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# The capture time of a graph

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A B S T R A C T

We consider the game of Cops and Robbers played on finite and countably infinite connected graphs. The length of games is considered on cop-win graphs, leading to a new parameter, the capture time of a graph. While the capture time of a cop-win graph on $n$ vertices is bounded above by $n-3$, half the number of vertices is sufficient for a large class of graphs including chordal graphs. Examples are given of cop-win graphs which have unique corners and have capture time within a small additive constant of the number of vertices. We consider the ratio of the capture time to the number of vertices, and extend this notion of capture time density to infinite graphs. For the infinite random graph, the capture time density can be any real number in $[0, 1]$. We also consider the capture time when more than one cop is required to win. While the capture time can be calculated by a polynomial algorithm if the number $k$ of cops is fixed, it is NP-complete to decide whether $k$ cops can capture the robber in no more than $t$ moves for every fixed $t$.



## 1. Introduction

*Cops and Robbers* is a vertex pursuit game played on a graph $G = (V, E)$ (we also use $V(G)$ and $E(G)$). All graphs we consider are simple, undirected, reflexive (that is, with a loop at each vertex) and countable (although usually finite) and we assume that they are connected since it is enough to consider connected components. We denote the (open) neighbourhood of a vertex $v \in V$ by $N(v)$ and write $N[v]$ for the closed neighbourhood of $v$, $N(v) \cup \{v\}$. Throughout this paper, $k$ and $\ell$ are positive integers. The game of cops-and-robbers with parameters $k$ and $\ell$ on a graph $G$ we consider has two teams of *players*, a set of $k$ *cops* $\mathcal{C}$ and a set of $\ell$ *robbers* $\mathcal{R}$. We think of the players (cops, robbers) as *occupying* vertices, or *being at* vertices; a vertex can be occupied by more than one player. Each player knows the others' current locations. The two teams play in *rounds*, each round consisting of two *moves* timed by a common clock, a first one by the cops, the second by the robbers. Each move takes one time unit and each player must move (some authors prefer to forgo the loops and allow the players to pass; the effect is the same, but reflexivity simplifies homomorphisms, which are essential to many proofs). At round 0, first the cops choose a set of at most $k$ vertices then the robbers choose at most $\ell$ vertices. At subsequent rounds, first each cop moves to a vertex adjacent to her current location, then each robber to a vertex adjacent to his. The cops win and the game ends if each robber is at a vertex occupied by some cop; otherwise, $\mathcal{R}$ wins. This general setting has not been studied much apart

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form a brief consideration in [13] and we shall revert to the usual setting of $\ell = 1$ for the rest of the paper. Throughout, $n$ is the number of vertices of the graph under discussion and $k$ is the number of cops.

As placing a cop on each vertex guarantees that the cops win, we may define the *cop number*,[1] denoted by $c(G)$, as the minimum number of cops that have a winning strategy on $G$. A *strategy* is a mapping $\sigma_p : V^k \times V \longrightarrow V$ which tells the player $p$ (either the robber, or the set of cops) what the next move is, based on the players' current positions. A strategy is *winning* if it allows the player to win, no matter what the opponent's moves are.

The graphs with cop number 1 are called *cop-win* and were first characterised in [17,18]; see the next section for more. There is no known structural characterisation of graphs with cop number 2 or higher. For a survey of results on the cop number and related search parameters for graphs, see [1,11]. For an annotated bibliography of cops-and-robbers literature, see [7].

If $c(G) = k$, how many moves does it take for the $k$ cops to win? To be more precise, say that the *length* of a game is $t$ if the robber is captured in $t$-th round. It is infinite if the robber can indefinitely evade capture. We say that a play of the game with $c(G)$ cops is *optimal* if its length is the minimum over all possible games, assuming the robber is trying to evade capture for as long as possible. Observe that if a game is optimal, then the robber always gets caught on the cops' move. There may be many optimal plays possible (for example, on $P_4$, one cop may start on either vertex of the centre), but the length of an optimal game is an invariant of $G$. We denote this invariant $\mathrm{capt}_k(G)$, and call it the *$k$-capture time* of $G$. For $k = 1$, we write $\mathrm{capt}(G)$ instead of $\mathrm{capt}_1(G)$. The $k$-capture time of $G$ may be viewed as the temporal counterpart of the cop number. The concept is in part motivated by the fact that in real-world networks with limited resources, not only the number of cops, but the time it takes to capture the robber on the network is of practical importance.

