Infinite random geometric graphs from the hexagonal metric

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Abstract. We consider countably infinite random geometric graphs, whose vertices are points in $\mathbb{R}^n$, and edges are added independently with probability $p \in (0,1)$ if the metric distance between the vertices is below a given threshold. Assume that the vertex set is randomly chosen and dense in $\mathbb{R}^n$. We address the basic question: for what metrics is there a unique isomorphism type for graphs resulting from this random process? It was shown in [7] that a unique isomorphism type occurs for the L_∞-metric for all $n \geq 1$. The hexagonal metric is a convex polyhedral distance function on $\mathbb{R}_2$, which has the property that its unit balls tile the plane, as in the case of the L_∞-metric. We may view the hexagonal metric be seen as an approximation of the Euclidean metric, and it arises in computational geometry. We show that the random process with the hexagonal metric does not lead to a unique isomorphism type.

1 Introduction

Geometric random graph models play an important role in the modelling of real-world networks [21] such as on-line social networks [6], wireless and ad hoc networks [3, 17, 19], and the web graph [1, 16]. In such stochastic models, vertices of the network are represented by points in a suitably chosen metric space, and edges are chosen by a mixture of relative proximity of the vertices and probabilistic rules. In real-world networks, the underlying metric space is a representation of the hidden reality that leads to the formation of edges. In the case of on-line social networks, for example, users are embedded in a high dimensional *social space*, where users that are positioned close together in the space exhibit similar characteristics.

Growth is a pertinent feature of most real-life networks, and most stochastic models take the form of a time process, where graphs increase in size over time. The limit of such a process as time goes to infinity is a countably infinite graph. This study of such limiting graphs is in part motivated by the large-scale nature of real-world complex networks. It is expected that the infinite limit will elucidate the structure that emerges when graphs generated by the process become large. The limiting graphs are also of considerable interest in their own right.

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The study of countably infinite graphs is further motivated by two major research directions within graph theory and theoretical computer science. First, there is a well-developed theory of the *infinite random graph*, or the *Rado* graph, written R. The investigation of R lies at the intersection of logic, probability theory, and topology; see the surveys [11, 12, 15] or Chapter 6 of [5].

Another line of investigation has focused on so-called *graph limits*, developed by Lovász and others [8, 9, 22]. A framework is given to define the convergence of sequences of graphs of increasing size. Convergence is based on *homomorphism densities*, and the limit object is a symmetric measurable function. Countably infinite graphs that arise as limits of such sequences can be interpreted as random graphs sampled from the limiting object.

In [7] we considered infinite limits of a simple random geometric graph model. In our model, vertices are chosen at random from a metric space, and if the distance between the two vertices is no larger than some fixed threshold, the vertices are adjacent with some fixed probability. More precisely, for a space S with metric d, consider a threshold $\delta \in \mathbb{R}^+$, a countable subset V of S, and a link probability $p \in [0,1]$. The *Local Area Random Graph* $\mathrm{LARG}(V, \delta, p)$ has vertices V, and for each pair of vertices u and v with $d(u,v) < \delta$ an edge is added independently with probability p. Note that V may be either finite or infinite. For simplicity, we consider only the case when $\delta = 1$; we write $\mathrm{LARG}(V, p)$ in this case. The LARG model generalizes well-known classes of random graphs. For example, special cases of the LARG model include the random geometric graphs (where $p = 1$), and the binomial random graph $G(n,p)$ (where S has finite diameter D, and $\delta \geq D$). We note that the theory of random geometric graphs has been extensively developed (see, for example, [2, 14] and the book [23]).

The basic question we consider is whether the graphs generated by this random process retain information about the metric space from which they are derived. In [7] we obtained a positive answer to this question for $\mathbb{R}^n$ with the L_∞-metric (for any $n \geq 1$). In particular, we showed that in this case, if V is countably infinite, dense in $\mathbb{R}^n$ and randomly chosen, then with probability 1, any two graphs generated by the $\mathrm{LARG}(V, \delta, p)$ (for any fixed δ and p) are isomorphic. Moreover, the isomorphism type is the same for all values of $\delta \in \mathbb{R}^+$ and $p \in (0,1)$. Thus, we can take the unique infinite graph resulting from this process to represent the geometry of this particular metric space.

The isomorphism result described above for the L_∞-metric lead us to consider following general question.

Geometric Isomorphism Dichotomy (GID): For which metrics on $\mathbb{R}^n$ do we have the property that graphs generated by the random process $\mathrm{LARG}(V, \delta, p)$, for V countably infinite, dense in $\mathbb{R}^n$ and randomly chosen, and any $\delta \in \mathbb{R}^+$ and $p \in (0,1)$ are isomorphic with probability 1?

