Large Classes of Infinite *k*-Cop-Win Graphs

Anthony Bonato,[1] Geňa Hahn,[2] and Claude Tardif[3]

[1]*DEPARTMENT OF MATHEMATICS*
RYERSON UNIVERSITY
TORONTO, ON, CANADA M5B 2K3
E-mail: abonato@ryerson.ca

[2]*DÉPARTEMENT D'INFORMATIQUE ET DE*
RECHERCHE OPÉRATIONNELLE, UNIVERSITÉ
DE MONTRÉAL, C.P. 6126 SUCCURSALE
CENTRE-VILLE, MONTRÉAL, QC, CANADA H3C 3J7
E-mail: hahn@iro.umontreal.ca

[3]*DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE*
ROYAL MILITARY COLLEGE OF CANADA
P.O. BOX 17000 STN FORCES
KINGSTON, ON, CANADA K7K 7B4
E-mail: Claude.Tardif@rmc.ca

Abstract: While finite cop-win finite graphs possess a good structural characterization, none is known for infinite cop-win graphs. As evidence that such a characterization might not exist, we provide as large as possible classes of infinite graphs with finite cop number. More precisely, for each infinite cardinal κ and each positive integer k, we construct 2^{κ} non-isomorphic k-cop-win graphs satisfying additional properties such

as vertex-transitivity, or having universal endomorphism monoid and automorphism group. © 2010 Wiley Periodicals, Inc. J Graph Theory 65: 334–342, 2010

Dedicated to Robert Woodrow on his 60th birthday

1. INTRODUCTION

Although vertex-to-vertex pursuit games have been actively studied for the last few decades, there has been a recent resurgence of interest in them in part due to their connections to various graph width parameters; see [8]. In *Cops and Robbers*, which is the original such pursuit game on graphs, we are given a graph $G=(V,E)$, a fixed natural cardinal k, and two players: a set of k *cops* $\mathcal{C}$, and a *robber* $\mathcal{R}$. The game is played in *rounds*, each round consisting of a *move* by $\mathcal{C}$ followed by a move by $\mathcal{R}$ (except possibly at the last round if the robber is caught on the cops' move). At round zero, the cops choose at most k vertices to occupy, then the robber chooses a vertex. Note that a vertex can accommodate any number of cops. On subsequent moves, taken alternately beginning with the cops, the players move to vertices adjacent to the ones currently occupied, or remain at their current positions (alternately, we may assume that the graph G is reflexive; that is, has a loop at every vertex). The cops win if after a finite number of rounds at least one of them occupies the same vertex as the robber; the robber wins otherwise. This is a game of perfect information as all players know the positions of all players at all times. To simplify notation, we will identify the players with the nodes they occupy. For a survey of results on Cops and Robbers, see [10].

As placing a cop on each vertex guarantees that the cops win, it makes sense to define the *cop number*, written $c(G)$, to be the infimum of the cardinalities of the sets of cops that win on G. The cop number is also referred to as the *search number*, written $sn(G)$, so as to distinguish it from the *sweep number*, written $sw(G)$, used when the robber might also be on the edges of the graph. The cop number was introduced by Aigner and Fromme [1] (see also [4] and [9]). If $c(G)=k$, then G is called *k-cop-win*. Finite *cop-win* graphs (that is, 1-cop-win graphs) were structurally characterized in [17, 21]; see also [20, 22]. For a vertex x, let the *open neighborhood* of x be the set $N(x)=\{y\in V:xy\in E\}$, and let the *closed neighborhood* of x be $N[x]=N(x)\cup\{x\}$. The finite cop-win graphs are exactly those graphs which are dismantlable. A finite n-vertex graph is *dismantlable* if there exists a linear ordering $(x_j:1\leq j\leq n)$ of V—called *a dismantling ordering*—so that for each $1\leq i\leq n-1$, there is a $i<j\leq n$ such that

$$N_i[x_i]\cap\{x_i,x_{i+1},\ldots,x_n\}\subseteq N[x_j]\cap\{x_i,x_{i+1},\ldots,x_n\}.$$

For instance, chordal and bridged graphs are cop-win (since they are dismantlable); see [2, 3]. Characterizations of graphs with cop number k, where $k>1$ is a fixed integer, were given recently in [7].

