Partitioning a Graph Into Two Isomorphic Pieces

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Abstract: A simple graph G has the neighbour-closed-co-neighbour property, or ncc property, if for all vertices x of G, the subgraph induced by the set of neighbours of x is isomorphic to the subgraph induced by the set of non-neighbours of x. We present characterizations of graphs with the ncc property via the existence of certain perfect matchings, and thereby prove that the ncc graphs can be recognized in polynomial time. Operations preserving the ncc property are discussed, and the graphs with the ncc property that are r-regular, where $r \leq 5$, are classified. The graphs with the

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ncc property that are locally H, for certain graphs H are classified. Infinite graphs with the ncc property are discussed. J Graph Theory 44: 1–14, 2003

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1. INTRODUCTION

Throughout, all our graphs are simple and finite, unless otherwise stated. We follow the notation of [2] with the following exceptions. For a graph G and $x \in V(G)$, define the *neighbour set of* x to be the set $N(x) = \{y \in V(G) : xy \in E(G)\}$, and the *closed co-neighbour set of* x to be the set $N^c[x] = \{y \in V(G) : xy \notin E(G)\}$. The *co-neighbour set of* x is the set $N^c(x) = N^c[x] \backslash \{x\}$. If S is a subset of $V(G)$, then the subgraph induced by S is written $G \restriction S$. If G and H are graphs, then the *Cartesian product of* G and H, written $G \,\square\, H$, has vertices $V(G) \times V(H)$ with (w, x) adjacent to (y, z), if and only if, $w = y$ and $xz \in E(H)$ or $wy \in E(G)$ and $x = z$. For a fixed $y \in V(H)$, the subgraph induced by the set $\{(x, y) : x \in V(G)\}$ in $G \,\square\, H$ is isomorphic to G and is called a G-layer. The disjoint union of graphs G and H is written $G + H$.

Let P_n be the path with n vertices, C_n be the circuit with n vertices, and K_n be the complete graph with n vertices. A set of vertices with no edges between them is *independent* and a graph with no edges is called *empty*.

The infinite random graph R is the unique countable graph, up to isomorphism, satisfying the *existentially closed* or *e.c. property:* for each pair of finite disjoint sets S and T of vertices, there is a vertex $z \notin S \cup T$ adjacent to each vertex of S, and not adjacent to any vertex of T. (See [3] for further information on R.) From the e.c. property of R follows a *fractal* or *partition property*, written $(\mathcal{N})$: for all $x \in V(R)$,

$$R \restriction N(x) \cong R \restriction N^c(x) \cong R.$$

We do not know of any other countable graphs with $(\mathcal{N})$ (see [1]). We may however, consider variants of $(\mathcal{N})$ for finite graphs. Among the finite graphs, K_1 is unique with the property that for all $x \in V(G)$, $G \restriction N(x) \cong G \restriction N^c(x)$. (We leave this as an exercise.) One intriguing variant of $(\mathcal{N})$, proposed by the second author, is the following. We say that a graph G has the *neighbour-closed-co-neighbour property*, or *ncc property*, if for all $x \in V(G)$, $G \restriction N(x) \cong G \restriction N^c[x]$. We say that G has *ncc* or is an *ncc graph*. For example, each of the graphs $K_{n,n}$ and $K_n \,\square\, K_2$ have ncc. A similar but unrelated property is that of *neighbourhood symmetry* studied in [6].

2. THE MAIN RESULT

On the surface, the graphs with the ncc property do not display any special structure. Since the ncc property is defined by finding certain isomorphisms, it would seem likely that the recognition problem for the ncc graphs would not be of polynomial time complexity. After a closer look, we see that both of these assertions are, perhaps surprisingly, false.

A graph G of even order has a *disjoint neighbour perfect*, or *dnp*, matching if there is some perfect matching $M = \{e_i : 1 \leq i \leq n\}$ of G, so that for all i, if $e = a_i b_i$, then $N(a_i) \cap N(b_i) = \emptyset$. We say that M is *dnp*. For example, every perfect matching of a bipartite graph is dnp. If G is any graph, then $G \square K_2$ has a dnp matching: the perfect matching joining the G-layers.

Theorem 2.1. *Let G be a graph. Then G has ncc, if and only if, there is a positive integer n so that*

1. *the order of G is $2n$,*
2. *G is n-regular, and*
3. *G has a dnp matching.*

Proof. Suppose that G is an ncc graph. For a fixed $x \in V(G)$, $|V(G)| = |N(x)| + |N^c[x]| = 2|N(x)|$, so property (1) holds, with $n = |N(x)|$. If $y \in V(G)$ is distinct from x, then $|V(G)| = 2|N(y)|$, so $|N(x)| = |N(y)|$ and hence, property (2) holds.