In [7], we showed that two graphs generated by $\mathrm{LARG}(V, \delta, p)$ in $\mathbb{R}^2$ with the L_2-(or Euclidean) metric are non-isomorphic with probability 1, thereby answering the GID in the negative. In the present work, we extend our understanding of the GID to include another metric on $\mathbb{R}^2$, the so-called *hexagonal* (or *honeycomb*) *metric*, written d_{hex}, which is defined by having hexagonal unit balls. This metric may be seen to lie in between the

L_∞-metric, where unit balls are squares, and the Euclidean metric, where unit balls are circles. Hexagons have the property that they tile the plane (as squares do), but on the other hand they can be seen as approximate circles. The precise definition of the d_{hex} metric is given in Section 3.

The hexagonal metric arises in the study of Voronoi diagrams and period graphs (see [18]) in computational geometry, which in turn have applications to nanotechnology. The hexagonal metric arises as a special case of *convex polygonal (or polygon-offset) distance functions*, where distance is in terms of a scaling of a convex polygon containing the origin; see [4, 13, 20]. A precise definition of this metric is given in Section 3. We note that *honeycomb networks* formed by tilings by hexagonal meshes have been studied as, among other things, a model of interconnection networks; see [24].

Our main result is the following theorem.

Theorem 1. *Let V be a randomly chosen, countable, dense set in $\mathbb{R}^2$ with the d_{hex}-metric. Let G and H be two graphs generated by the model* $\text{LARG}(V, p)$*, where $0 < p < 1$. Then with probability 1, G and H are not isomorphic.*

The theorem answers the *GID* in the negative for $\mathbb{R}^2$ with the hexagonal metric. We devote the present article to a sketch of a proof of Theorem 1. Our techniques are largely geometric and combinatorial (such as Hall's condition), and appear in Section 3. In Section 2, we introduce the hexagonal metric and review some of the concepts developed in [7] which are needed to obtain the non-isomorphism result. We conclude with a conjecture on the GID for a wide family of metrics defined by other polygons.

All graphs considered are simple, undirected, and countable unless otherwise stated. Let $\mathbb{N}$, $\mathbb{N}^+$, $\mathbb{Z}$, and $\mathbb{R}$ denote the non-negative integers, the positive integers, the integers, and real numbers, respectively. Vectors are written in **bold**. For a reference on graph theory the reader is directed to [25], while [10] is a reference on metric spaces.

2 Conditions for isomorphism

2.1 Hexagonal metric

We now formally define the hexagonal metric. Consider the vectors

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{a}_2 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2}\sqrt{3} \end{pmatrix}, \text{ and } \mathbf{a}_3 = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2}\sqrt{3} \end{pmatrix}.$$

These are the normal vectors to the sides of a regular hexagon, as shown in Figure 1.

For $\mathbf{x} \in \mathbb{R}^2$ define the *hexagonal norm* of $\mathbf{x}$ as follows:

$$\|\mathbf{x}\|_{\text{hex}} = \max_{i=1,2,3} |\mathbf{a}_i \cdot \mathbf{x}|,$$

where "$\cdot$" is the dot product of vectors. The *hexagonal metric* in $\mathbb{R}^2$ is derived from the hexagonal norm, and defined by

$$d_{\text{hex}}(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|_{\text{hex}}.$$

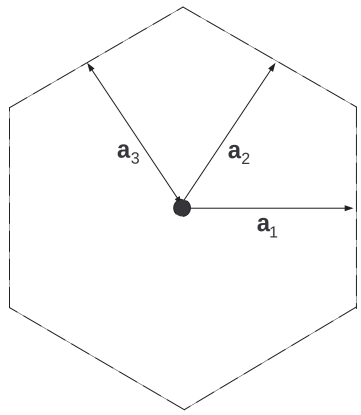

Fig. 1. The vectors $\mathbf{a}_i$.

We may drop the subscript "hex" when it is clear from context. Note that the unit balls with the hexagonal metric are regular hexagons as in Figure 1. We denote the metric space consisting of $\mathbb{R}^2$ with the hexagonal metric by $(\mathbb{R}^2, d_{\text{hex}})$.

2.2 Step-isometries

For the proof of Theorem 1, we rely on the following geometric theorem. Given metric spaces (S, d_S) and (T, d_T), sets $V \subseteq S$ and $W \subseteq T$, a *step-isometry* from V to W is a surjective map $f : V \to W$ with the property that for every pair of vertices $u, v \in V$,

$$\lfloor d_S(u, v) \rfloor = \lfloor d_T(f(u), f(v)) \rfloor.$$

Every isometry is a step-isometry, but the converse is false, in general. For example, consider $\mathbb{R}$ with the Euclidean metric, and let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = (\lfloor x \rfloor + x)/2$ is a step-isometry, but not an isometry.

