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Almost all cop-win graphs contain a universal vertex

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ABSTRACT

We consider cop-win graphs in the binomial random graph $G(n, 1/2)$. We prove that almost all cop-win graphs contain a universal vertex. From this result, we derive that the asymptotic number of labelled cop-win graphs of order n is equal to $(1 + o(1)) n2^{n^2/2-3n/2+1}$.

1. Introduction

Cops and Robbers is vertex-pursuit game played on a reflexive graph. There are two players, consisting of a set of *cops* and a single *robber*. The game is played over a sequence of discrete time-steps or *rounds*, with the cops going first in the first round and then playing alternate time-steps. The cops and the robber occupy vertices. When a player is ready to move in a round they must move to a neighbouring vertex. Because of the loops, players can *pass*, or remain on their own vertex. Observe that any subset of cops may move in a given round. The cops win if, after some finite number of rounds, one of them can occupy the same vertex as the robber. This is called a *capture*. The robber wins if he can evade capture indefinitely. A *winning strategy for the cops* is a set of rules that, if followed, result in a win for the cops. A *winning strategy for the robber* is defined analogously.

If we place a cop at each vertex, then the cops are guaranteed to win. Therefore, the minimum number of cops required to win in a graph G is a well-defined positive integer, named the *cop number* (or *copnumber*) of the graph G. We write $c(G)$ for the cop number of a graph G. If $c(G) = k$, then we say that G is *k-cop-win*. In the special case $k = 1$, we say that G is *cop-win* (or *copwin*). Nowakowski and Winkler [10], and independently Quilliot [13], considered the game with one cop only; the introduction of the cop number came in [1]. Many papers have now been written on cop number since these three early works; see the surveys [2,8,9].

Since their introduction, the structure of cop-win graphs has been relatively well understood. In [10,13,14], a kind of ordering of the vertex set – now called a cop-win or elimination ordering – was introduced which completely characterizes such graphs. If u is a vertex, then the *closed neighbour set of* u, written $N[u]$, consists of u along with the neighbours of u. A vertex u is a *corner* if there is some vertex v, $v \neq u$, such that $N[u] \subseteq N[v]$. We say that v is the *parent* of u, and that u is the *child* of v. A graph is *dismantlable* if some sequence of deleting corners results in the graph with a single vertex. For example, each tree is dismantlable, and, more generally, so are chordal graphs (that is, graphs with no induced cycles of length more than 3). To prove the latter fact, note that a chordal graph contains a vertex whose neighbour set is a clique; see, for example, West [16].

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The following theorem gives the main results characterizing cop-win graphs.

Theorem 1.1 ([10,13,14]).

(1) *If u is a corner of a graph G, then G is cop-win if and only if* $G - u$ *is cop-win.*
(2) *A graph is cop-win if and only if it is dismantlable.*

From Theorem 1.1, cop-win (or dismantlable) graphs have a recursive structure, made explicit in the following sense. A permutation $v_1, v_2, \ldots, v_n$ of the vertices of G is a *cop-win ordering* if there exist vertices $w_1, w_2, \ldots, w_n$ such that, for all $i \in [n] = \{1, 2, \ldots, n\}$, $N[v_i] \subseteq N[w_i]$ in $V(G) \setminus \{v_j : j < i\}$ and $v_i \neq w_i$. We use the notation **v** for a cop-win ordering, and **w** for its *parent sequence*. Cop-win orderings are sometimes called *elimination orderings*, as we delete the vertices from lower to higher index until only vertex v_n remains.

We say that an event holds asymptotically almost surely (*a.a.s.*) if it holds with probability tending to 1 as n tends to infinity. The probability of an event A is denoted by $\mathbb{P}(A)$.