For property (3), fix $x \in V(G)$. Since $G \restriction N(x) \cong G \restriction N^c[x]$ and x is isolated in $G \restriction N^c[x]$, there exist isolated vertices in $G \restriction N(x)$. Let

$$\{y_i : 1 \leq i \leq k_x\}$$

be the isolated vertices in $N(x)$. Therefore, there exist k_x-many isolated vertices in $G \restriction N^c[x]$; call these $x = x_1, \ldots, x_{k_x}$. Since each y_i is isolated in $N(x)$, by n-regularity, we must have that $N(y_i) = N^c[x]$. Similarly, for each x_i, we must have that $N(x_i) = N(x)$. Hence, each vertex x_i is adjacent to each y_j. Define M_x to be the matching $\{x_i y_i : 1 \leq i \leq k_x\}$. We will define $M_{x_i} = M_{y_i}$ for all $1 \leq i \leq k_x$. Choose any vertex z not in $\{x_i, y_i : 1 \leq i \leq k_x\}$ and proceed inductively to obtain matchings M_z for every vertex $z \in V(G)$. Define

$$M = \bigcup_{z \in V(G)} M_z.$$

We claim that M is the desired dnp matching of G. By construction, each vertex of G is an endpoint of some edge in M. To see that M is a matching, suppose that x is a vertex that is incident with two distinct edges of M, say xy and xy'. By construction of M, y and y' are isolated in $N(x)$, and so are in the set $\{y_i : 1 \leq i \leq k_x\}$. But x is matched by M to a unique element of $\{y_i : 1 \leq i \leq k_x\}$, which gives a contradiction. Hence, M is a perfect matching of G. The matching M is dnp by construction.

For the converse, suppose that G satisfies properties (1), (2), and (3). Let $M = \{e_i = a_i b_i : 1 \leq i \leq n\}$ be a dnp matching of G.

Claim. The map

$$f : G \restriction (\{a_i : 1 \le i \le n\}) \to G \restriction (\{b_i : 1 \le i \le n\})$$

defined by $f(a_i) = b_i$ is an isomorphism.

For the proof of the claim, suppose that $ab \in M$. Since $N(a) \cap N(b) = \emptyset$, we have that $N(b) \subseteq N^c[a]$. Since $|N^c[a]| = |N(b)| = n$ by n-regularity, we have that

$$\begin{aligned} N(b) &= N^c[a], \\ N(a) &= N^c[b], \end{aligned} \tag{2.1}$$

the second equality holding by symmetry. See Figure 1. (The equations (2.1) and the diagrams in Figure 1 will play an important role in the remainder of the article.)

We must show that $a_ia_j \in E(G)$, if and only if, $b_ib_j \in E(G)$. Without loss of generality, suppose that $i = 1$ and $j = 2$. Suppose that $a_1a_2 \in E(G)$. By (2.1), $a_2b_1 \notin E(G)$ and $a_1b_2 \notin E(G)$. As $b_1 \in N^c[a_2]$ and by (2.1), we have that $b_1b_2 \in E(G)$. The converse holds similarly, which proves the Claim.

For a fixed $x \in V(G)$, we show that $G \restriction N(x) \cong G \restriction N^c[x]$. Without loss of generality, $x = a_1$. Suppose that $N(a_1) \cap \{a_i : 2 \le i \le n\} = A_1$ and $N^c[a_1] \cap \{a_i : 2 \le i \le n\} = A_2$. For $i = 1, 2$, define B_i to be the vertices of $\{b_i : 1 \le i \le n\}$ that are ends of edges in M beginning with a vertex of A_i.

Since f is an isomorphism, $f(a_i) = b_i$, and by (2.1), we have that

$$\begin{aligned} N(a_1) \cap \{b_i : 2 \le i \le n\} &= B_2, \\ N^c[a_1] \cap \{b_i : 2 \le i \le n\} &= B_1. \end{aligned}$$

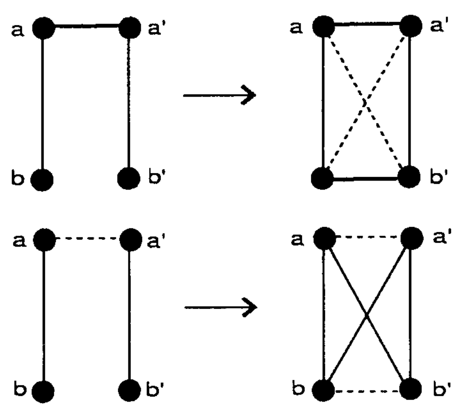

FIGURE 1. The two configurations arising in (2.1). The edges *ab*, *a′b′* are in the dnp matching *M*. The dotted lines are non-edges.