A subset V is *dense* in S if for every point $x \in S$, every ball around x contains at least one point from V. We refer to $u \in S$ as points or vertices, depending on the context. A crucial step in the proof of isomorphism results of graphs produced by $\text{LARG}(V, p)$ when V is dense in the underlying metric space (S, d), is that any isomorphism must be a step-isometry. For a rough sketch of this fact, observe that if G is a graph produced by $\text{LARG}(v, d)$, and if u and v are two vertices in V so that $k < \lfloor d(u, v) \rfloor < k+1$, then with probability 1, there exists a path of length k from u to v in G. Since no edge can connect vertices at distance 1 or higher, no path of length less than $k-1$ can exist between u and v. Thus, the graph distance equals the floor of the distance. Since an isomorphism must preserve graph distance, it therefore must also preserve the distance, up to its integer multiple, and thus, be a step-isometry.

We will use this fact in the form of the following lemma, adapted from [7], whose proof is omitted.

Lemma 1. *Let V be a countable set dense in $\mathbb{R}^2$ with the d_{hex}-metric, randomly chosen, and let G and H be two graphs generated by the model* $\text{LARG}(V, p)$*, where $0 < p < 1$. If G and H are isomorphic via f, then with probability* 1 *we have that f is a step-isometry.*

The following theorem is central to our proof of Theorem 1.

Theorem 2. *Let V and W be dense subsets of $\mathbb{R}^2$ with the d_{hex}-metric, with the property that V contains two points $\mathbf{p}_1, \mathbf{p}_2$ so that their distance $d_{hex}(\mathbf{p}_1, \mathbf{p}_2)$ is irrational. Then every step-isometry from V to W is uniquely determined by the images of $\mathbf{p}_1$ and $\mathbf{p}_2$.*

The next section will be devoted to the proof of Theorem 2. Using Theorem 2, we may prove Theorem 1. As the proof of Theorem 1 is analogous to the proof of Theorem 15 in [7], it is omitted.

3 Proof of Theorem 2

To prove Theorem 2, we show first that each finite set of points in the plane introduces a set of lines (which will be recursively defined below). These lines will fall into three parallelism classes, defined by their normals $\mathbf{a}_i$, for $i = 1, 2, 3$. More precisely, for a fixed $i \in \{1, 2, 3\}$, define $\mathcal{F}_i$ to be the family of lines in the plane with normal vector $\mathbf{a}_i$. For $r \in \mathbb{R}$, let $\mathcal{F}_i(r)$ be the family of lines in $\mathcal{F}_i$ which are at integer distance from the line $\mathbf{a}_i \cdot \mathbf{x} = r$. Thus, $\mathcal{F}_i(r)$ contains the lines with equations

$$\mathbf{a}_i \cdot \mathbf{x} = r + z, \text{ for some } z \in \mathbb{Z}.$$

We will show that any step-isometry f between two graphs with dense vertex sets must be "consistent" with these lines (that is, a point in the domain framed by a set of lines must be mapped to a point which is framed by a corresponding set of lines in the range); see Lemma 2. We then show that we can choose these lines to be dense in Lemma 3, and thereby prove that f is in fact uniquely determined by a finite set of points.

We now define this notion of consistency in a more precise fashion. Let $V \subseteq \mathbb{R}^2$ and $f : V \to \mathbb{R}^2$ be an injective map. Let $r \in [0, 1)$ and $i \in \{1, 2, 3\}$, and let σ be a permutation of the index set $\{1, 2, 3\}$. The map f is *consistent* with the family of lines $\mathcal{F}_i(r)$ *with respect to the permutation* σ if there exists $r' \in [0, 1)$ so that for all $\mathbf{x} \in V$, for all $z \in \mathbb{Z}$,

$$\mathbf{a}_i \cdot \mathbf{x} < z + r \text{ if and only if } \mathbf{a}_{\sigma(i)} \cdot \mathbf{x}' < z + r',$$

where $\mathbf{x}' = f(\mathbf{x})$.

In the following, we will assume that all sets contain the origin $\mathbf{0}$, and we will assume without loss of generality that any map preserves the origin. Note that we can always replace any set V by an equivalent, translated set $V + \mathbf{b} = \{\mathbf{v} + \mathbf{b} : \mathbf{v} \in V\}$ so that this is indeed the case.

Lemma 2. *Suppose that V and W are dense in $\mathbb{R}^2$, and $f : V \to W$ is a bijection. If f is a step-isometry, then there exists a permutation σ of the index set $\{1, 2, 3\}$ such that f is consistent with the family of lines $\mathcal{F}_i(0)$ with respect to σ for $i = 1, 2, 3$.*

Proof. For $1 \leq i \leq 6$, let S_i be the six sections partitioning $\mathbb{R}^2$ formed by the lines $\ell_j : \mathbf{a}_j \cdot \mathbf{x} = 0$, where $j = 1, 2, 3$. See Figure 2.

