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Continuum many universal Horn classes of graphs of bounded chromatic number

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In [1], X. Caicedo proved, among other things, that there are continuum many universal Horn classses of graphs. His proof rested on the existence of a countably infinite family, $\mathscr{F}$, of pairwise homomorphically independent finite graphs (a pair of graphs is homomorphically independent if there is no homomorphism between them). $\mathscr{F}$ contains graphs of arbitrarily large chromatic number. However, the question, also posed in [1], as to whether there are continuum many universal Horn classes whose members are of bounded chromatic number was left open. We exhibit continuum many universal Horn classes between every interval of colour families above the class of 2-colourable graphs.

Throughout, a graph is considered a structure with one binary, irreflexive, and symmetric relation E. The complete graph on n vertices, for $n \geq 1$, is denoted $\mathbf{K}_n$. The n-cycle, for $n \geq 3$, is denoted $\mathbf{C}_n$. If $\mathbf{G}$ and $\mathbf{H}$ are graphs, a map $f\colon \mathbf{G} \rightarrow \mathbf{H}$ is a homomorphism if $aE^{\mathbf{G}}b$ implies $f(a)E^{\mathbf{H}}f(b)$. A graph $\mathbf{G}$ is an induced subgraph of the graph $\mathbf{H}$ if $G \subseteq H$ and $E^{\mathbf{G}} = E^{H} \upharpoonright G$. If $\{\mathbf{G}_i \colon i \in I\}$ is a set of graphs, define the product $\Pi_{i \in I} \mathbf{G}_i$ to be the graph $\mathbf{P}$ with vertices $\Pi_{i \in I} G_i$ and with edges defined so that $aE^{\mathbf{P}}b$ if and only if $a(i)E^{\mathbf{G}_i}b(i)$, for all $i \in I$.

DEFINITION 1. Let $\mathbf{G}$ be a finite graph. Define $\mathscr{C}_{\mathbf{G}}$ to be the class of graphs that map homomorphically to $\mathbf{G}$. After [6], we call this the *colour family* determined by $\mathbf{G}$.

For $n \geq 1$, and $\mathbf{G} = \mathbf{K}_n$, $\mathscr{C}_{\mathbf{G}}$ is the class of n-colourable (or n-partite) graphs. For $n \geq 2$, the class of n-chromatic graphs is the class $\mathscr{C}_{\mathbf{K}_n} - \mathscr{C}_{\mathbf{K}_{n-1}}$. There are non-trivial colour families not equal to the classes of n-colourable graphs. For

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example, $\{\mathscr{C}_{\mathbf{C}_{2n+1}}: n \geq 2\}$, the colour families determined by the odd cycles, form a strictly descending chain of colour families that are 3-chromatic: $\mathscr{C}_{\mathbf{C}_5} \supset \mathscr{C}_{\mathbf{C}_7} \supset \mathscr{C}_{\mathbf{C}_9} \supset \cdots$.

DEFINITION 2. Let $\mathbf{G}$ and $\mathbf{H}$ be graphs. We write $\mathbf{G} \to \mathbf{H}$ if there is a homomorphism from $\mathbf{G}$ to $\mathbf{H}$.

If $\mathbf{G}$ and $\mathbf{H}$ are finite graphs, $\mathbf{G} \to \mathbf{H}$ if and only if $\mathscr{C}_{\mathbf{G}} \subseteq \mathscr{C}_{\mathbf{H}}$.

DEFINITION 3. A class $\mathscr{K}$ of graphs is a *finitely generated universal Horn class* if there is a finite set of finite graphs $\mathscr{C}$ so that $\mathscr{K} = ISP(\mathscr{C})$; that is, $\mathscr{K}$ is the class of isomorphic copies of induced subgraphs of products of members of $\mathscr{C}$.

The following Proposition was first proved in [4].

PROPOSITION 4. *$\mathscr{C}_{\mathbf{G}}$ is a finitely generated universal Horn class.*

Proof. Let $\mathbf{G}' = \langle (\{0\} \times G) \cup (\{1\} \times G), E^{\mathbf{G}'} \rangle$, with $(i, x) E^{\mathbf{G}'} (j, y)$ if and only if $x E^{\mathbf{G}} y$ and i and j are both not equal to 1. Then $\mathscr{C}_{\mathbf{G}} = ISP(\mathbf{G}')$. To see this, first note that $ISP(\mathbf{G}') \subseteq \mathscr{C}_{\mathbf{G}}$.

Now, let $\mathbf{H} \in \mathscr{C}_{\mathbf{G}}$, witnessed by a homomorphism $f: \mathbf{H} \to \mathbf{G}$.

Let $I = \{(x, y): x, y \in H \text{ and } x \neq y\}$. For each $(x, y) \in I$, define a map $f_{xy}: \mathbf{H} \to \mathbf{G}'$ so that

$$f_{xy}(z) = \begin{cases} (0, f(z)) & \text{if } z \neq y \\ (1, f(z)) & \text{if } z = y \text{ or } z = x, \neg x E^{\mathbf{H}} y, \text{ and } f(x) \neq f(y). \end{cases}$$

Then f_{xy} is a homomorphism so that $f_{xy}(x) \neq f_{xy}(y)$, and if x and y are not adjacent in $\mathbf{H}$ then $f_{xy}(y)$ not adjacent in $\mathbf{G}'$. Let F be the product map $\Pi_{i \in I} f_{xy}: \mathbf{H} \to \mathbf{G}'^{I}$. Then F is an embedding. □

If $\mathbf{G}$ is infinite, the class of graphs that map homomorphically to $\mathbf{G}$ may not even be an elementary class. For example, consider $\mathbf{K}_{\aleph_0}$, the complete graph on $\aleph_0$ many vertices. If $\mathscr{C}_{\mathbf{K}_{\aleph_0}}$ were elementary, each ultrapower of $\mathbf{K}_{\aleph_0}$ would be in $\mathscr{C}_{\mathbf{K}_{\aleph_0}}$. However, every ultrapower of $\mathbf{K}_{\aleph_0}$ is a complete graph. This gives rise to a contradiction, since there is no homomorphism from $\mathbf{K}_{\lambda}$ to $\mathbf{K}_{\aleph_0}$, if $\lambda > \aleph_0$.