Our goal is to investigate the structure of *random cop-win* graphs. The random graph model we use is the familiar $G(n, 1/2)$ probability space of all labelled graphs on n vertices, where each pair of vertices is joined with probability 1/2, independently of the events for other pairs of vertices. Note that a given graph G on e_G edges occurs with probability

$$\mathbb{P}(G \in G(n, 1/2)) = \left(\frac{1}{2}\right)^{e_G} \left(1 - \frac{1}{2}\right)^{\binom{n}{2} - e_G} = \left(\frac{1}{2}\right)^{\binom{n}{2}},$$

which does not depend on G. Thus, $G(n, 1/2)$ is in fact a uniform probability space over all labelled graphs on n vertices. We heavily use this interpretation of $G(n, 1/2)$ in the proof of our main result, Theorem 2.1, stated below. We expect results analogous to Theorem 2.1 (that is, with 2 replaced by $1/p$) for other constants $p \in (0, 1)$ and $p = p(n)$ tending to zero with n. (The argument for $p = p(n)$ tending to 1 needs to be modified when the expected number of universal vertices is $\Omega(1)$; see [11].) However, this seems not to be an interesting research direction in the theory of random graphs, where we usually focus on investigating typical properties that hold a.a.s. in $G(n, p)$. Therefore, studying bounds for the cop number that hold a.a.s. are of interest, and a number of papers have been published on this topic (see, for example [3,6,15,12] and the recent monograph [5]). We focus on $G(n, 1/2)$ in this paper since it gives the typical structure of a cop-win graph. Therefore, from now on our probability space is always taken to be $G(n, 1/2)$.

2. Main results

A vertex is *universal* if it is joined to all others. Let **cop-win** be the event that the graph is cop-win, and let **universal** be the event that there is a universal vertex. If a graph has a universal vertex w, then it is cop-win; in a certain sense, graphs with universal vertices are the simplest cop-win graphs. The probability that a random graph is cop-win can be estimated as follows:

$$\begin{aligned}\mathbb{P}(\textbf{cop-win}) \geq \mathbb{P}(\textbf{universal}) &= n2^{-n+1} - O(n^2 2^{-2n+3}) \\ &= (1 + o(1))n2^{-n+1}. \end{aligned} \tag{2.1}$$

Surprisingly, this lower bound is in fact the correct asymptotic value for $\mathbb{P}(\textbf{cop-win})$. Our main result is the following theorem.

Theorem 2.1. *In* $G(n, 1/2)$, *we have that*

$$\mathbb{P}(\textbf{cop-win}) = (1 + o(1))n2^{-n+1}.$$

Using Theorem 2.1, we derive the asymptotic number of labelled cop-win graphs.

Corollary 2.2. *The number of cop-win graphs on n labelled vertices is*

$$(1 + o(1))2^{\binom{n}{2}} n2^{-n+1} = (1 + o(1))n2^{n^2/2 - 3n/2 + 1}.$$

It also follows that almost all cop-win graphs contain a universal vertex, a fact not obvious a priori.

Corollary 2.3.

$$\mathbb{P}(\textbf{universal} \mid \textbf{cop-win}) = 1 - o(1).$$

We prove Theorem 2.1 in the next section. We finish this section with some notation that will be used in the proof. The *degree* of a vertex u is written $\deg(u)$. We let $\Delta(G)$ denote the *maximum degree* of G (or just Δ if G is clear from context). The *co-degree* of a vertex u in a graph of order n is $n - 1 - \deg(u)$.

3. Proofs of main results

To prove Theorem 2.1, we bound the probability of **cop-win** for graphs of maximum degree at most $n-2$. Since the proof for $\Delta = n - 2$ has a different flavour than the one for $\Delta \leq n - 3$, we prove it independently.

Theorem 3.1.

(a) *For some* $\epsilon > 0$, *we have that*

$$\mathbb{P}(\textbf{cop-win} \textit{ and } \Delta \le n-3) \le 2^{-(1+\epsilon)n}.$$

(b) $\mathbb{P}(\textbf{cop-win} \textit{ and } \Delta = n-2) \le 2^{-(3-\log_2 3)n+o(n)}$.

Theorem 2.1 follows immediately from Theorem 3.1 and (2.1).