The dismantling characterization for finite cop-win graphs does not apply to infinite cop-win graphs; for example, an infinite one-way ray has an dismantling ordering, but is not cop-win. See [6, 18] for discussions of dismantling in infinite graphs. Nowakowski

and Winkler [17] nevertheless gave the following characterization of cop-win graphs that works for all cardinalities, and is in terms of a relation $\preceq$ on vertices. The relation is defined recursively, with $x \leq_0 x$ for all vertices x. For an ordinal α, define $x \leq_\alpha y$ if and only if for each $u \in N[x]$ there exists a $v \in N[y]$ such that $u \leq_\beta v$ for some $\beta < \alpha$ (think of α as the number of rounds the cop needs to win). Let ρ be the least ordinal such that $\leq_\rho = \leq_{\rho+1}$ and set $\preceq = \leq_\rho$ (clearly $\rho \leq |V|$). It is shown in [17] that a graph G is cop-win if and only if the relation $\preceq$ is trivial, that is, if $x \preceq y$ for all $x, y \in V(G)$. While this characterization is useful for finite graphs—see [12]—it is less so for infinite graphs. In particular, unlike the dismantling order for finite cop-win graphs, it makes transparent neither the properties nor the recognition of infinite cop-win graphs. Indeed, relatively little work has been done after [17] on infinite cop-win graphs, with the exceptions of [5, 6, 11, 18, 19]. For example, in [11] it was shown that, in stark contrast to the situation for finite graphs, there are infinite chordal graphs of diameter two that are not cop-win.

A class of graphs is *large* if for each infinite cardinal κ there are 2^κ many non-isomorphic graphs of order κ in the class. In other words, a large class contains as many as possible non-isomorphic graphs of each infinite cardinality. For example, the classes of all graphs, all trees, and all k-chromatic graphs for k finite are large. Large classes definable in first-order logic have so-called unclassifiable first-order theories (for further reading, see, for example, [13]). A graph G is *vertex-transitive* if for each pair of vertices x and y there is an automorphism of G mapping x to y. We show that for any integer $k > 0$ there are large families of k-cop-win graphs that are vertex-transitive (see Theorem 1), or that have universal monoids and groups (see Theorem 6). These results suggest that the class infinite cop-win graphs—or, more generally, the classes of infinite k-cop-win graphs for k finite—do not possess a good structural characterization. Further, they reinforce the division between the theories of finite and infinite graphs. Our proofs rely on a novel application of infinite strong products, and a new kind of non-commutative graph summation.

All graphs we consider are undirected and simple, and are usually infinite unless otherwise stated. We work in ZFC set theory; no other set-theoretic axioms will be used unless stated. An *end-vertex* is a vertex of degree one. A *ray* in a graph is a one-way infinite path (that is, a graph isomorphic to the graph on the vertex set $\mathbb{N}$, with edges $\{i(i+1) : i \in \mathbb{N}\}$). A graph is *rayless* if it does not contain a ray. The automorphism group of a graph is denoted by $\mathrm{Aut}(G)$, while its endomorphism monoid is denoted by $\mathrm{End}(G)$.

2. INFINITE VERTEX-TRANSITIVE COP-WIN GRAPHS

The class of *pointed trees*—graphs formed from a tree with the addition of one universal vertex—forms a large class of cop-win graphs. However, such graphs are far from transitive, and from the point of view of our game also far from interesting. Our goal is to provide large classes of non-trivial cop-win graphs that are also vertex-transitive.

Theorem 1.   *The class of cop-win, vertex-transitive graphs with the property that the cop can win in two moves is large.*

Theorem 1 has no analogue for finite graphs, where vertex-transitive cop-win graphs are rare. In fact, as noted in [17], the only regular cop-win graphs are complete graphs.