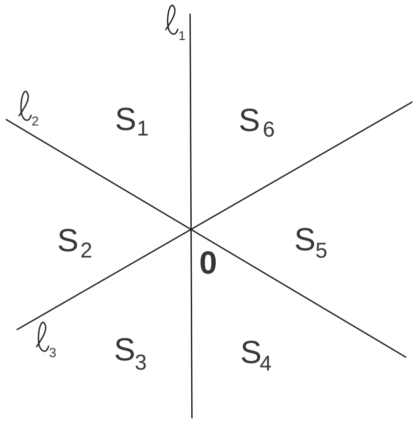

Fig. 2. The sectors S_i and lines $\ell_j : \mathbf{a}_i \cdot \mathbf{x} = 0$.

Note that each of the sectors is uniquely defined by its two bounding lines. For example, sector S_1 contains all points $\mathbf{x}$ for which $\mathbf{a}_1 \cdot \mathbf{x} < 0$ and $\mathbf{a}_2 \cdot \mathbf{x} > 0$, and sector S_2 contains all points $\mathbf{x}$ for which $\mathbf{a}_2 \cdot \mathbf{x} < 0$ and $\mathbf{a}_3 \cdot \mathbf{x} > 0$. Thus, any permutation σ of the index set $\{1, 2, 3\}$ induces a permutation σ^* of the sectors. For example, the permutation $\sigma = (1, 2, 3)$ induces the permutation $\sigma^* = (1, 2, 3, 4, 5, 6)$ of the sector indices.

Claim. There exist six points $v_i \in V$, $1 \leq i \leq 6$, and a permutation σ of the index set $\{1, 2, 3\}$ so that for all i we have that $\mathbf{v}_i \in S_i$ and $f(\mathbf{v}_i) \in S_{\sigma^*(i)}$, where σ^* is the permutation of the sectors induced by σ.

Proof. Let $k = 26$. Choose points $\mathbf{u}_i$, where $i = 0, \ldots, 6k - 1$, such that the following four conditions hold.

1. All points lie in a band at distance between $k-1$ and k from the origin. More precisely, for all i, $0 \leq i < 6k$, $k - 1 < d(\mathbf{0}, \mathbf{u}_i) < k$.
2. Any two consecutive points lie at less than unit distance from each other. More precisely, for all i, $0 \leq i < 6k$, $d(\mathbf{u}_i, \mathbf{u}_{i+1}) < 1$, where addition in the index is taken modulo $6k$.
3. Any two points that are not consecutive are further than unit distance apart. More precisely, for all i and j, $0 \leq i < 6k$, $2 \leq j \leq 6k - 2$, $d(\mathbf{u}_i, \mathbf{u}_{i+j}) > 1$, where addition in the index is taken modulo $6k$. Note that if these points are adjacent if and only if their distance is less than 1, then the points would form a cycle.
4. For all j, $1 \leq j \leq 6$, the points $\mathbf{u}_{(j-1)k+i}$, $0 \leq i < k$ lie in sector S_j.

For all i, $0 \leq i < 6k$, let $\mathbf{u}'_i = f(\mathbf{u}_i)$, and let $U' = \{\mathbf{u}'_i : 0 \leq i < 6k\}$. Since f is a step-isometry, the points $\mathbf{u}'_i$ must satisfy conditions (1), (2), and (3) for the given

$\mathbf{u}_i$. Note that we cannot conclude that (4) holds as well. However, we can deduce that each sector must contain at least $k-1$ vertices from U'. Namely, let $\mathbf{u}'_s \in U'$ be in sector S_2. Let $\mathbf{u}'_{t_1}$ and $\mathbf{u}'_{t_2}$ be the vertices with lowest index and highest index in $S_2 \cap U'$, respectively. Then $\mathbf{u}'_{t_1-1}$ and $\mathbf{u}'_{t_2+1}$ are in the two sectors adjacent to S_2; without loss of generality $\mathbf{u}'_{t_1-1} \in S_1$ and $\mathbf{u}'_{t_2+1} \in S_3$. By condition (1) and by the geometry of the sectors, $d(\mathbf{u}'_{t_1-1}, \mathbf{u}'_{t_2+1}) \geq k-1$, and by condition (2), $d(\mathbf{u}'_{t_1-1}, \mathbf{u}'_{t_2+1}) < (t_2+1)-(t_1-1)$. Thus,

$$(t_2+1)-(t_1-1) > k-1,$$

so $t_2 - t_1 + 1 \geq k-1$. Since the points $\mathbf{u}'_{t_1}, \ldots, \mathbf{u}'_{t_2}$ are in S_2, it follows that S_2 contains at least $k-1$ points. The same conclusion holds for the other sectors.