As first noted in [2],

$$\mathrm{capt}(G) \leq |V(G)|^{c(G)+1}.$$

This upper bound is far from sharp in general; for example, it is not hard to see that the capture time of a tree on $n$ vertices is at most $\left\lfloor \frac{n}{2} \right\rfloor$. In this paper we focus primarily on the capture time of cop-win graphs as they are better understood. For cop-win graphs we prove, in Theorem 2, that $\mathrm{capt}(G) \leq |V(G)| - 3$ if $|V(G)| \geq 5$ and in Theorem 3, we show that the upper bound of $\left\lfloor \frac{|V(G)|}{2} \right\rfloor$ applies to a large class of graphs including connected chordal graphs. Perhaps surprisingly, there are cop-win graphs of order $n$ whose capture time is within an additive constant of $n$. These graphs exhibit the interesting structural property of possessing a unique corner (that is, a vertex whose closed neighbourhood is contained in the closed neighbourhood of some other vertex). Using the results of [13], we show in Section 2 that if the number of cops $k$ is fixed, then it is polynomial time computable to determine if $\mathrm{capt}(G) \leq t$, for a given $t$. We consider the capture time density for the infinite random graph; see Theorem 6. In the final section, we consider the case when more than one cop is allowed. We introduce the parameter $c_t(G)$ which is the smallest number of cops needed for capturing the robber in no more than $t$ steps. We prove in Theorem 7 that it is NP-complete to determine if $c_t(G) \leq k$ if $t$ is fixed, even if $G$ is planar or chordal.

## 2. Cop-win graphs

Before proceeding, it is useful to introduce some notation. Consider a graph $G$ whose vertex set is $\{v_1, v_2, \ldots, v_n\}$. For $i = 1, \ldots, n$, define $G_i$ to be the subgraph induced by $\{v_i, \ldots, v_n\}$; thus $G_1 = G$. Similarly, for a vertex $v_j$, define $N_i[v_j] = N[v_j] \cap \{v_i, \ldots, v_n\}$. It is shown in [17,18] that the finite cop-win graphs are exactly those graphs that have a *dismantling ordering*; that is, a linear ordering $(x_j : 1 \leq j \leq n)$ of the vertices so that for each $1 \leq i \leq n$, there is a $i < j \leq n$ such that $N_i[x_i] \subseteq N_i[x_j]$. The characterisation comes from the observation that the robber's last move must be from a vertex whose neighbourhood is contained in that of the cop's current vertex, together with the fact that a retract of a cop-win graph is cop-win. From the characterisation we can naturally define a *corner* of a graph $G$ as a vertex $u \in V(G)$ such that $N[u] \subseteq N[v]$ for some $v \in V(G),\ v \neq u$. We say in this case that $u$ is *dominated* by $v$ and that $v$ *dominates* $u$. Observe that if $G$ is cop-win and $u$ is a corner in $G$, then $G - u$ is cop-win since it is a retract of $G$.

We pause to prove the claim on retracts since both the result and the argument (mutatis mutandis) are used several times in what follows. Recall that a *homomorphism* from a graph $G$ to a graph $H$ is an edge-preserving mapping $h : V(G) \longrightarrow V(H)$ (that is, an edge must be mapped onto an edge). A *retraction* of a graph $G$ is homomorphism $\rho : V(G) \longrightarrow V(G)$ which is the identity on its image. A *retract* of $G$ is the image of a retraction. See [15] or [14] for more.