Hence,

$$N(a_1) = \{b_1\} \cup A_1 \cup B_2,$$
$$N^c[a_1] = \{a_1\} \cup B_1 \cup A_2.$$

Define the map $F : G \restriction N(a_1) \to G \restriction N^c[a_1]$ as follows:

$$F(b_1) = f^{-1}(b_1) = a_1, F \restriction A_1 = f \restriction A_1, F \restriction B_2 = f^{-1} \restriction B_2.$$

The map F is seen to be an isomorphism by applications of (2.1). ∎

For a vertex $x \in V(G)$, define $N[x] = N(x) \cup \{x\}$. We say that G has the *closed-neighbour-co-neighbour property*, or *cnc property*, if for all $x \in V(G)$, $G \restriction N[x] \cong G \restriction N^c(x)$. We say that G has *cnc*. It is not hard to see that a graph has cnc, if and only if, its complement has ncc. Hence, Theorem 2.1 supplies a characterization of the cnc graphs.

The next two corollaries follow straightforwardly from Theorem 2.1.

Corollary 2.1. *The bipartite ncc graphs are* $K_{n,n}$, *where n is a positive integer.*

We remark that there are non-regular graphs with dnp matchings. By Corollary 2.1, any bipartite graph with a perfect matching that is distinct from $K_{n,n}$ for all $n \geq 1$, is such an example. Also, a $2n$-vertex, n-regular graph need not have a dnp matching. For example, see Figure 2. (Any such graph has at least 8 vertices.)

The *distance* between vertices x and y is written $d(x, y)$. The *diameter* of a connected graph G is the maximum integer in the set $\{d(x, y) : x, y \in V(G)\}$.

Corollary 2.2. *If G is an ncc graph, then G is connected of diameter* 2, *and if* $|V(G)| = 2n$, *then* $|E(G)| = n^2$.

We emphasize that the Claim in the proof of Theorem 2.1 holds no matter which "orientation" the edges of M are given. Hence, if there are n edges in the matching M, then each edge $xy \in M$ can be oriented in two ways: with x as an "a" vertex and y as a "b" vertex, or the opposite. Call the isomorphism

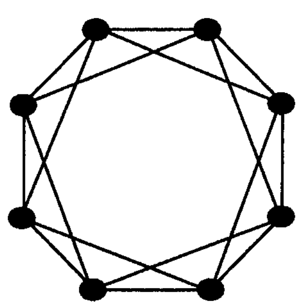

FIGURE 2. An 8-vertex, 4-regular graph with no dnp matching.

$f(a_i) = b_i$ described in the proof of the Claim an *M-isomorphism*. Hence, given M of order n, there are 2^n such M-isomorphisms. The ncc property is therefore, quite a powerful regularity condition. We will often abuse notation and write $M = A \cup B$, where A is one set of ends of the edges of the matching and B is the other. We say that M is *an isomorphism from A to B*.

A consequence of Theorem 2.1 is the following.

Corollary 2.3. *If G is an ncc graph, then G has a spanning subgraph consisting of a disjoint union of graphs of the form* $K_i \square K_2$, *where* $i \geq 1$.

Given a graph G, let $E^{\triangle} = \{e \in E(G) : e$ is an edge of some K_3 in $G\}$. Define $G^{-\triangle} = G - E^{\triangle}$. A *Tutte set* in G is a set T of vertices so that the number of odd order connected components of $G - T$ has cardinality strictly greater than $|V(T)|$. J. Edmonds pointed out that recognizing ncc graphs can be done in the following way.

Corollary 2.4. *Let G be a graph of order* $2n$ *that is n-regular. The following are equivalent.*

1. *The graph G has the ncc property.*
2. *The graph* $G^{-\triangle}$ *has a perfect matching.*
3. *The graph* $G^{-\triangle}$ *has no Tutte set.*

Proof. A classic theorem of Tutte [7] says that a graph G has a perfect matching, if and only if, G has no Tutte set. If G has a dnp matching M, then M is a perfect matching of $G^{-\triangle}$. Conversely, if M^- is a perfect matching of $G^{-\triangle}$, then M^- is a dnp matching of G. ■

Given G as input, the graph $G^{-\triangle}$ clearly may be constructed in polynomial time. By a well-known result of Edmonds [5], checking for a perfect matching is in P, so the recognition problem for ncc graphs is in P.