Before we prove Theorem 1, we recall some properties of the strong product of a set of graphs. Let I be an index set. The *strong product* of a set $\{G_i : i \in I\}$ of graphs is the graph $\boxtimes_{i \in I} G_i$ defined by

$$V(\boxtimes_{i \in I} G_i) = \left\{ f : I \to \bigcup_{i \in I} V(G_i) : f(i) \in V(G_i) \text{ for all } i \in I \right\},$$

$$E(\boxtimes_{i \in I} G_i) = \{ fg : f \neq g \text{ and for all } i \in I, \, f(i) = g(i) \text{ or } f(i)g(i) \in E(G_i) \}.$$

Strong products exhibit peculiar properties if there are infinitely many factors. For example, if I is infinite, the graph $\boxtimes_{i \in \mathbb{N} \setminus \{0\}} P_i$ (where P_i is the path with i edges) is not connected even though each factor is. To overcome this, we allow vertices to differ in only finitely many components. Fix a vertex $f \in V(\boxtimes_{i \in I} G_i)$ and define the *weak strong product* of $\{G_i : i \in I\}$ *with base f* as the subgraph $\boxtimes_f^I G_i$ of $\boxtimes_{i \in I} G_i$ induced by the set of all $g \in V(\boxtimes_{i \in I} G_i)$ such that $\{i \in I : g(i) \neq f(i)\}$ is finite. Observe that $\boxtimes_f^I G_i$ is connected if each factor is, and if $|I| \leq \kappa$ and $|V(G_i)| \leq \kappa$ for each $i \in I$, then $|V(\boxtimes_f^I G_i)| \leq \kappa$. For $i \in I$, the *projection mapping* $\pi_i : \boxtimes_f^I G_i \to G_i$ is defined by $\pi_i(g) = g(i)$. For background on strong products, we refer the reader to [14].

When all the factors are isomorphic to some fixed graph G, we refer to a *power* of G (with base f if needed). More precisely, let $\{G_i : i \in I\}$ be a set of isomorphic copies of G (they need not be disjoint). Denote by $\boxtimes^I G$ the strong product $\boxtimes_{i \in I} G_i$. If $f \in V(\boxtimes^I G)$ is fixed, denote by G_f^I the weak strong product $\boxtimes_f^I G$ with base f. One particular power of a graph is of special interest to us and will be used several times. It allows us to construct vertex-transitive graphs out of non-transitive ones. Let κ be a cardinal, and let H be a graph of order κ. Let $I = \kappa \times V(H)$ and define $f : I \to V(H)$ by $f(\beta, v) = v$. The power H_f^I of H with base f will be called the *canonical power* of H and will be denoted by H^H.

As automorphisms of the factors applied coordinate-wise yield an automorphism of the product, it follows that the weak strong product of vertex-transitive graphs is vertex-transitive. However, the following lemma demonstrates the unusual property that if there are infinitely many factors, the weak strong product may be vertex-transitive even if none of the factors are.

Lemma 2.   *If H is an infinite graph, then the canonical power of H^H is vertex-transitive.*

   Proof.   We prove that for all $g \in V(H^H)$ there is an automorphism ψ_g of H^H which maps f to g. The lemma follows from this claim.

Fix $g \in V(H^H)$. Without loss of generality, we may assume that $g \neq f$ as in that case ψ is the identity map. The families $\{g^{-1}(v)\}_{v \in V(H)}$ and $\{f^{-1}(v)\}_{v \in V(H)}$ partition I and $|g^{-1}(v)| = |f^{-1}(v)| = \kappa$. It follows that for each $v \in V(H)$ there exists a bijection $\phi_v : g^{-1}(v) \longrightarrow f^{-1}(v)$ (indeed, such a bijection may be chosen as the identity map except on a finite set). Using the maps ϕ_v, we define a map $\phi : I \longrightarrow I$ by

$$\phi(\beta, v) = \phi_{g(\beta, v)}(\beta, v).$$

Then ϕ is a bijection since for each (β, v), $\phi_{g(\beta, v)}$ is a bijection, and the sets $g^{-1}(v)$ for $v \in V(H)$ partition I. Observe that $g(\beta, v) = f(\phi(\beta, v))$.

We now define ψ_g as follows. For a vertex $h \in V(H^H)$, set $\psi_g(h) = \hat{h}$ with $\hat{h}(\beta, v) = h(\phi(\beta, v))$. We claim first that ψ_g is a bijection. To see this, note that if $h \neq h'$, then $h(\beta, v) \neq h'(\beta, v)$ for some $(\beta, v) \in I$. Hence,

$$\hat{h}(\phi^{-1}\beta, v) = h(\beta, v) \neq h'(\beta, v) = \hat{h}'(\phi^{-1}(\beta, v)).$$

Also note that for any h, if h' is defined by $h'(\beta, v) = h(\phi^{-1}(\beta, v))$ then $\psi_g(h') = h$.