**Theorem 1** (*Nowakowski and Winkler, Quilliot*). *A retract of a cop-win graph is cop-win.*

**Proof.** Let $G$ be a cop-win graph and let $H$ be a retract of $G$, with a retraction $\rho$. We describe a winning strategy for the cop on $H$. Since the cop has a winning strategy on $G$, she can use it as long as both players are on $H$ since $H$ is an induced subgraph of $G$. If the strategy requires that the cop move to a vertex $u \notin V(H)$, the move is to $\rho(u)$. Thus, the cop captures the robber at the latest when she would have captured him on $G$. □

[1] The cop number is also referred to more recently as the *search number*, denoted by $sn(G)$. The latter name better differentiates between our discrete version — *searching a graph* — and the also studied continuous version, *sweeping a graph*. We use "cop number" and $c(G)$ here for primarily historical reasons.

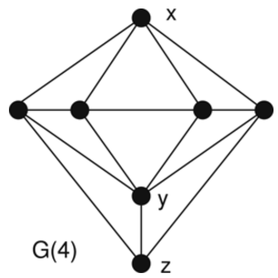


**Fig. 1.** The graph $G(4)$.

The parameter $\mathrm{capt}_k(G)$ was first considered (not with this notation) in [12] in an analysis of lengths of games for chordal graphs. For more general cop-win graphs, we have the following upper bound.

**Theorem 2.** *If $G$ is cop-win then* $\mathrm{capt}(G) \leq |V(G)| - 3$, *provided that* $|V(G)| \geq 5$.

**Proof.** As $\mathrm{capt}(G)$ is finite, the upper bound is trivial in the case when $G$ is infinite. For $G$ finite, the proof is by induction on $|V(G)|$, with the case $|V(G)| = 5$ following by a direct check of all cop-win graphs of order 5. Assume that the theorem holds for graphs with $n \geq 5$ vertices and consider a cop-win graph $G$ with $n + 1$ vertices. Hence, $G$ contains a corner and, since $G - u$ is a retract of $G$, it is cop-win.

Each optimal game on $G - u$ has length $n - 3$ and the cop has a strategy to win it. The cop plays her winning strategy on $G - u$ and whenever $\mathcal{R}$ is on $u$, $\mathcal{C}$ plays as if he were on $v$. After at most $n - 3$ moves, either the robber is caught on $G - u$, or $\mathcal{R}$ is on $u$ and $\mathcal{C}$ is on $v$. But then $\mathcal{C}$ can win in one more move, and so this strategy uses at most $n - 2 = (n + 1) - 3$ moves. As $\mathrm{capt}(G)$ is the length of an optimal game, we have that $\mathrm{capt}(G) \leq (n + 1) - 3$. □

This improves the result of [5] (Theorem1.2.3) that if $G$ is cop-win, then $\mathrm{capt}(G) \leq |V(G)| - 1$. For many graphs (such as trees) the bound in Theorem 2 is not sharp. We introduce a new graph class where the bound for trees applies.

Two corners $a, b$ in a cop-win graph $G$ are *separate* if neither is dominated only by the other. Observe that $a$ is a corner in $G - u$ and $b$ is one in $G - a$; this means that the dismantling order can begin (without loss of generality) with $(a, b, \ldots)$. We say that a graph $G$ is 2-*dismantlable* if it is cop-win and if it has two separate corners $a, b$ and if $G - \{a, b\}$ either has two separate corners or has fewer than 7 vertices. Note that in either case, $G - \{a, b\}$ is cop-win.

Thus, deleting two corners from a 2-dismantlable graph with at least eight vertices leaves another 2-dismantlable graph. Each connected chordal graph is 2-dismantlable (as chordal graphs contain at least two simplicial vertices). However, the 4-wheel ($C_4$ plus a universal vertex) is 2-dismantlable but not chordal.

**Theorem 3.** *If $G$ is a finite 2-dismantlable graph, then* $\mathrm{capt}(G) \leq \left\lfloor \frac{|V(G)|}{2} \right\rfloor$.