Corollary 2.5. *The following problem is in P.*

Instance: A graph G.
Question: Does G have the ncc property?

3. OPERATIONS PRESERVING THE NCC PROPERTY

Suppose that G and H are two graphs with dnp matchings $M = A \cup B$ and $M' = A' \cup B'$, respectively. Define the *relative join of G and H with respect to* M, M' to be the graph J consisting of $G \cup H$ along with the following edges: each vertex of A is adjacent to each vertex of B', and each vertex of B is adjacent to each vertex of A'. We will write $G_M \vee H_{M'}$ for J. If e is an edge of G, then $G - [e]$ consists of the induced subgraph of G that results by deleting e and its ends from G.

Lemma 3.1. *Let G be an ncc graph with a dnp matching $M = A \cup B$, and let H be an ncc graph with a dnp matching $M' = A' \cup B'$. Then the following hold.*

1. *The graph $\bar{G} \cup M$ with vertices $V(G)$ and edges $E(\bar{G}) \cup M$ has ncc.*
2. *The graph $G_M \vee H_{M'}$ has ncc.*
3. *If e is an edge of M, then $G - [e]$ has ncc, and $M \backslash \{e\}$ is a dnp matching of $G - [e]$.*

Proof. For item (1), we need only check properties (2) and (3) of Theorem 2.1. However, by Theorem 2.1, if $|V(G)| = 2n$, then $\bar{G}$ is regular of degree $n - 1$, and so property (2) holds for $\bar{G} \cup M$. For property (3), consider the matching M, and an edge $e = ab$ of M. The set $N(a)$ in $\bar{G} \cup M$ equals $N^c(a) \cup \{b\}$ in G. Similarly, the set $N(b)$ in $\bar{G} \cup M$ equals $N^c(b) \cup \{a\}$ in G. But $N^c(a) \cap N^c(b) \subseteq N^c[a] \cap N^c[b] = \emptyset$ in G, and so M is dnp in $\bar{G} \cup M$.

For item (2) of this Lemma, note that $G_M \vee H_{M'}$ has order $|V(G)| + |V(H)|$, and is regular of degree $|M| + |M'| = \frac{1}{2}(|V(G)| + |V(H)|)$. It is not hard to see that $M \cup M'$ is a dnp matching of $G_M \vee H_{M'}$. The rest follows by Theorem 2.1.

For item (3), let $e = ab$, and suppose that $|V(G)| = 2n$, where $n \geq 2$. By (2.1), $G - [e]$ is $(n-1)$-regular. As $M \backslash \{e\}$ is a dnp matching in $G - [e]$, the result follows by Theorem 2.1. ■

Lemma 3.1 (3) allows for inductive proofs: the class of ncc graphs is closed under the deletion of edges (with ends) from any dnp matching. As one application, we consider the following. Recall that if M is a matching, then an *M-alternating path* is a path whose edges alternate between edges in M and edges not in M. We will assume that two paths are distinct if the symmetric difference of their edge sets is non-empty.

Theorem 3.1. *Let G be an ncc graph of order $2n$, and suppose that M is a dnp matching of G. Then G contains at least 2^{n-1} distinct Hamilton M-alternating paths that begin and end in an edge of M.*

Proof. We prove the theorem by induction on the cardinality of M. If $|M| = 1$, then $G = K_2$ which has exactly one M-alternating path. Suppose that $|M| = n + 1$. Fix an edge e of M and consider $G - [e]$. By Lemma 3.1 (3), the order $2n$ graph $G - [e]$ has ncc and $M \backslash \{e\}$ is a dnp matching. Hence, there are 2^{n-1} distinct $(M \backslash \{e\})$-alternating Hamilton paths P in $G - [e]$ that begin and end with edges of $M \backslash \{e\}$. Let $e = ab$, and suppose that P has initial vertex x and final vertex y. Since $N(a) \cap N(b) = N^c[a] \cap N^c[b] = \emptyset$, there are exactly two edges with one end in $\{x, y\}$ and the other in $\{a, b\}$. Suppose that a is adjacent to both x and y (the other cases are similar). Then ax and ay are not in M as $e \in M$ and M is a matching. Let P_1 be the path with initial vertex b, followed by the edge ab, then a, then ax, then the path P traversed from x to y; let P_2 be the path with initial vertex b, followed by the edge ab, then a, then ay, then the path P traversed from

y to x. The paths P_1 and P_2 are two distinct M-alternating Hamilton paths of G that begin and edge with an edge of M. ■

Corollary 3.1. *If G is an ncc graph of order $2n$, then G has at least 2^{n-1} distinct Hamilton paths.*

By n-regularity and Dirac's Theorem [4], each ncc graph has a Hamilton circuit. However, an ncc graph need not have a Hamilton circuit that goes through every edge of some dnp matching. For example, the graphs $K_{2n+1} \square K_2$ have no such Hamilton circuit.