Proof of Theorem 3.1(a). Let G be a random graph drawn from the $G(n, 1/2)$ distribution. We study the probability that $\Delta \le n-3$, and that there exists a permutation $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ and a sequence of vertices $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ which are a cop-win ordering and associated parent sequence for G, respectively. We show that this event holds *with extreme probability* (*wep*), which means that the probability it holds is at most $2^{-(1+\epsilon)n}$ for some $\epsilon > 0$. Observe that, if we can show that a polynomial number of events holds *wep*, then the same is true for a union of these events.

We partition the set of all such pairs $(\mathbf{v}, \mathbf{w})$ into a four groups, and show that for each group the property holds *wep*. Actually, we refine some of the groups to specify the degree sequence of the parents. Moreover, we usually focus on the initial segments of $\mathbf{v}$ and $\mathbf{w}$ of length cn, where $c \in (0, 1)$ is a constant. We state the partition we consider before moving to the proof that each group yields an event that holds *wep*.

(1) There exist $\mathbf{v}$ and $\mathbf{w}$ having

$$s = |\{w_i : i \le 0.05n\}| > s_0 = 17.$$

In other words, there is a cop-win ordering whose vertices in their initial segments of length $0.05n$ have more than 17 parents.

(2) There exist $\mathbf{v}$ and $\mathbf{w}$ having $s \le s_0$ and w_i with co-degree $d_i > n^{2/3}$ for all $i \le 0.05n$. That is, there is a cop-win ordering whose vertices in their initial segments of length $0.05n$ have at most 17 parents, each of which has co-degree more than $n^{2/3}$.

(3) There exist $\mathbf{v}$ and $\mathbf{w}$ having $2 \le s \le s_0$, and at least one parent has co-degree $d_i \le n^{2/3}$ for some $i \le 0.05n$. That is, there is a cop-win ordering whose initial segments of length $0.05n$ have between 2 and 17 parents, and at least one parent has co-degree at most $n^{2/3}$.

(4) There exist $\mathbf{v}$ and $\mathbf{w}$ having $s = 1$. In other words, there exists $w \in V(G)$ with co-degree between 2 and $n^{2/3}$, such that $w_i = w$ for $i \le 0.05n$.

Group (1). Set $c = 0.05$, and suppose that there exist $\mathbf{v}$ and $\mathbf{w}$ with the property we consider in this group. Let $w_{a_1}, w_{a_2}, \ldots, w_{a_s}$ (where $w_{a_i} \in [n]$ for $i \in [s]$) be $s > s_0$ distinct parents of corresponding children $v_{a_1}, v_{a_2}, \ldots, v_{a_s}$ (where $v_{a_i} \in [cn]$ for $i \in [s]$). We would like to have the set of all of those vertices (both parents and children) distinct. All parents and all children are different but, of course, it can happen that $v_{a_i} = w_{a_j}$ for some $i \ne j$. However, since each parent can be a child only once (recall that all children are distinct), we must have at least $\lceil s/2 \rceil$ disjoint parent/child pairs. Let

$$X = V(G) \setminus (\{v_i : i \in [cn]\} \cup \{w_i : i \in [cn]\}).$$

Note that X contains at least $(1-2c)n$ vertices. Since $N[v_i] \subseteq N[w_i]$ in $G \setminus \{v_1, v_2, \ldots, v_{i-1}\}$, the following event $Q(v_i, w_i, X)$ holds: no vertex $x \in X$ is adjacent to v_i but not adjacent to w_i.

Thus, this implies that there exist s (where $s_0 < s \le n$), a set C of cn vertices, a function $p : C \to V$ (we interpret $p(v)$ as telling us the parent of v), and $\lceil s/2 \rceil$ mutually disjoint pairs $(z_i, p(z_i))$ such that $Q(z_i, p(z_i), X)$ holds for $i \in \{1, 2, \ldots, \lceil s/2 \rceil\}$, with

$$X = V(G) \setminus (C \cup \{p(z_i) : i \in \{1, 2, \ldots, \lceil s/2 \rceil\}\}).$$

The number of configurations we need to consider is at most

$$n \binom{n}{cn} \binom{n}{s} s^{cn} \le n 2^n 2^n 2^{cn \log_2 s} = 2^{(2 + c \log_2 s)n + o(n)}. \tag{3.1}$$