**Proof.** Let $n$ be the number of vertices of the 2-dismantlable graph $G$ and let $u_0, u_1$ be two separate corners of $G$. We induct on $n$. The statement is clearly true for $n \leq 6$, by Theorem 2, so assume that $n \geq 7$. For $i = 0, 1$, let $u_i' \neq u_{(i+1)(\mathrm{mod}\ 2)}$ be a vertex that dominates $u_i$ in $G$ (and, hence, in $G - u_{(i+1)(\mathrm{mod}\ 2)}$). Consider now $H = G - \{u_0, u_1\}$, a retract of $G$. By assumption, the cop has a winning strategy on $H$ that gives $\mathrm{capt}(H) \leq \lfloor \frac{n-2}{2} \rfloor$. To win on $G$, the cop stays on the retract $H$, playing the strategy as long as the robber stays on $H$. If the robber is on $u_i$, $i \in \{0, 1\}$, the cop plays as if the robber were on $u_i'$. After $\lfloor \frac{n-2}{2} \rfloor$ rounds at most, the cop either occupies the same vertex of $H$ as the robber, or is at $u_i'$ while the robber is at $u_i$. One more move by the cop finishes the game in at most $\left\lfloor \frac{n-2}{2} \right\rfloor + 1 = \left\lfloor \frac{n}{2} \right\rfloor$ moves. □

Not every cop-win graph has two corners. For an integer $n \geq 4$, define $G(n)$ by adjoining two vertices $x$ and $y$ joined to each vertex of a path $P$ with $n$ vertices. Add a vertex $z$ that is joined to $y$ and the endpoints of $P$. Then $G(n)$ is cop-win but $z$ is the unique corner of $G(n)$. See Fig. 1.

Using the graph $G(4)$ as a template, we construct an infinite family of graphs whose capture time is within an additive constant of the upper bound $|V(G)|$ of Theorem 2. For $n \geq 7$, the graph $H(n)$ has vertices $1, \ldots, n$, where $1, 2, 3, 4, 5, 6, 7$ induce $G(4)$ (so that $x = 5$, $y = 3$, $z = 7$, and the remaining vertices on the path joined to $x$ and $y$ are (from left to right) $6, 2, 1, 4$.) For $i > 7$, the vertex $i$ is joined to $j < i$ if $j$ equals one of $i - 4$, $i - 3$, and $i - 1$. We name the vertices $7, 8, \ldots, n$ *special*. See Fig. 2 for $H(11)$.

**Theorem 4.** *For a fixed integer $n \geq 7$, the graphs $H(n)$ have the following properties.*

(1) *The graph $H(n)$ is planar.*
(2) *The graph $H(n)$ is cop-win and has a unique corner.*
(3) $\mathrm{capt}(H(n)) = n - 4$.

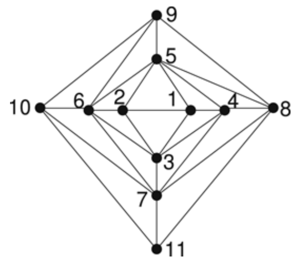


**Fig. 2.** The graph $H(11)$.

**Proof.** To see item (1), we describe a planar drawing of $H(n)$. The rough idea is to spiral the special vertices around $G(4)$ in an anti-clockwise fashion. To be more precise, first draw $G(4)$ as in Fig. 1. Embed this drawing of $G(4)$ in the unit square in any fixed way, so that vertex 4 has coordinates $(1, 0)$, 5 has coordinates $(1, 0)$, 6 has coordinates $(-1, 0)$, and 7 has coordinates $(0, -1)$.

For each vertex $i$, $i \geq 8$, write $i$ as $4r + s$, where $r \geq 2$ and $s$ is one of 0, 1, 2, 3. Place $4r + s$ at coordinate $(r, 0)$ if $s = 0$, at $(0, r)$ if $s = 1$, at $(-r, 0)$ if $s = 2$, and $(0, -r)$ if $s = 3$. Hence, we place the special vertices $u$ around $G(4)$, so $u$ is on the positive or negative arms of the $x$- or $y$-axes depending on the residue of $x(\text{mod } 4)$. It is easy to see that this is a planar drawing of $H(n)$.