Suppose that H has a dnp matching $M = A \cup B$, and ab, $a'b'$ are edges of M. Suppose that aa' and bb' are edges. By property (2.1), neither ab' nor $a'b$ are edges. Define a new graph by deleting the edges aa', bb' and adding the edges $ab', a'b$. If, instead, ab', ba' are edges of H, then delete these and add the edges aa', bb'. In either case, let the new graph be named H'. Note that M is a dnp matching in H'. It is not hard to see, using Theorem 3.1, that H is an ncc graph, if and only if, H' is an ncc graph. We write $H \sim_M H'$. We write $H \sim_M^* H'$ if there is a dnp matching M of H and an integer $n \geq 0$ and graphs $H_0 = H, H_1, \ldots,$ $H_n = H'$ so that $H_i \sim H_{i+1}$ for all $0 \leq i \leq n-1$. The relation $\sim_M^*$ gives an alternative characterization of the ncc graphs.

Corollary 3.2. *A graph G is an ncc graph, if and only if, G is of order $2n$, and there is a dnp matching of G so that $G \sim_M^* K_n \square K_2$.*

Proof. By an inductive argument, each ncc graph G satisfies $G \sim_M^* K_n \square K_2$. The converse follows since $K_n \square K_2$ is an ncc graph and by the remarks in the paragraph preceding the corollary. ■

4. CLASSIFICATION AND LOCAL PROPERTIES

We begin with a straightforward lemma.

Lemma 4.1. *Let G be an ncc graph.*

1. *If G has a vertex x with $G \restriction N(x)$ independent, then $G \cong K_{n,n}$ for some integer $n \geq 1$.*
2. *If G has a vertex x with $G \restriction N(x)$ isomorphic to $K_{n-1} + K_1$, then $G \cong K_n \square K_2$ for some integer $n \geq 2$.*

Proof. For part (1), since $G \restriction N(x) \cong G \restriction N^c[x]$, we have that $V(G)$ can be partitioned into two independent sets. Therefore, G is bipartite, and the conclusion follows from Corollary 2.1. For part (2), suppose that z is the unique isolated vertex in $G \restriction N(x)$. If $|V(G)| = 2n$, then by n-regularity, each vertex y of $N(x)\backslash\{z\}$ is adjacent to a unique vertex y' of $N^c[x]\backslash\{x\}$. Then z is adjacent to each y' and G is isomorphic to $K_n \square K_2$ with K_n-layers $\{x\} \cup \{y : y \in N(x)\backslash\{z\}\}$ and $\{z\} \cup \{y' : y' \in N^c[x]\backslash\{x\}\}$. ■

Theorem 4.1. *If G is a graph of order n, then G is isomorphic to an induced subgraph of an ncc graph of order at most $2n$.*

Proof. We proceed by induction on n to show that if G is of order n, then there is an ncc graph H of order $2n$ with a dnp matching $M = A \cup B$ so that $H \restriction A \cong H \restriction B \cong G$. If G has order 1, then choose H to be K_2.

Suppose that $G = G' \cup \{x\}$, where G' has order n. We can find an ncc graph H' of order $2n$ with a dnp matching $M' = A' \cup B'$ so that $H' \restriction A' \cong H' \restriction B' \cong G$. For ease of notation, let $H' \restriction A' = G_1$, $H' \restriction B' = G_2$, let $C_1 = N(x) \cap A'$, $D_1 = A' \setminus C_1$, and let C_2, D_2 be the images of C_1 and D_1, respectively, under the isomorphism M'. Add a new vertex x'. Join x to D_2 and join x' to C_2, D_1, and x. Name the resulting graph H. The perfect matching $M = M' \cup \{xx'\}$ of H is dnp, $|V(H)| = 2n + 2$, and H is $(n+1)$-regular. Thus, the graph H has ncc by Theorem 2.1. Further, if $A = A' \cup \{x\}$, $B = B' \cup \{x'\}$, then $M = A \cup B$ and $H \restriction A \cong H \restriction B \cong G$. ■

The proof of Theorem 4.1 supplies a method of classifying the r-regular ncc graphs, where $r \geq 2$. We call this *the ncc method.*

Step 1. Fix an r-vertex graph G with vertices $\{1, \ldots, r\}$. Let G' be a disjoint copy of G with vertices $\{1', \ldots, r'\}$. Match G with G' via $M = \{xx' : 1 \leq x \leq r\}$ to obtain a graph H^*.