A simple counting argument shows that each sector can contain at most $6k-5(k-1) = k+5$ points from U' (as there are $6k$ points $\mathbf{u}_i$). For $j = 1, 2, \ldots, 6$, let $U'_j = \{f(\mathbf{u}) : \mathbf{u} \in U \cap S_j\}$ be the collection of images of points from u that lie in sector S_j. By condition (4), $|U'_j| = k$ for all j. Note that k is chosen large enough so that $tk > (t-1)(k+5)$ for any t, $1 \leq t \leq 6$. Thus, by the pigeonhole principle the tk vertices from t of the sets U'_j cannot be contained in less than t sectors. Thus, Hall's condition (see for example, [25]) holds, and we can find a permutation σ^* of the index set $\{1, 2, \ldots, 6\}$ so that $U'_i \cap S_{\sigma^*(i)} \neq \emptyset$. Moreover, because of the cyclic structure of the points in U' implied by conditions (2) and (3), adjacent sets U'_j and U'_{j+1} must be mapped by σ^* to adjacent sectors. This guarantees that σ^* is compatible with a permutation σ of the index set $\{1, 2, 3\}$ of the lines that define the sectors.

For $j = 1, \ldots, 6$, we can then choose $\mathbf{v}_j \in U_j = U \cap S_j$ so that $f(\mathbf{v}_j) \in U'_j \cap S_{\sigma^*(j)}$. This completes the proof of the claim. □

To complete the proof of the lemma, fix the $\mathbf{v}_i$ and σ as in Claim 3. Let

$$A = \{\mathbf{v}_i : 1 \leq i \leq 6\}.$$

Without loss of generality, we set σ to be the identity permutation. For a contradiction, assume there exists $\mathbf{u} \in V$ and $i \in \{1, 2, 3\}$ so that $\mathbf{a}_i \cdot \mathbf{u} < 0$ but $\mathbf{a}_i \cdot \mathbf{u}' > 0$, where $\mathbf{u}' = f(\mathbf{u})$. Let $L = \{\mathbf{v} \in A : \mathbf{a}_i \cdot \mathbf{v} < 0\}$, and $R = A - L = \{\mathbf{v} \in A : \mathbf{a}_i \cdot \mathbf{v} > 0\}$. Let $L' = \{f(\mathbf{v}) : \mathbf{v} \in L\}$ and $R' = \{f(\mathbf{v}) : \mathbf{v} \in R\}$. By definition, each of the vertices of A lies in a different sector, so we must have that $|L| = |R| = 3$. See Figure 3; we assume in the figure that $i = 1$, so f is not consistent with ℓ_1.

Choose $\mathbf{w}$ so that for some $b \in \mathbb{Z}^+$,

$$d_{\text{hex}}(\mathbf{w}, \mathbf{x}) > b \text{ for all } \mathbf{x} \in R \cup \{\mathbf{0}\}, \text{ and}$$
$$d_{\text{hex}}(\mathbf{w}, \mathbf{x}) < b \text{ for all } \mathbf{x} \in L \cup \{\mathbf{u}\}.$$

More precisely, the ball with radius b centered at $\mathbf{w}$ contains $L \cup \{\mathbf{u}\}$, and is disjoint from $R \cup \{\mathbf{0}\}$.

Since f is a step-isometry, if we set $\mathbf{w}' = f(\mathbf{w})$, then

$$d_{\text{hex}}(\mathbf{w}', \mathbf{x}) > b \text{ for all } \mathbf{x} \in R' \cup \{\mathbf{0}\}, \text{ and}$$
$$d_{\text{hex}}(\mathbf{w}', \mathbf{x}) < b \text{ for all } \mathbf{x} \in L' \cup \{\mathbf{u}'\}.$$