LEMMA 5. *Colour families, ordered by* $\subseteq$, *form a distributive lattice* $\mathscr{L}_{\mathscr{C}}$, *with* $\mathscr{C}_{\mathbf{G}} \vee \mathscr{C}_{\mathbf{H}} = \mathscr{C}_{\mathbf{G} \uplus \mathbf{H}}$ (*where* $\uplus$ *is disjoint union*) *and* $\mathscr{C}_{\mathbf{G}} \wedge \mathscr{C}_{\mathbf{H}} = \mathscr{C}_{\mathbf{G} \times \mathbf{H}}$ (*where* $\times$ *is the usual Cartesian product*). $\mathscr{L}_{\mathscr{C}}$ *is the* lattice of colour families.

Proof. Clearly, $\mathscr{C}_{\mathbf{G} \uplus \mathbf{H}}$, and $\mathscr{C}_{\mathbf{G} \times \mathbf{H}}$ are upper and lower bounds, respectively of $\mathscr{C}_{\mathbf{G}}$ and $\mathscr{C}_{\mathbf{H}}$. If $\mathscr{C}_{\mathbf{A}}$ is an upper bound of $\mathscr{C}_{\mathbf{G}}$ and $\mathscr{C}_{\mathbf{H}}$, for some finite graph $\mathbf{A}$, then $\mathbf{G}, \mathbf{H} \to \mathbf{A}$. Then $\mathbf{G} \uplus \mathbf{H} \to \mathbf{A}$, by taking the union of the two homomorphisms, so that $\mathscr{C}_{\mathbf{G} \uplus \mathbf{H}} \subseteq \mathscr{C}_{\mathbf{A}}$. Hence, $\mathscr{C}_{\mathbf{G} \uplus \mathbf{H}}$ is a least upper bound of $\mathscr{C}_{\mathbf{G}}$ and $\mathscr{C}_{\mathbf{H}}$. Similarly, if $\mathbf{A} \to \mathbf{G}, \mathbf{H}$, then $\mathbf{A} \to \mathbf{G} \times \mathbf{H}$, by considering the product of the two maps. Thus, $\mathscr{C}_{\mathbf{G} \times \mathbf{H}}$ is a greatest lower bound of $\mathscr{C}_{\mathbf{G}}$ and $\mathscr{C}_{\mathbf{H}}$. Distributivity follows since for finite graphs, $\mathbf{G}$, $\mathbf{H}_1$, and $\mathbf{H}_2$, $\mathbf{G} \times (\mathbf{H}_1 \uplus \mathbf{H}_2) = \mathbf{G} \times \mathbf{H}_1 \uplus \mathbf{G} \times \mathbf{H}_2$. □

PROPOSITION 6. *There are* $2^{\aleph_0}$ *many universal Horn classes of graphs of bounded chromatic number. Indeed, the conclusion holds for every interval in* $\mathscr{L}_{\mathscr{C}}$ *above* $\mathscr{C}_{\mathbf{K}_2}$.

Proof. The ordering of $\mathscr{L}_{\mathscr{C}}$ is dense above $\mathscr{C}_{\mathbf{K}_2}$ (which is a cover of $\mathscr{C}_{\mathbf{K}_1}$, the zero of $\mathscr{L}_{\mathscr{C}}$) (see [6]). Let $\mathbf{I}$ be an interval in $\mathscr{L}_{\mathscr{C}}$ above $\mathscr{C}_{\mathbf{K}_2}$. Let $\mathbf{I}'$ be a nonempty chain in $\mathbf{I}$ without endpoints. Then $\mathbf{I}'$ has the order type of $\mathbb{Q}$. $\mathscr{L}_{\mathscr{C}}$ is a suborder, with respect to $\subseteq$, of the complete lattice of all universal Horn classes of graphs, $\mathscr{L}_{\mathscr{U}\mathscr{H}}$. Hence, the corresponding Dedekind-MacNeille completion $DM(\mathbf{I}')$ (see, for example, [2]) of each such $\mathbf{I}'$ has the order type of $\mathbb{R} \cup \{-\infty, \infty\}$ in $\mathscr{L}_{\mathscr{U}\mathscr{H}}$. In this way, we obtain $2^{\aleph_0}$ many distinct universal Horn classes of graphs in $DM(\mathbf{I}') \subseteq \mathbf{I}$ as an interval of $\mathscr{L}_{\mathscr{U}\mathscr{H}}$. □

The proof of Proposition 6 generalizes to digraph colour families. There, the question of density of colour families is more problematic (see [5]). However, every interval above $\mathscr{C}_{\mathbf{K}_2}$ (where $\mathbf{K}_2$ is considered a digraph) is again dense ([5]), so an analogous conclusion as in Proposition 6 will apply there.

REMARK 7. Proposition 6 has also been proven (independent of the author's proof above) in [3].

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