For item (2), note that the graph $H(n)$ is cop-win, since we may dismantle the special vertices in reverse order $n, n-1, \ldots, 2, 1$. By a straightforward (and so omitted) induction argument, $n$ is the unique corner of $H(n)$.

For item (3), we present a strategy $\mathcal{S}$ for the cop to win which always results in a game of length at most $n - 4$. First note that each vertex $5 \leq x \leq n - 4$ has neighbours exactly $\{x - 4, x - 3, x - 1, x + 1, x + 3, x + 4\}$. So the cop and robber may move to vertices with index 1, 3 or 4 more or less than their current index.

The strategy $\mathcal{S}$ has three parts, with the third part repeated until the robber is captured (which we will demonstrate eventually happens).

(S1) In the 0-th round, place the cop on vertex 1.
(S2) After robber places himself on $i > 1$ in the 0-th round, in the first round the cop moves to $j \in V(G(4))$ with $j \in \{2, 3, 4, 5\}$ so that such that $i \equiv j(\text{mod } 4)$.
(S3) Repeat the following steps until the robber is eventually caught.
  (a) If the robber moves from $i$ to $i + k$, then the cop moves from $j$ to $j + k$, where $k = 1, 3$, or 4.
  (b) If the robber moves from $i$ to $i - 1$, then the cop moves from $j$ to $j + 3$.
  (c) If the robber moves from $i$ to $i - 3$, then the cop moves from $j$ to $j + 1$.
  (d) If the robber moves from $i$ to $i - 4$, then the cop moves from $j$ to $j + 4$.

Let the cop play with $\mathcal{S}$, and let $\text{cop}(t)$ and $\text{robber}(t)$ be the positions of the cop and robber after round $t$ in this game. Note that for all $t \geq 0$, $\text{cop}(t + 1) > \text{cop}(t)$. We prove by induction that for all $t \geq 1$,

$$\text{cop}(t) \equiv \text{robber}(t - 1)(\text{mod } 4). \tag{2.1}$$

The base case of (2.1) follows by $(S1)$ and $(S2)$. Suppose that (2.1) holds for a fixed $t \geq 1$. Suppose that $\text{cop}(t) = j$, with $\text{robber}(t - 1) = i$. At time $t$, the robber moves to $i + m$, where $m \in \{-4, -3, -1, 1, 3, 4\}$.

Then the cop moves at round $t + 1$ to $j + m'$ for some $m'$ using $(S3)$. It is straightforward to check that $i + m \equiv j + m'(\text{mod } 4)$ holds for all possible moves of the robber. Hence, the induction step follows.

It follows that the difference of the indices of the cop and robber is kept $0(\text{mod } 4)$, and when the robber goes to a higher or lower index, the difference is monotonically decreasing. Over time the cop gets closer and closer to the robber. Note that for all rounds except for the last one where the robber is captured, $\text{cop}(t) < \text{robber}(t)$. Eventually $\text{robber}(t - 1) = i$ and $\text{cop}(t) = j$ so that $i = j + 4$. In that case, the robber can only evade capture by moving to $i + 1$ (which is not joined to $j$). The cop then moves to $j + 5$ following $(S3a)$. The index of the robber will then eventually increase to $n$, with the robber in the unique corner $n$ of $H(n)$, with the cop in $n - 4$ (by repeated applications of $(S3a)$). The cop will then capture the robber in the next move.