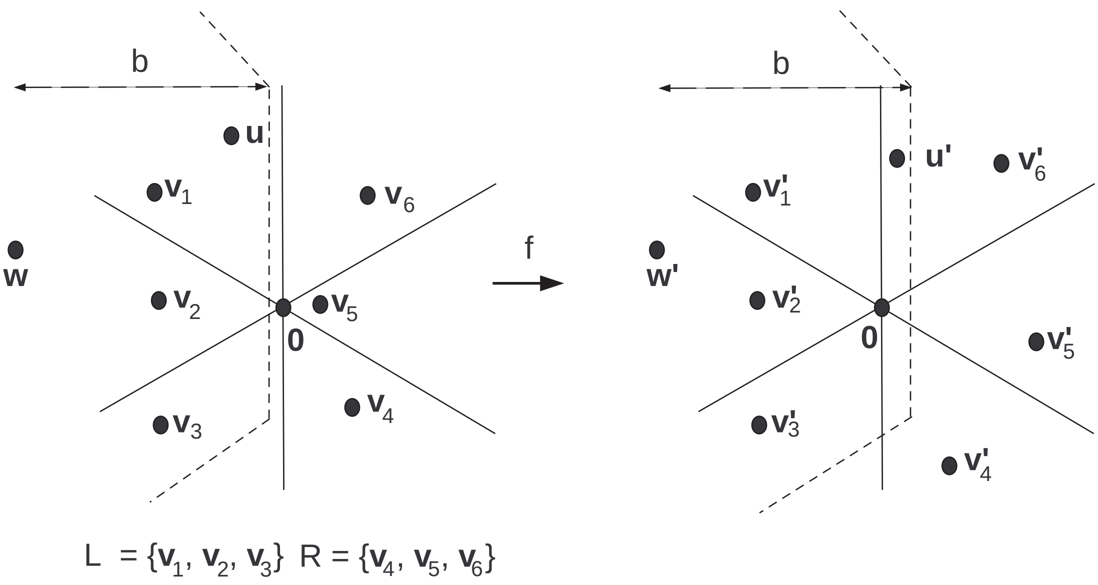

Fig. 3. The original configuration and its image under f. Dotted lines correspond to the boundaries of the (partially shown) balls of radius b centred around $\mathbf{w}$ and $\mathbf{w}'$.

By the definition of A and the assumption that σ is the identity, for each $\mathbf{v} \in A$, if $\mathbf{v}$ is in a certain sector S_j then so is $f(\mathbf{v})$. Thus, $\mathbf{a}_i \cdot \mathbf{v}' < 0$ for all $\mathbf{v}' \in L'$. However, $\mathbf{a}_i \cdot \mathbf{u}' > 0$, so the vertices in $L' \cup \{\mathbf{u}'\}$ lie in four different sectors. Thus, the ball of radius b around $\mathbf{w}'$ must intersect at least four sectors (see Figure 3) and so contains the origin. Since $d_{\text{hex}}(\mathbf{w}', \mathbf{0}) > b$, this ball cannot contain $\mathbf{0}$, which gives a contradiction. $\square$

Let B be a finite subset of $\mathbb{R}^2$. Define a collection of lines $\mathcal{L}(B)$ inductively as follows. To define $\mathcal{L}_0(B)$: for each $\mathbf{w} \in B$, add the three lines through $\mathbf{w}$ in the families $\mathcal{F}_i$, $(i = 1, 2, 3,)$ along with their integer parallels. Specifically, these are all the lines with equation $\mathbf{a}_i \cdot \mathbf{x} = \mathbf{a}_i \cdot \mathbf{w} + z$, where $z \in \mathbb{Z}$.

For the inductive step, assume that $\mathcal{L}_j(B)$ has been defined for some $j \geq 0$. To define $\mathcal{L}_{j+1}(B)$, for each point $\mathbf{p} \in \mathbb{R}^2$ which lies on the intersection of two lines in $\mathcal{L}_j(B)$ (which must belong to different, non-parallel families), add the unique line belonging to the third family. See Figure 4. More precisely, if $\mathbf{p}$ is at the intersection of lines ℓ and m, where $l \in \mathcal{F}_{i_\ell}$ and $m \in \mathcal{F}_{i_m}$ (with $i_\ell \neq i_m$), then add to $\mathcal{L}_{j+1}(B)$ the line with equation $\mathbf{a}_j \cdot \mathbf{x} = \mathbf{a}_j \cdot \mathbf{p}$, where j is the unique element in $\{1, 2, 3\} \backslash \{i_\ell, i_m\}$.

Finally, define

$$\mathcal{L}(B) = \bigcup_{i \in \mathbb{N}} \mathcal{L}_i(B).$$

We need the following lemma.

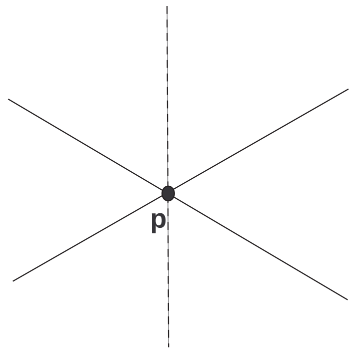

Fig. 4. The solid lines are in $\mathcal{L}_i$ and the dotted line is in $\mathcal{L}_{i+1}$.

Lemma 3. *Let* $B = \{0, \mathbf{p}\}$, *where* $\mathbf{a}_i \cdot \mathbf{p} = r \in (0, 1)$. *Then the family* $\mathcal{L}(B)$ *contains all the following lines, where* $i \in \{1, 2, 3\}$, $z_1, z_2 \in \mathbb{Z}$*:*

$$\mathbf{a}_i \cdot \mathbf{x} = z_1 r + z_2.$$

Moreover, if r *is irrational, the set of lines* $\mathcal{L}(B)$ *is dense in* $\mathbb{R}^2$.