Step 2. Complete H^* to an ncc graph by using the following pattern: if $e = aa'$ and $f = bb'$ are edges of M, and if $ab \in E(G)$, then we have an induced 4-circuit with edges: $\{ab, bb', b'a', a'a\}$. If $ab \notin E(G)$, then we have an induced 4-circuit with edges $\{ab', b'b, ba', a'a\}$. See Figure 1.

Therefore, once G has been chosen in Step 1, the graph H^* is determined by adding M and the edges in Step 2. By Lemma 4.1, we may assume that G is neither complete nor empty. The ncc method immediately gives an upper bound on the number of ncc graphs of a given order.

Corollary 4.1. *For $n \geq 2$ an integer, the number of non-isomorphic $2n$-vertex ncc graphs is at most the number of non-isomorphic n-vertex graphs.*

As an application of the ncc method, we list the ncc graphs that are r-regular, with $r \leq 5$. For this, we use the four non-isomorphic 3-vertex graphs and the eleven non-isomorphic 4-vertex graphs. Define the graphs H_i, where $1 \leq i \leq 7$, as in Figure 3.

We leave it as an exercise to check that each of the graphs H_i are non-isomorphic. Each of the H_i graphs are ncc graphs by the ncc method or by Corollary 3.2.

Theorem 4.2. 1. *The 3-regular ncc graphs are, up to isomorphism, $K_{3,3}$, and $K_3 \square K_2$.*

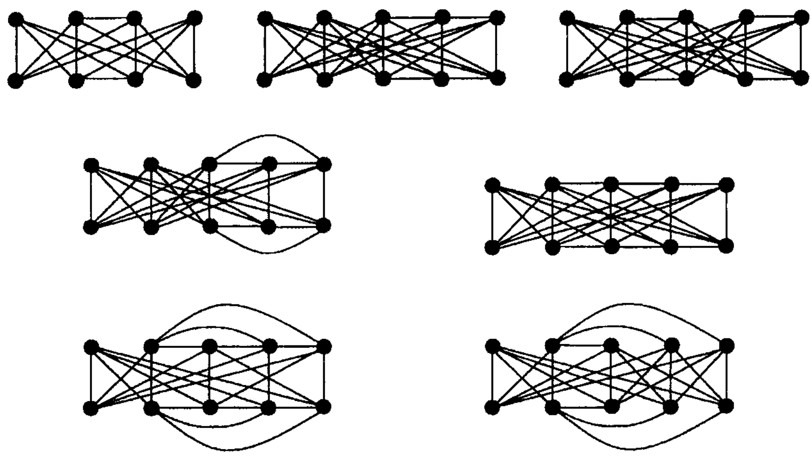

FIGURE 3. The graphs $H_1,\ldots,H_7$.

2. *The* 4*-regular ncc graphs are, up to isomorphism,* $K_{4,4}$, $K_4 \square K_2$, *and* H_1.
3. *The* 5*-regular ncc graphs are, up to isomorphism,* $K_{5,5}$, $K_5 \square K_2$, *and* H_i, *where* $2 \leq i \leq 7$.

We list one consequence of Theorem 4.2.

Theorem 4.3. *The only planar ncc graphs are, up to isomorphism,* K_2, C_4, *and* $K_3 \square K_2$.

Proof. If G is planar ncc graph of order $2n$, where $n \geq 2$, then

$$n^2 = |E(G)| \leq 3|V(G)| - 6 = 6n - 6.$$

Hence, $n = 1$, 2, 3, or 4. The proof now follows by Theorem 4.2, and by the facts that $K_4 \square K_2$ has a K_5 minor and H_1 has a $K_{3,3}$ minor. ■

The graph H_5 defined earlier is unusual since the subgraph induced by each neighbour set is isomorphic to $K_1 + P_4$. If G and H are graphs, then G is a *locally* H graph if for all $x \in V(G)$, $G \restriction N(x) \cong H$. Hence, H_5 is a locally $(K_1 + P_4)$ graph. We will say that H is *good* if there is some ncc graph G which is locally $K_1 + H$. We now investigate which graphs are good. The maximum degree of a vertex in a graph H is written Δ_H.

Lemma 4.2. *Let G be an ncc graph of order* $2n$ *that is locally* $(K_1 + H)$, *where H has at least one edge. Then* $\Delta_H \geq \frac{n}{3}$.