As $\psi_g(f) = g$, the claim of the lemma follows once we verify that ψ_g preserves adjacencies and non-adjacencies. Let h, $h' \in V(H^H)$. We have that $hh' \in E(H^H)$ if and only for any $(\beta, v) \in I$, either $h(\beta, v) = h'(\beta, v)$, or the two images of (β, v) are adjacent in H. Since ϕ is a permutation of I, and by definition of ψ, this happens if and only if $h(\phi(\beta, v)$ and $h'(\phi(\beta, v)$ are either identical or adjacent for every $(\beta, v) \in I$. In particular, $hh' \in E(H^H)$ if and only if $\hat{h}\hat{h}' \in E(H^H)$. ∎

Proof of Theorem 1. As is well-known, for each infinite cardinal κ there are 2^κ many non-isomorphic trees of order κ. Fix a tree T of order κ, and let $\hat{T}$ be the pointed tree formed by adding a universal vertex u to T. The canonical power $\hat{T}^{\hat{T}}$ of $\hat{T}$ has order κ and is vertex-transitive by Lemma 2.

A single cop may win on $\hat{T}^{\hat{T}}$ by playing his/her strategies on each projection. More precisely, the cop $\mathcal{C}$ will initially occupy the base vertex $c_0 = f$ and the robber $\mathcal{R}$ some vertex r_0. For all but finitely many indices $(\beta, v) \in I$, we have that $r_0(\beta, v) = c_0(\beta, v)$. We define the next position c_1 of the cop coordinate-wise: if $c_0(\beta, v)$ is joined or equal to $r_0(\beta, v)$, let $c_1(\beta, v) = c_0(\beta, v)$; otherwise, let $c_1(\beta, v) = u$. When the robber moves from r_0 to r_1, we have $c_1(\beta, v)$ joined or equal to $r_1(\beta, v)$ for every $(\beta, v) \in I$; so the cop captures the robber by moving to $c_2 = r_1$.

It remains to show that if T and T' are not isomorphic, then $\hat{T}^{\hat{T}}$ and $\hat{T}'^{\hat{T}'}$ are not isomorphic. We cannot apply a unique factorization theorem in the context of weak strong products as we could in the case of the strong product with finitely many factors (see [14]), but the structure of the factors allows for a direct proof. First note that the maximal cliques in $\hat{T}^{\hat{T}}$ are products of triangles in every factor. Formally, if K is a maximal clique in $\hat{T}^{\hat{T}}$, then for every $(\beta, v) \in I$, there exist two adjacent vertices $s_{(\beta, v)}, t_{(\beta, v)}$ of T such that

$$K = \{g \in V(\hat{T}^{\hat{T}}) : g(\beta, v) \in \{u, s_{(\beta, v)}, t_{(\beta, v)}\} \text{ for all } (\beta, v) \in I\}.$$

Consequently, if $P = K \cap K'$ is a maximal proper intersection of maximal cliques in $\hat{T}^{\hat{T}}$, there exists an index $(\beta_0, v_0) \in I$ and a vertex s_0 of T such that

$$P = \left\{ g \in V(\hat{T}^{\hat{T}}) : g(\beta, v) \in \left\{ \begin{array}{l} \{u, s_0\} \text{ if } (\beta, v) = (\beta_0, v_0) \\ \{u, s_{(\beta, v)}, t_{(\beta, v)}\} \text{ otherwise} \end{array} \right\} \right\}.$$

Let $\mathcal{V}(T)$ be the set of a maximal proper intersections of maximal cliques in $\hat{T}^{\hat{T}}$. We construct a graph $H(T)$ with vertex set $\mathcal{V}(T)$ by putting an edge between two cliques $P, Q \in \mathcal{V}(T)$ if $P \cup Q$ is a maximal clique of $\hat{T}^{\hat{T}}$. By the above characterization, $H(T)$

consists of disjoint copies of T, one for each index (β_0, v_0) in I. Thus, if T and T' are not isomorphic, $H(T)$ and $H(T')$ are not isomorphic. ∎

With Theorem 1 in hand, we may extend it to the case for $k \geq 2$ via the following lemma.