Proof. Consider the triangular lattice formed by all lines in $\mathcal{L}_0(\{\mathbf{0}\})$; that is, all lines with equations $\mathbf{a}_i \cdot \mathbf{x} = z$, where $i = 1, 2, 3$ and $z \in \mathbb{Z}$. See Figure 5. Consider the triangle

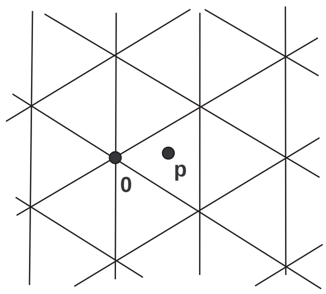

Fig. 5. A triangular lattice.

that contains $\mathbf{p}$. We assume that this triangle is framed by the lines with equations $\mathbf{a}_1 \cdot \mathbf{x} = 1$, $\mathbf{a}_2 \cdot \mathbf{x} = 0$ and $\mathbf{a}_3 \cdot \mathbf{x} = 0$, as shown in Figure 6. (The proof can easily be adapted to cover all other possibilities.) By definition, the line ℓ_1 defined by $\mathbf{a}_1 \cdot \mathbf{x} = r$ is part of $\mathcal{L}_0(B)$.

The line ℓ_1 intersects the two sides of the triangle in $\mathbf{p}_1$ and $\mathbf{p}_2$. The point $\mathbf{p}_1$ lies on the intersection of the lines from the families $\mathcal{F}_1$ and $\mathcal{F}_3$. Thus, $\mathcal{L}_1(\mathbf{w})$ contains the line ℓ_2 in $\mathcal{F}_2$ through $\mathbf{p}_1$ which has equation $\mathbf{a}_2 \cdot \mathbf{x} = \mathbf{a}_2 \cdot \mathbf{p}_2$. Similarly, $\mathcal{L}_1(\mathbf{w})$ contains the line ℓ_3 in $\mathcal{F}_3$ through $\mathbf{p}_2$ which has equation $\mathbf{a}_3 \cdot \mathbf{x} = \mathbf{a}_3 \cdot \mathbf{p}_2$. See Figure 6.

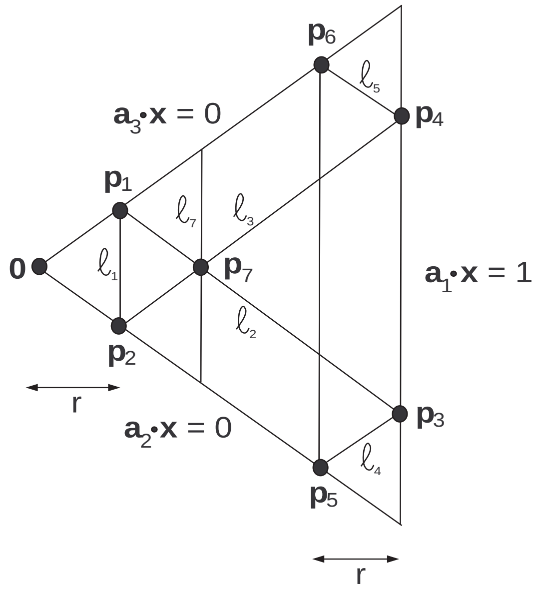

Fig. 6. Generating the line $\mathbf{a}_3 \cdot \mathbf{p}_2 = 1 - r$.

The lines ℓ_2 and ℓ_3 intersect the third side of the triangle in $\mathbf{p}_3$ and $\mathbf{p}_4$, generating two lines ℓ_4 and ℓ_5 in $\mathcal{L}_2(B)$ with equations $\mathbf{a}_3 \cdot \mathbf{x} = \mathbf{a}_3 \cdot \mathbf{p}_3$ and $\mathbf{a}_2 \cdot \mathbf{x} = \mathbf{a}_2 \cdot \mathbf{p}_4$, respectively. The lines ℓ_4 and ℓ_5 intersect with the sides of the triangle in $\mathbf{p}_5$ and $\mathbf{p}_6$, generating one line ℓ_6 in $\mathcal{L}_3(B)$ with equation $\mathbf{a}_1 \cdot \mathbf{x} = \mathbf{a}_1 \cdot \mathbf{p}_5 = \mathbf{a}_1 \cdot \mathbf{p}_6$.

From the fact that the triangle formed by $\mathbf{0}$, $\mathbf{p}_1$, and $\mathbf{p}_2$ is isosceles, it follows that $r = \mathbf{a}_1 \cdot \mathbf{p} = \mathbf{a}_2 \cdot p_1 = \mathbf{a}_3 \cdot \mathbf{p}_2$. By the comparison of similar triangles, we obtain that

$$\mathbf{a}_1 \cdot \mathbf{p}_5 = \mathbf{a}_2 \cdot \mathbf{p}_4 = \mathbf{a}_3 \cdot \mathbf{p}_2 = 1 - r.$$

Now the parallel lines $\mathbf{a}_i \cdot \mathbf{x} = r + z_2, -r + z_2$, $z_2 \in \mathbb{Z}$, may be generated from all similar triangles in the lattice in an analogous fashion.