To prove that $\text{capt}(H(n)) = n - 4$, we need to show that the robber can survive this many steps against any strategy of the cop (including $\mathcal{S}$). To see this, note that the robber may use the lowest indexed vertex that is available. In round 0, the robber may be placed on a vertex $1, \ldots, 6$, since no vertex of $H(n)$ dominates all six of these vertices. In round $t \geq 0$, if $i = \text{robber}(t - 1) \in \{1, 2, 3, 4\}$, then in such a case the robber can always move to a vertex of label at most 7 (we leave the verification to the reader). If $5 \leq i \leq n - 1$, then the robber's moves are given by the following table.

| $\text{cop}(t)$ | $\text{robber}(t)$ |
|---|---|
| $i - 4$ | $i + 1$ |
| $i - 3$ | $i - 1$ |
| $i - 1$ | $i - 3$ |
| $i + 1$ | $i - 4$ |
| $i + 3$ | $i - 4$ |
| $i + 4$ | $i - 4$ |

Hence, robber$(t) \leq 6 + t$ for every $t \geq 1$, and so the robber cannot be caught in fewer than $n - 4$ moves of the cop. □

Recently (2007), Gavenčiak [9] looked (using a computer) closely at the graphs $H(n)$ defined earlier and characterised cop-win graphs $G$ on $n$ vertices such that $\text{capt}(G) = n - 4$. Further, for each $n > 7$, he showed that $\text{capt}(G) \leq n - 4$ and found exponentially many graphs for which equality holds. It is an open problem to characterise graphs, called *k-cop-win*, on which $k \geq 2$ cops are necessary and sufficient to catch a robber. Almost nothing is known about directed graphs and cop-and-robber games. In an attempt to close these knowledge gaps, Hahn and MacGillivray [13] provide an algorithmic characterisation whose byproduct is a polynomial (in the number of vertices) time algorithm for determining if $\text{capt}_k(G) \leq t$ for a given (di)graph $G$ and a given $t$. It is based (in retrospect) on the characterisation of cop-win graphs given in [17] that also applies to infinite graphs. The idea is to assign an ordinal to each pair of vertices $(u, v)$, indicating how many rounds the cop at $u$ needs to win if the robber is at $v$.

Let $G$ be a graph (finite or infinite). Define recursively a sequence of order relations $\preceq_\alpha$ indexed by ordinals $\alpha$ as follows.

- $u \preceq_0 u$ for all $u \in V(G)$;
- $u \preceq_\alpha v$ if for each $x \in N[u]$ there is a $y \in N[v]$ and a $\beta < \alpha$ such that $x \preceq_\beta y$.

Since $\preceq_\beta \subseteq \preceq_\alpha$ whenever $\beta \leq \alpha$, and, clearly, $\preceq_{|V(G)|+1} = \preceq_{|V(G)|}$, there is a least $\alpha$ such that $\preceq_\gamma = \preceq_\alpha$ for all $\gamma \geq \alpha$. Set $\preceq = \preceq_\alpha$.

**Theorem 5** *([17]). A graph $G$ is cop-win if and only if the relation $\preceq$ is trivial, that is, $u \preceq v$ for all $u, v \in V(G)$*

In [13], an algorithm based on this idea is developed that decides, in time polynomial in the number of vertices of the (obviously finite) input graph $G$, whether or not the graph is $k$-cop-win ($k$ fixed). From the labelling (really by the appropriate $\beta$) of the *vertices* one can read off the length of an optimal game. The problem of deciding whether the capture time of a graph is at most $t$ is, therefore, polynomial in the number of vertices of the graph, provided the number $k$ of cops is fixed.

One absence that motivates the present paper is that of a good (that is, achievable) bound on the capture time in terms of some known graph parameters. For example, the diameter might seem a likely candidate, but it is shown in [12] that for every natural number $k$ there is a chordal, diameter 2 finite cop-win graph with capture time $k$ (and, by compactness, that there are chordal, diameter 2 infinite graphs which are not cop-win). Other parameters do not bound the capture time, such as the length of a longest path (consider the complete graph), or the length of a longest chordless path (consider a graph obtained from a path by adding a new universal vertex). One might hope to get a bound in terms of the least $l$ such that $v_l, \ldots, v_n$ induce a complete graph in a finite cop-win graph $G$ with an enumeration of its vertices as guaranteed by the structural characterisation theorem, but the last-mentioned counterexample works here as well. The question is as interesting — and as open — for infinite graphs. Let $S$ be the graph obtained from $(\mathbb{N}, \{0i : 0 \neq i \in \mathbb{N}\})$ by replacing each edge $0i$ by a path $0v_i^1v_i^2 \ldots v_i^i$ of length $i$; this is a star with 0 as its centre and paths of length $i$ (one for each $i > 0$) as rays. Clearly the robber determines the length of the game by his choice of his starting vertex. Let $SP$ be the graph obtained from the ray (one-way infinite path) $v_0v_1v_2 \ldots$ by the addition of a new vertex $v$ adjacent to all the vertices of the ray (call $SDP$ the graph obtained from the double ray by the same method). The two graphs $SP$ and $SDP$ provide counterexamples to hypotheses as to the bounds on the number of moves in terms of the diameter or of the length of a longest chordless path.