For each configuration, we need to estimate the probability that the corresponding event holds. The probability that, for a given i, the event $Q(z_i, p(z_i), X)$ holds is at most

$$\left(\frac{3}{4}\right)^{n-2cn} = 2^{-\log_2(4/3)(1-2c)n}.$$

Moreover, since parent/child pairs are mutually disjoint, the events $Q(z_i, p(z_i), X)$ are mutually independent. Hence, the probability in question can be estimated from above by

$$2^{-\log_2(4/3)(1-2c)(s/2)n + o(n)}. \tag{3.2}$$

Thus, by (3.1) and (3.2), the property holds *wep* if, say, s is chosen such that

$$2 + c \log_2 s - \log_2(4/3)(1-2c)(s/2) < -1.1;$$

that is, s is a sufficiently large constant that depends on c but does not depend on n. In particular, if $c = 0.05$, then $s \geq 18$ works.

Group (2). As we mentioned above, a polynomial multiplicative term is not going to affect a result that holds *wep*. Thus, there is no problem at this point to introduce co-degrees of all parents for the initial segment under consideration in this group.

We estimate the number of configurations we consider in this group. Fix parents for the initial segment of length $0.05n$ ($O(n^{s_0})$-many) and their degrees and neighbourhoods ($O((2^n)^{s_0})$-many). Fix the initial segment of $\mathbf{v}$ of length $0.05n$ ($O(n!)$-many) and assignment of parents ($O(s_0^n)$-many). The total number of configurations is, therefore,

$$2^{O(n \log_2 n)}. \tag{3.3}$$

Since every parent has co-degree larger than $n^{2/3}$, v_1 cannot be adjacent to at least $n^{2/3}$ non-neighbours of w_1, v_2 has to avoid $n^{2/3} - 1$ non-neighbours of w_2 (note that v_1 is perhaps a non-neighbour of w_2), v_3 avoids $n^{2/3} - 2$ vertices, and so on. Note that edges of all parents are exposed at this point, so we should focus on children in the first segment that are not parents for any other child in this segment. Since there are at most $s_0 = O(1)$ parents, this causes no problem; we do not consider these vertices. For a given configuration, the probability is at most

$$2^{-(n^{2/3}-s_0)-(n^{2/3}-s_0-1)-(n^{2/3}-s_0-2)-\cdots-2-1} = 2^{-\Omega(n^{4/3})},$$

which tends to zero fast enough when compared to (3.3) so that the property we consider holds *wep*.

Group (3). Set $c = 0.05$, and suppose that there exist $\mathbf{v}$ and $\mathbf{w}$ with the property we consider in this group. This implies that there exists a vertex $w_i \in V$ with small co-degree ($d_i \leq n^{2/3}$). Moreover, there exists a set of vertices C with $|C| = cn$, $v_j \in C$, and $w_j \in V(G) \setminus \{v_j, w_i\}$ such that the event $Q(v_j, w_j, V(G) \setminus (C \cup \{w_j, w_i\}))$ holds. (See the argument for Group (1) for the definition of $Q(\cdot, \cdot, \cdot)$.) There are only two differences between the argument here and the one used in Group (1). First, this time the size of X is $(1 - c)n + O(1)$, not $(1 - 2c)n$ as before. Second, we only investigate neighbourhoods of vertices v_j, w_j, w_i in order to estimate the probability that the event in question holds, not $\lceil s/2 \rceil$ pairs as before.