To complete the proof, consider that the lines ℓ_2 and ℓ_3 intersect in point $\mathbf{p}_7$, which generates a line $\ell_7 \in \mathcal{F}_1$ as indicated in Figure 6. Since the triangle formed by $\mathbf{p}_1, \mathbf{p}_2$, and $\mathbf{p}_7$ is a reflection of the triangle formed by $\mathbf{0}, \mathbf{p}_1$, and $\mathbf{p}_2$, it follows that ℓ_7 has equation $\mathbf{a}_1 \cdot \mathbf{x} = 2r$. This process can be repeated to obtain all the lines $\mathbf{a}_i \cdot \mathbf{x} = z_1 r + z_2$, $z_1, z_2 \in \mathbb{Z}$, $i = 1, 2, 3$.

If r is irrational, then the set $\{z_1 r + z_2 : z_1, z_2 \in \mathbb{Z}\}$ is dense in $\mathbb{R}$ (this is a result from folklore which can be proved by using the pigeonhole principle). That completes the proof of the lemma. □

We now give a proof of the main theorem in this section.

Proof of Theorem 2. Let $f : V \to W$ be a step-isometry. By Lemma 2 there exists a permutation σ of the index set $\{1, 2, 3\}$ such that f is consistent with the family of lines $\mathcal{F}_i(0)$ for $i = 1, 2, 3$ with respect to σ. Without loss of generality, we assume that σ is the identity. Assume that V has two points whose distance is irrational; assume without loss of generality that a one of them is the origin. Choose $B = \{\mathbf{0}, \mathbf{p}\} \subseteq V$ so that $d_{hex}(\mathbf{0}, \mathbf{p})$ is irrational. By Lemma 3, the set of lines $\mathcal{L}(B)$ is dense in $\mathbb{R}^2$. Thus, any of the points in V is uniquely defined by their position with respect to all lines in $\mathcal{L}(B)$. Moreover, the same is true for W, and the set of lines generated by the images of $\mathbf{0}$ and $\mathbf{p}$. Therefore, once the images of $\mathbf{0}$ and $\mathbf{p}$ are given, each other vertex has a unique image under f. □

Note that it is a crucial point in the proof of Theorem 2 that V must contain two points whose distance is irrational. Since the set of rationals has measure 0 in the reals, if V is chosen randomly, then such two points will exist with probability 1. If V does not contain such points (for example, if V is chosen to contain all rational points in $\mathbb{R}^2$ that lie at rational hexagonal distance from the origin), then for any finite set B the family $\mathcal{L}(B)$ will contain only a finite number of line families, and thus, cannot not lead to a dense grid. In that case, we expect that methods similar to those used to prove uniqueness for the L_∞-metric can be used to show that LARG(V, p) results in a unique isomorphism type.

4 Conclusion and further work

We have shown that the hexagonal metric on a randomly chosen, dense subset of $\mathbb{R}^2$ can lead to non-isomorphic limit graphs in the LARG random process. Our main tool was Theorem 2 which proves that a step-isometry f between randomly chosen dense sets is, with probability 1, determined by the image of two points. Theorem 2 was proven by exploiting that f is consistent with a dense set of lines as proved in Lemmas 2 and 3.

The methods described in this article should extend to other polygonal metrics and higher dimensions. A *polygonal metric* is one where the unit ball is a (convex) point-symmetric polygon. The distance between two points $\mathbf{a}$ and $\mathbf{b}$ given as follows. Translate the polygon until it is centered at $\mathbf{a}$. Let $\mathbf{v}$ be the unique point on the intersection of the ray from $\mathbf{a}$ to $\mathbf{b}$ with the boundary of the polygon. Then the distance is given by the ratio of the (Euclidean) distance between $\mathbf{a}$ and $\mathbf{b}$ to the distance between $\mathbf{a}$ and $\mathbf{v}$. Alternatively, it is the factor by which the polygon centered at $\mathbf{a}$ would have to be enlarged until it touches $\mathbf{b}$.

To be a metric, the polygon needs to be point-symmetric, and thus, has an even number of sides. If the number of sides equals $2n$ for a metric defined in $\mathbb{R}^n$, then the polygon can be transformed into an n-dimensional hypercube by rescaling of the coordinates, and thus the metric is equivalent to the L_∞-metric. We conjecture that this is the only case for which the GID is answered in the affirmative.

Conjecture 1. For all $n \geq 2$, for all convex polygonal distance functions where the polygon has at least six sides, two graphs generated by LARG(V, p), with V randomly chosen and dense in $\mathbb{R}^n$, and $p \in (0, 1)$, are non-isomorphic with probability 1.

We showed earlier that the GID is answered in the negative for the Euclidean metric and $\mathbb{R}_2$. We think that the analogous statement is true for the Euclidean metric in higher dimensions, and that the methods in this paper may suggest a suitable approach to proving this fact.

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