## 3. Capture time density and infinite graphs

In this section we introduce a new parameter which measures limits of the ratio of the capture time to the number of vertices over chains of induced subgraphs. A similar approach was given in [3] for the cop number.

As proven by Erdős and Rényi [6], with probability 1, a countably infinite random graph has a unique isomorphism type written $R$. The (deterministic) graph $R$ is called the *infinite random graph* or the *Rado* graph (see [4] for more on the random graph). The graph $R$ is the unique isomorphism type of countable graph satisfying the *e.c. property*: for all finite disjoint sets of vertices $A, B$, there is a vertex $z \notin (A \cup B)$ such that $z$ is joined to each vertex of $A$ and to no vertex of $B$. From the results of [3], $c(R) = \aleph_0$; that is, $R$ is not $k$-cop-win for any finite $k$. By [6], it follows that almost no countably infinite graphs are $k$-cop-win for a finite $k$.

Similar to [3] we consider the density of the capture time parameter, relative to the number of vertices. A *chain of graphs* is a sequence $(G_n : n \in \mathbb{N})$, each $G_n$ is an induced subgraph of $G_{n+1}$, for all $n \in \mathbb{N}$. Given a chain $(G_n : n \in \mathbb{N})$ of induced subgraphs of $G$, we write $G = \lim_{n\to\infty} G_n$ if $V(G) = \bigcup_{n\in\mathbb{N}} V(G_n)$ and $E(G) = \bigcup_{n\in\mathbb{N}} E(G_n)$. Note that every countable graph $G$ is the limit of a chain of finite graphs, and there are infinitely many distinct chains with limit $G$. Suppose that $G = \lim_{n\to\infty} G_n$, for a fixed chain $C = (G_n : n \in \mathbb{N})$ of induced subgraphs of $G$. We say that $C$ is a *full chain for G*. For $G$ a finite graph, define the *capture time density* by

$$D_{st}(G) = \frac{\text{capt}(G)}{|V(G)|}.$$

By Theorem 2, $D_{st}(G)$ is a rational number in $[0, 1)$. We may extend the definition of $D_{st}$ as follows. For a full chain $C = (G_n : n \in \mathbb{N})$ in $G$, define

$$D_{st}(G, C) = \lim_{n\to\infty} D_{st}(G_n).$$

We refer to this as the *capture time density of* $G$ *relative to* $C$. For simplicity, we will always consider $G$ and $C$ where this limit exists. Note $D_{st}(G, C)$ is a real number $[0, 1]$.

For example, let $C$ be the chain with $G_n$ isomorphic to $P_n$, and so the limit graph $G$ is the infinite one-way path. In this case, $D_{st}(G, C) = \frac{1}{2}$. If we let our chain consist of the graphs $H(n)$, for $n \geq 7$ (where $H(n)$ is embedded in obvious way in $H(n+1)$), then the capture time density of the limit graph relative to this chain is 1.

For the infinite random graph, we obtain the surprising result that the capture time density can be any real number in $[0, 1]$.