Using the Stirling formula (which states that $n! \sim \sqrt{2\pi n}(n/e)^n$), we find that the number of configurations is at most

$$nn^{2/3}\binom{n}{n^{2/3}}\binom{n}{cn}n^2 = 2^{-\log_2(c^c(1-c)^{1-c})n+o(n)}. \tag{3.4}$$

Suppose first that $v_j \neq w_i$ (that is, all the vertices v_j, w_j, w_i are distinct). Hence, the desired probability can be estimated from above by

$$2^{-n+1}2^{-\log_2(4/3)(1-c)n+O(1)}, \tag{3.5}$$

where 2^{-n+1} corresponds to the edges incident with the vertex w_i. Observe that the term in (3.5) tends to zero fast enough when compared to (3.4) for the event to hold *wep*. (Note that not every $c \in (0, 1)$ works this time. However, one can check that it is the case for, say, $c < 0.07$.) If $v_j = w_i$, then the situation is even better. Since w_j has to be adjacent to all neighbours of $v_j = w_i$ in $V(G) \setminus (C \cup \{w_j, w_i\})$ (that is, all vertices in $V(G) \setminus (C \cup \{w_j, w_i\})$ but, perhaps, $O(n^{2/3})$ of them), we estimate the probability by

$$2^{-n+1}2^{-(1-c)n+O(n^{2/3})} < 2^{-n+1}2^{-\log_2(4/3)(1-c)n+O(1)},$$

and the assertion holds in this case.

Group (4). Suppose that $w_i = w$ for $i \leq cn(1 + o(1))$ for some $c \in [0.05, 1]$, and the first child $\bar{v}$ with parent $\bar{w} = w_j \neq w$ occurs if $j = cn(1 + o(1))$). (Note that we cannot have one parent only, since $\Delta \neq n - 1$.) Moreover, we can insist that w is used as a parent for as long as possible; that is, until we get the property that all remaining vertices that are in $N[w]$ are adjacent to at least one of the non-neighbours $\bar{N}[w]$ of w, so that the new parent, $\bar{w}$, has to be introduced.

This implies that there exist a set C of $cn(1 + o(1))$ vertices and vertices $w, \bar{v}, \bar{w} \in V(G) \setminus C$ (possibly, $\bar{v} = w$) such that the following events hold. The co-degree of w is d, $2 \leq d \leq n^{2/3}$, every vertex in C is adjacent to w but not to any of co-neighbours of w, and every vertex in $V(G) \setminus (C \cup \bar{N}[w] \cup \{\bar{v}, \bar{w}\})$ is adjacent to at least one co-neighbour of w. Moreover, the event $Q(\bar{v}, \bar{w}, V(G) \setminus (C \cup \{w, \bar{v}, \bar{w}\}))$ holds. (See the argument for Group (1) for the definition of $Q(\cdot, \cdot, \cdot)$.) The number of configurations to consider can be bounded from above by

$$2^{-\log_2(c^c(1-c)^{1-c})n+o(n)}. \tag{3.6}$$

Suppose first that $\bar{v} \neq w$. The probability can be bounded from above by

$$2^{-n}2^{-dcn(1+o(1))}2^{-\log_2(4/3)(1-c)n+o(n)},$$

which is enough when compared to (3.6) for the event to hold *wep* for any $c \in (0.05, 0.25) \cup (0.25, 1]$ or $d \geq 3$. (Note that for $c = 1/4$ and $d = 2$ we get that the event holds with probability at most $2^{-n+o(n)}$ only, not *wep*.) Similarly as in Group (3), the situation is better if $\bar{v} = w$, since then $\bar{w}$ has to be adjacent to almost all vertices in $V(G) \setminus (C \cup \{w, \bar{v}, \bar{w}\})$.

It remains to consider the case $c = 1/4$ and $d = 2$. Using our extra knowledge that every vertex in $V(G) \setminus (C \cup \bar{N}[w] \cup \{\bar{v}, \bar{w}\})$ is adjacent to at least one of the two co-neighbours of w, we obtain the extra multiplicative factor of $(3/4)^{3n/4+o(n)}$, which is enough for the event to hold *wep*. □

Now, we are ready to prove the second part of Theorem 3.1.