Proof. Let $V(G) = \{0, \ldots, n-1\} \cup \{0', \ldots, (n-1)'\}$. Let $M = \{ii' \mid i = 0, 1, \ldots, n-1\}$ be a dnp matching of G. Let $N(0) = \{i' \mid i = 0, 1, \ldots, n-1\}$ and thus, $N(0') = \{i \mid i = 0, 1, \ldots, n-1\}$. Without loss of generality, both of the subgraphs induced by $\{i \mid i = 1, \ldots, n-1\}$ and $\{i' \mid i = 1, \ldots, n-1\}$ are isomorphic to H. We may assume that vertices 1 and 2 are adjacent. In $N(0')$, let J be the set of vertices not adjacent to 1 nor adjacent to 2. It follows that

$$|J| \geq n - 3 - 2(\Delta_H - 1) = n - 1 - 2\Delta_H.$$

In $G\restriction N(1) \cong H$, by (2.1), 2 is adjacent to $0'$ and the vertices in $J' = \{i' \mid i \in J\}$. Therefore, we have that

$$\Delta_H \geq d_{GN(1)}(2) \geq 1 + |J'| = 1 + |J| \geq n - 2\Delta_H. \qquad \blacksquare$$

By Lemma 4.2, if H is a r-regular good graph of order $n-1$, then $r \geq n/3$. For triangle-free graphs, more can be said.

Corollary 4.2. *Let G be an locally $(K_1 + H)$ ncc graph of order $2n$, where H is an r-regular, triangle-free graph. Then $r = \frac{n}{3}$.*

Proof. In the proof Lemma 4.2, we have that $|J| = n - 1 - 2r$ and also that $r = d_{G\restriction N(1)}(2) = n - 2r$. $\blacksquare$

The *n-dimensional hypercube* Q_n is defined as the Cartesian product of K_2 with itself n times. The graph Q_n is n-regular, triangle-free, and order 2^n; hence, by Corollary 4.2, Q_n is not good for all $n \geq 4$. As another application of Corollary 4.2, observe that H is a good circuit of order n only if $n = 5$. Indeed, C_5 is good, as is witnessed by the graph H_8 as in Figure 4, which is seen to be an ncc graph by the ncc method.

Theorem 4.4. *If G is a locally $(K_1 + T)$ ncc graph, where T is a tree with at least three vertices, then $T \cong P_4$ and $G \cong H_5$.*

Proof. We proceed by contradiction. If there exists another graph G with this property and $|V(G)| = 2n$, then $n \geq 6$ by Theorem 4.2. Let G have vertices $\{i : 0 \leq i \leq n-1\} \cup \{i' : 0 \leq i \leq n-1\}$ so that $M = \{ii' : 0 \leq i \leq n-1\}$ is a dnp matching. Suppose $G\restriction N(0) \cong K_1 + T$ with vertices $\{i' : 0 \leq i \leq n-1\}$, $G\restriction N^c[0] \cong K_1 + T'$ with vertices $\{i : 0 \leq i \leq n-1\}$, and $T \cong T'$ via the isomorphism M. Without loss of generality, suppose that 1 is of degree 1 in T and $12 \in E(T)$. Then the subgraph induced by $N(1)$ consists of $0'$, $1'$, 2, and a tree S' which is a subtree of T'. The order of S' is $n-3$. As $1'$ is the unique isolated vertex in $G\restriction N(1)$, the vertex 2 is adjacent to $2'$ and to a unique vertex of S', say y'. By (2.1), $12' \notin E(G)$, and so $2' \notin S'$. Hence, $2' \neq y'$. The vertex 2 is

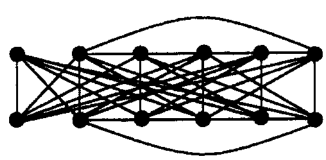

FIGURE 4. The locally $(K_1 + C_5)$ graph H_8.

therefore adjacent only to $0', 2'$, and y' in $\{i' : 0 \leq i \leq n-1\}$. Hence, for n-regularity, the vertex 2 has degree $n-3$ in T. Hence, $\Delta_T \geq n-3 \geq 3$.

The subgraph $T-1$ is isomorphic to the star $K_{1,n-4}$ with one edge subdivided. So $T-1$ consists of 2 which is adjacent to an independent set U of order $n-4$, and a vertex $y \neq 2$ adjacent to a unique vertex z of U. Without loss of generality, let $U = \{3, \ldots, n-2\}$, with $z = 3$ and $y = n-1$. Since $n-4 \geq 2$, vertex 4 is in U and is therefore distinct from $n-1$. Now, $N(4)$ consists of $0'$, 2, and the $(n-2)$-set $V' = \{1', 3', 4', \ldots, (n-1)'\}$. Since M is an isomorphism, V' has only one edge between its vertices: $3'(n-1)'$. As 2 is only adjacent to $(n-1)'$ among the vertices of V', each vertex in $G \restriction N(4)$ has degree at most 2, which contradicts that $\Delta_T \geq 3$. ■

We do not know which graphs are good, but we conjecture that there are only finitely many regular good graphs distinct from $\bar{K}_n$, K_n, and C_5. Theorem 4.4 supplies evidence for this, as does the following corollary.

