implies that $\lfloor \|f_i - f_k\| \rfloor \geq N$ and $\lfloor \|f_j - f_k\| \rfloor < N$. By Corollary 1, we have $\lfloor \|g_i - g_k\| \rfloor \geq N$ and $\lfloor \|g_j - g_k\| \rfloor < N$ also.

By(4), we have that

$$\begin{aligned} -\kappa < g_n - f_n \circ \varphi < \kappa \text{ for all } n; \text{ and so} \\ -\kappa - \tfrac{\Delta}{2} < g_k - h \circ \varphi < \kappa + \tfrac{\Delta}{2} \end{aligned}$$

Combining this with (5), it follows that $g_i - g_k$ and $g_j - g_k$ are non-negative.

We also see that

$$\begin{aligned} g_i(x) - g_k(x) &\leq f_i(\varphi(x)) - h(\varphi(x)) + 2\kappa + \tfrac{\Delta}{2} \\ &\leq N + \Delta + 2\kappa - l|\varphi(x) - x_0| \end{aligned}$$

In particular, $g_i(x) - g_k(x) < N$ whenever $|\varphi(x) - x_0| > (\Delta + 2\kappa)/l$. Since $\|g_i - g_k\| \geq N$, it follows that there exists y_l such that $|\varphi(y_l) - x_0| \leq (\Delta + 2\kappa)/l$ with $g_i(y_l) - g_k(y_l) \geq N$. On the other hand, since $\|g_j - g_k\| < N$, we see $g_j(y_l) - g_k(y_l) < N$. In particular, we deduce that $g_i(y_l) > g_j(y_l)$. We now have a sequence of points y_l where $g_i(y_l) > g_j(y_l)$. We also have $|\varphi(y_l) - x_0| \to 0$ as $l \to \infty$, which implies $y_l \to y_0 = \psi(x_0)$. By continuity of g_i and g_j, we deduce that $g_i(\psi(x_0)) \geq g_j(\psi(x_0))$. □

Corollary 19. *The graphs $GR(\text{ICD})$ and $GR(\text{SD})$ are non-isomorphic.*

Proof. Let (f_n) be IC-dense and (g_n) be a smoothly dense sequence of polynomials (such a sequence exists by Theorem 8). Let G_1 and G_2 be LARG graphs with vertices (f_n) and (g_n), respectively, so that G_1 is almost surely isomorphic to $GR(\text{ICD})$ and G_2 is almost surely isomorphic to $GR(\text{SD})$. Suppose for a contradiction that G_1 and G_2 are isomorphic.

We now apply Theorem 18, and assume without loss of generality that we are in the orientation-preserving case. Thus, there exists a permutation $i \mapsto n_i$ of $\mathbb{N}$ and a homeomorphism ψ of $[0,1]$ such that $f_i(x) > f_j(x)$ implies $g_{n_i}(\psi(x)) \geq g_{n_j}(\psi(x))$ for all x. By the infinite crossing dense condition, there must be two functions that intersect. That is, there exist $i \neq j$ so that $f_i(x) = f_j(x)$ for some $x \in (0,1)$. By the IC-dense property (4), there exists a strictly monotonic sequence $x_m \to x$ such that $f_i(x_m) = f_j(x_m)$ and between any pair x_m and x_{m+1}, there exist y_m and z_m such that $f_i(y_m) > f_j(y_m)$ and $f_i(z_m) < f_j(z_m)$. It follows that $g_{n_i}(\psi(y_m)) \geq g_{n_j}(\psi(y_m))$ and $g_{n_i}(\psi(z_m)) \geq g_{n_j}(\psi(z_m))$. In particular, by the Intermediate Value Theorem, there exists w_m between y_m and z_m such that $g_{n_i}(\psi(w_m)) = g_{n_j}(\psi(w_m))$. Hence, the two polynomials g_{n_i} and g_{n_j} agree at infinitely many points, and so are equal, which is a contradiction. □

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