Proof of Theorem 3.1(b). First, we deduce from Theorem 3.1(a) a rough upper bound for the probability that the graph is cop-win: there exists $\epsilon > 0$ such that

$$\begin{aligned}\mathbb{P}(\textbf{cop-win}) &\le \mathbb{P}(\textbf{cop-win} \text{ and } \Delta \le n-3) + \mathbb{P}(\Delta \ge n-2)\\ &\le 2^{-(1+\epsilon)n} + n^2 2^{-n+1}\\ &\le 2^{-n+o(n)}.\end{aligned} \tag{3.7}$$

The vertex set of every cop-win graph with $\Delta = n-2$ can be partitioned as follows: it must have a vertex w of degree $n-2$, a (unique) vertex v which is not adjacent to w, a set B of vertices adjacent to v (and also to w), and a set A of vertices that are not adjacent to v (but adjacent to w). We claim that the graph induced by B is cop-win.

By Theorem 1.1, we can dismantle all vertices in A (using w as a parent), leaving us with the cop-win subgraph H induced by v, w, and B. If B contains one vertex only, then the graph induced by B is clearly cop-win. Otherwise, either B has a universal vertex in G (and so the graph induced by B is cop-win and we can dismantle all remaining vertices of B), or B must have a corner (since if there is no universal vertex in B, you cannot dismantle either v or w but H is cop-win). In either case, we can dismantle some vertex x in B, so that the following properties hold.

(1) $H - x$ is a cop-win subgraph induced by v, w, and $B \setminus \{x\}$,
(2) v and w are joined to all of $B \setminus \{x\}$.

Hence, by (1) and (2) we may use induction to dismantle B starting from the subgraph H. Moreover, the same sequence of vertices can be used to dismantle the graph induced by the set B, since all the parents were in B. Therefore, B is cop-win.

Finally, we estimate the number of labelled cop-win graphs with $\Delta = n-2$. There are n choices for w, $n-1$ choices for v, and $2^{n-2} = \sum_{i=0}^{n-2}\binom{n-2}{i}$ choices for A. The probability that w and v have the correct neighbourhoods is $2^{-n+1}2^{-n+1}$. If $|A| = i$, then the probability that the graph induced by B is cop-win is at most $2^{-n+2+i+o(n)}$ using (3.7) (note that $|B| = n-2-i$, and that there are no other restrictions on B except that the subgraph it induces is cop-win). Thus,

$$\begin{aligned}\mathbb{P}(\textbf{cop-win} \text{ and } \Delta = n-2) &\le n^2 \sum_{i=0}^{n-2}\binom{n-2}{i} 2^{-n+1}2^{-n+1}2^{-n+2+i+o(n)}\\ &= 2^{-3n+o(n)}\sum_{i=0}^{n-2}\binom{n-2}{i}2^i\\ &= 2^{-3n+o(n)}(1+2)^{n-2}\\ &= 2^{-(3-\log_2 3)n+o(n)}. \quad \square\end{aligned}$$

4. Discussion

For an integer $k > 1$, determining the asymptotic behaviour of the function $F_k(n)$, the number of labelled k-cop-win graphs of order n, remains an open problem. The limited current understanding of graphs with cop number 2 or higher is the main stumbling block. For example, there are no elementary analogues of cop-win orderings for higher k. An elimination ordering characterization of k-cop-win graphs for $k > 1$ was given in [7], although it becomes exponentially more complex as k increases (in particular, an ordering is given of vertices in the $(k+1)$th strong power of the graph). Nevertheless, we may conjecture that almost all k-cop-win graphs have a dominating set of cardinality k, which would generalize our Theorem 2.1, and imply that

$$F_k(n) = 2^{o(n)}(2^k-1)^{n-k}2^{\binom{n-k}{2}} = 2^{n^2/2-(1/2-\log_2(1-2^{-k}))n+o(n)}.$$

In [4,6], it has been shown that the cop number of $G \in G(n, 1/2)$ is a.a.s. equal to $(1+o(1))\log_2 n$. Hence, we have that $F_k(n) = o(2^{\binom{n}{2}})$ unless $k = (1+o(1))\log_2 n$. Another problem is whether $F_k(n)$ is *unimodal*: is there a function $K = K(n) = (1+o(1))\log_2 n$ such that, for n large enough, $F_k(n) \le F_{k+1}(n)$ for $k \le K$, and $F_k(n) \ge F_{k+1}(n)$ for $k > K$?

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