Lemma 3. *If k is a positive integer, $c(G)=k$, and H is cop-win, then $c(G \boxtimes H)=k$.*

Proof. For the upper bound, use Theorem 4.1 of [16]. For the lower bound, suppose that $j < k$ cops can win on $G \boxtimes H$. For a fixed $y \in V(H)$, define the *G-layer* $G_y = \{(x, y): x \in V(G)\}$. Note that $G_y \cong G$. If C is a cop on vertex (x, z), then write C' for its projection onto the vertex (x, y) in G_y. The strategy of R is to remain in G_y and to use his/her strategy in G to avoid capture by C': whenever a cop moves to (x, z), the robber moves as if the cop was on (x, y). As $c(G)=j$, the robber can always avoid capture by the cops projection in G_y. If the cops can win in $G \boxtimes H$ with the proposed robber's strategy, then consider the second-last move of the cops. Then $N[R]$ is contained in the union of the sets $N[C]$ for the cops C. Hence, $N[R] \cap V(G_y)$ is contained in the union of the sets $N[C']$, and so we derive the contradiction that the cops win in G_y in the next round. ∎

The *Cartesian product* of graphs G and H has vertices $V(G) \times V(G)$, with (a, b) joined to (c, d) if a is joined to c and $b=d$, or $a=c$ and b is joined to d.

Corollary 4. *For $k > 1$, the class of vertex-transitive k-cop-win graphs is large.*

Proof. Let $H(k)$ be the Cartesian product of $k-1$ cycles of length 4. Then $H(k)$ is vertex-transitive, and by Theorem 2.6 of [16], $c(H(k))=k$. Let

$$J(T)=H(k) \boxtimes \hat{T}^{\hat{T}},$$

where $\hat{T}^{\hat{T}}$ are the cop-win graphs from the proof of Theorem 1. Then $J(T)$ has order κ. As both factors are vertex-transitive, so is $J(T)$. By Lemma 3, $c(J(T))=k$. By the unique factorization theorem for Cartesian products (see Theorem B.9 of [14]) if $\hat{T}^{\hat{T}} \ncong \hat{T'}^{\hat{T'}}$, then $J(T) \ncong J(T')$. The assertion follows by Theorem 1. ∎

The large classes described in Theorem 1 and Corollary 4 have infinite clique number and hence, infinite chromatic number. One problem is to find large classes of cop-win graphs whose members are k-chromatic, where $k \geq 2$ is an integer. For infinite bipartite cop-win graphs, we can make a few straightforward observations which we assemble in the following theorem. This leaves the question open for $k > 2$. As the proof of the theorem is straightforward, we omit it.

Theorem 5. *For a graph G, the following are equivalent*:

(1) *The graph G is a cop-win tree.*
(2) *The graph G is a tree with finite cop number.*
(3) *The graph G is a rayless tree.*
(4) *The graph G is cop-win and bipartite.*
(5) *The graph G is cop-win and triangle-free.*

3.   INFINITE *k*-COP-WIN GRAPHS

Besides vertex-transitivity, we give examples of large families of infinite k-cop-win graphs with $k>1$ satisfying several interesting properties. The main result of this section is the following theorem:

Theorem 6. *For every infinite cardinal κ and each finite positive integer k, there are 2^{κ} many non-isomorphic connected graphs G of order κ which satisfy the following properties:*

(1) *The graph G is k-cop-win.*
(2) *Each group of order at most κ embeds (as a group) in $\mathrm{Aut}(G)$.*
(3) *Each monoid of order at most κ embeds (as a monoid) in $\mathrm{End}(G)$.*
   If we assume the Generalized Continuum Hypothesis (GCH), then G satisfies the following additional property.
(4) *Each graph of order at most κ embeds in G.*