Corollary 4.3. *If $g \in O(n)$, then, up to isomorphism, there are only finitely many connected regular good graphs H of order n with genus g. In particular, there are, up to isomorphism, only finitely many planar regular good graphs.*

Proof. Let H be such a graph. Since H is connected, we have that $g \geq \frac{|E(H)|}{6} - \frac{n}{2} + 1$, and so

$$\frac{(n^2+n)}{36} \leq g + \frac{n}{2} - 1.$$

Since $g \in O(n)$, there are only finitely many nonnegative integers n satisfying this inequality. ■

5. THE INFINITE CASE AND ANOTHER PARTITION PROPERTY

For infinite ncc graphs G of order α, it is not hard to see that G is α-regular. In particular, for $x \in V(G)$,

$$|N(x)| = |N^c[x]| \quad (5.1)$$

$$|N(x)| + |N^c[x]| = \alpha. \quad (5.2)$$

If $|N(x)| = \beta < \alpha$, then by (5.1) β is infinite. But then $|N(x)| + |N^c[x]| = \beta + \beta = \beta < \alpha$, contradicting (5.2). For infinite graphs, however, the analogue of Theorem 2.1 fails. Consider the complete graph of order $\aleph_0$, written $K_{\aleph_0}$, and suppose that the vertices of $K_{\aleph_0}$ are the natural numbers. Delete from $K_{\aleph_0}$ the infinite one way path with edges $n(n+1)$, for $n \geq 0$, to obtain the graph G. Define G' similarly, but the vertices are indexed by $\{n' : n \in \mathbb{N}\}$. Define the graph H by forming $G \cup G'$ and adding the perfect matching $M = \{nn' : n \in \mathbb{N}\}$. See Figure 5.

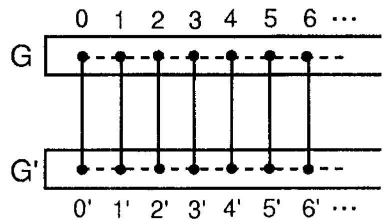

FIGURE 5. The graph *H*.

It is not hard to see that M is dnp, and that H is regular of infinite degree. However, the subgraph induced by $N(0)$ is isomorphic to $K_1 + G$ and the subgraph induced by $N^c[0]$ is isomorphic to $K_1 + G$ along with a single vertex of degree 1 adjacent to 0.

We will say that a perfect matching $M = \{e_i : 1 \leq i \leq n\}$ is *strongly dnp* if for all i, if $e = a_i b_i$, then $N(a_i) = N^c[b_i]$ and $N(b_i) = N^c[a_i]$. In other words, (2.1) holds for each edge e in M. For finite ncc graphs, the notion of a strongly dnp and dnp matching coincide. If G is an α-regular graph of order α with a strongly dnp matching, then the proof of the converse of Theorem 2.1 also gives that G has ncc. We only know that the converse holds under an additional assumption. We say that G is *special* if for each $x \in V(G)$, if y is isolated in $N(x)$, then $N(y) = N^c[x]$. It is not hard to see that each graph with a strong dnp matching M is special. To see this, suppose that y is isolated in $N(x)$ but is not adjacent to $z \in N^c[x] \backslash \{x\}$, and z is matched with z' by M. Now, $yz' \in E(G)$ since M is strongly dnp; but $xz' \in E(G)$, contradicting that y is isolated in $N(x)$. Following the proof of Theorem 2.1, we have the following.

Theorem 5.1. *Let G be an infinite graph of order α. Then G has ncc and is special, if and only if, G is α-regular and has a strongly dnp matching.*

We do not know a characterization of the infinite ncc graphs. Variants of the ncc property arise by considering other subsets of $V(G)$. We say that a connected graph has the *ncc(e) property* if for each vertex x, the subgraph induced by the set of vertices of even distance from x is isomorphic to the subgraph induced by the set of vertices of odd distance from x. (Note that x is considered to have distance 0 from x and hence, even distance from x.) The ncc graphs (which are of diameter 2), and the graphs C_{2n}, where $n \geq 1$, have ncc(e). We do not have a characterization of the ncc(e) graphs like the ones of the ncc graphs in Theorem 2.1, Corollary 2.4, and Corollary 3.2.

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