To prove Theorem 6 we use the following graph summation. For a sequence $(G_i : 1 \leq i \leq n)$ of graphs with disjoint vertex sets, define the graph

$$\bigoplus_{i=1}^{n} G_i$$

as follows: first form the disjoint union of the G_i, and then add all edges between G_i and G_j for distinct i and j. Each vertex of G_i receives exactly i additional end-vertices. Note that if one of the G_i is infinite, then the graph $\bigoplus_{i=1}^{n} G_i$ has order

$$\max\{|V(G_i)| : 1 \leq i \leq n\}.$$

While the operation $\bigoplus$ is not commutative in general, it has the following useful cancelation property:

Lemma 7. *For a fixed integer $n \geq 2$, suppose that $(G_i : 1 \leq i \leq n)$ and $(H_i : 1 \leq i \leq n)$ are sequences of graphs with some G_i and H_j infinite. If*

$$f : \bigoplus_{i=1}^{n} G_i \to \bigoplus_{i=1}^{n} H_i$$

is an isomorphism, then for all $1 \leq i \leq n$, $f \restriction G_i$ is an isomorphism onto H_i.

Proof. For a fixed i, as each vertex of G_i and H_i has infinite degree and is joined to exactly i end-vertices, $f \restriction G_i$ maps isomorphically onto H_i. ∎

Proof of Theorem 6. We first consider the case $k=1$. Fix an infinite cardinal κ and a graph H of order κ. Form the graph $G(H)$ defined as $\bigoplus_{i=1}^{3} G_i$ where $G_1 \cong K_1$, $G_2 \cong \overline{K_\kappa}$, and $G_3 \cong H$ (recall that the G_i are disjoint).

By Lemma 7, $G(H) \not\cong G(J)$ if $H \not\cong J$. Thus, there are 2^{κ} many non-isomorphic graphs $G(H)$. It is not difficult to see that $G(H)$ is connected and is of order κ. It is also cop-win. To see this, place the cop $\mathcal{C}$ on the vertex of G_1. The cop wins immediately if $\mathcal{R}$ is on any vertex of G_i for some i. Hence, the robber must choose some end-vertex as his initial move, and so $\mathcal{R}$ is captured in at most two moves of the cop. Item 1 follows.

For item 2, it is sufficient to prove that $\text{Sym}(\kappa)$ (the symmetric group on κ letters) embeds as a group in $\text{Aut}(G(H))$. The result then follows by Cayley's theorem: $\text{Sym}(\kappa)$ embeds all groups of order κ. The proof of item 3 is similar and so is omitted (using the full transformation monoid $T(\kappa)$, rather than $\text{Sym}(\kappa)$).

Clearly, $\text{Sym}(\kappa)$ embeds in $\text{Aut}(\overline{K_\kappa})$. Each automorphism f of K_κ faithfully extends to an automorphism of G_1 by mapping end-vertices in the obvious way. Let $\beta: \text{Sym}(\kappa) \to G_1$ be a fixed injective group homomorphism.

We define an injective group homomorphism

$$\alpha: \text{Aut}(G_1) \to \text{Aut}(G(H)).$$

Once the existence of α has been established, then $\alpha\beta: \text{Sym}(\kappa) \to \text{Aut}(G(H))$ is an embedding of groups.

Fix f an automorphism of G, and define F as follows:

$$F(z) = \begin{cases} f(z) & \text{if } z \in V(G_1); \\ z & \text{otherwise.} \end{cases}$$

It is straightforward to check that F is an automorphism of $G(H)$. As distinct f defines distinct F, the map $\alpha: \text{Aut}(G_1) \to \text{Aut}(G(H))$ is well-defined and injective. We now prove that α is a homomorphism of groups. This fact follows since it is straightforward to check that for f, g in $\text{Aut}(G)$ and $z \in V(G(H))$,

$$\alpha(fg)(z) = \alpha(f)\alpha(g)(z).$$

If we assume GCH, then there is a graph J of order κ that embeds each graph of order at most κ as an induced subgraph; see [15]. The proof of item 4 then follows by considering $\bigoplus_{i=1}^{4} G_i$ with G_1, G_2, and G_3 as before, with $G_4 \cong J$.

Now consider the case $k > 1$. Fix $G(k)$ a finite k-cop-win graph. Consider the graph $G'(H)$ formed from $G(H)$ by adding a disjoint copy of $G(k)$ attached via a bridge uv to a fixed (but arbitrary) vertex v of G_3. Then add exactly 4 end-vertices to each of the vertices of $G(k)$.

The graph $G'(H)$ is k-cop-win. To see this, suppose that there are $j < k$ cops. The robber may remain in a copy of $G(k)$ to win. If there are exactly k cops, then let them first move to u. If the robber stays in $G(k)$, then he will lose. Hence, he must eventually move to $G(H)$. In that case, one cop remains on u, while the rest move in $G(H)$. As $G(H)$ is cop-win by the previous discussion and the robber can never reenter $G(k)$, the $k - 1 \geq 1$ cops will eventually catch the robber in $G(H)$ and so win in $G(H)$.

As $G'(H) \not\cong G'(J)$ if $H \not\cong J$, this gives 2^κ many non-isomorphic k-cop-win graphs of order κ. Minor modifications to the above proofs of items 2–4 finish the proof. ∎

REFERENCES

[1] M. Aigner and M. Fromme, A game of cops and robbers, Discrete Appl Math 8 (1984), 1–12.

[2] B. Alspach, Sweeping and searching in graphs: A brief survey, Matematiche 59 (2006), 5–37.

[3] R. P. Anstee and M. Farber, On bridged graphs and cop-win graphs, J Combin Theory Ser B 44 (1988), 22–28.

[4] A. Berarducci and B. Intrigila, On the cop number of a graph, Adv Appl Math 14 (1993), 389–403.

[5] A. Bonato, G. Hahn, and C. Wang, The cop density of a graph, Contrib Discrete Math 2 (2007), 133–144.

[6] M. Chastand, F. Laviolette, and N. Polat, On constructible graphs, infinite bridged graphs and weakly cop-win graphs, Discrete Math 224 (2000), 61–78.

[7] N. Clarke and G. MacGillivray, Characterizations of k-cop-win graphs, Preprint, 2009.

[8] F. V. Fomin and D. Thilikos, An annotated bibliography on guaranteed graph searching, Theoret Comput Sci 399 (2008), 236–245.

[9] A. S. Goldstein and E. M. Reingold, The complexity of pursuit on a graph, Theoret Comput Sci 143 (1995), 93–112.

[10] G. Hahn, Cops, robbers and graphs, Tatra Mt Math Publ 36 (2007), 163–176.

[11] G. Hahn, F. Laviolette, N. Sauer, and R. E. Woodrow, On cop-win graphs, Discrete Math 258 (2002), 27–41.

[12] G. Hahn and G. MacGillivray, A characterization of k-cop-win graphs and digraphs, Discrete Math 306 (2006), 2492–2497.

[13] W. Hodges, Model Theory, Cambridge University Press, Cambridge, 1993.

[14] W. Imrich and S. Klavžar, Product Graphs-Structure and Recognition, Wiley-Interscience Series in Discrete Mathematics and Optimization, Wiley-Interscience, New York, 2000.

[15] P. Komjáth and J. Pach, Universal elements and the complexity of certain classes of infinite graphs, In: Directions in Infinite Graph Theory and Combinatorics, Topics in Discrete Mathematics 3 (R. Diestel, Ed.), Elsevier, North Holland, 1992.

[16] S. Neufeld and R. Nowakowski, A game of cops and robbers played on products of graphs, Discrete Math 186 (1998), 253–268.

[17] R. Nowakowski and P. Winkler, Vertex-to-vertex pursuit in a graph, Discrete Math 43 (1983), 235–239.

[18] N. Polat, Asymmetrization of infinite trees, In: Directions in Infinite Graph Theory and Combinatorics, Topics in Discrete Mathematics 3 (R. Diestel, Ed.), Elsevier, North Holland, 1992.

[19] N. Polat, Retract-collapsible graphs and invariant subgraph properties, J Graph Theory 19 (1995), 25–44.

[20] A. Quilliot, Jeux et Points Fixes sur les graphes, Thèse de 3ème Cycle, Université de Paris VI, 1978, pp. 131–145.

[21] A. Quilliot, Problèmes de jeux, de point fixe, de connectivité et de représentation sur des graphes, des ensembles ordonnés et des hypergraphes, Thèse d'Etat, Université de Paris VI, 1983.

[22] A. Quilliot, A short note about pursuit games played on a graph with a given genus, J Combin Theory (B) 38 (1985), 89–92.