**Theorem 6.** *For all* $r \in [0, 1]$, *there is a full chain* $C$ *in* $R$ *such that* $D_{st}(G, R) = r$.

**Proof.** Let $(p_n : n \in \mathbb{N})$ be a sequence of rationals such that $p_n \in [0, 1)$ if $n \geq 1$, $p_0 = 0$, and $\lim_{n\to\infty} p_n = r$. We construct a chain $C = (G_n : n \in \mathbb{N})$ in $G = R$ such that $G = \lim_{n\to\infty} G_n$, and with the property that $D_{st}(G) = p_n$, and each $G_n$ is cop-win. Without loss of generality, assume that $V(G) = \{x_n : n \in \mathbb{N}\}$.

We proceed inductively on $n$. For $n = 0$, let $G_0$ be the subgraph induced by $x_0$. Then $D_{st}(G_0) = 0 = p_0$.

Consider an $n \geq 1$, and suppose that the induction hypothesis holds for all $k \leq n$,. Let $p_{n+1} = \frac{a}{b}$, where $a, b$ are positive integers. Further suppose for an inductive hypothesis that $\{x_0, \ldots, x_n\} \subseteq V(G_n)$. Without loss of generality, we may assume that $a < b$ and that $\gcd(a, b) = 1$.

We add vertices to $G_n$ in several ways. First, if necessary, add $x_{n+1}$ to $G_n$, to form the graph $G''_{n+1}$. Next, form $G'_{n+1}$ by adding a universal vertex $u$ ($u$ exists by the e.c. property). Then $D_{st}(G'_{n+1}) = 1$, and say that $|V(G'_{n+1})| = b'$, where $b'$ is a positive integer. If $\frac{1}{b'} = \frac{a}{b}$, then let $G_{n+1} = G'_{n+1}$. Otherwise, we add some new vertices to adjust the parameter $D_{st}(G'_{n+1})$.

We may add an arbitrary finite number $y$ of vertices of degree one adjacent to $u$. For some $x \geq 7$, by the e.c. property add a copy of $H(x)$ by identifying the vertex $u$ with 1 and keeping all other vertices disjoint from the existing ones. This gives the graph $G_{n+1}$, with $x \geq 5$ and $y$ to be determined. The graph $G_{n+1}$ is cop-win, since $H(x)$ can be dismantled to $1 = u$, and all of $G'_{n+1}$ can be dismantled to 1.

We claim that $\mathrm{capt}(G_{n+1}) = x - 4$. To see this place the cop first at 1. If the robber is in $G'_{n+1}$, then he is caught in the next round. If the robber is in $H(x)$, then play $\mathcal{S}$ as in the proof of Theorem 4. The optimal play for the robber is then also is as in the proof of Theorem 4. As the robber is always on a vertex $i \geq 6$, by properties of $\mathcal{S}$ he can never "escape" to 1 and $G'_{n+1}$ without being captured in less than $x - 4$ moves. Note that the $y$ vertices of degree one adjacent to 1 do not increase the capture time. In this way, the robber can play so that the game takes $x - 4$ moves.

Thus,

$$D_{st}(G_{n+1}) = \frac{x-4}{b' + x + y} = \frac{a}{b} \tag{3.1}$$

where $x, y$ are positive integers. We must find positive integer solutions $(x, y)$ with $x \geq 5$ to the Diophantine equation

$$(b - a)x - ay = b'a + 4b.$$

As $\gcd(b - a, -a) = \gcd(a, b)$ and $b - a$ and $-a$ are opposite signs, we may find infinitely many of the desired $(x, y)$. This completes the induction step in constructing $G_{n+1}$.

As $\{x_0, \ldots, x_n\} \subseteq V(G_n)$ for all $n \in \mathbb{N}$, we have that $C = (G_n : n \in \mathbb{N})$ is a full chain for $G$. Further,

$$D(G, C) = \lim_{n\to\infty} p_n = r. \quad \square$$

## 4. More than one cop

The general situation for graphs with cop number greater than 1 appears to be more complex. For example if $n \geq 3$, then it can be shown that

$$\mathrm{capt}_2(C_n) = \begin{cases} \dfrac{n-i}{4} & \text{if } n \equiv i \pmod 4, \quad i = 0, 1, 2; \\ \dfrac{n+1}{4} & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

To study the case for more than one cop, we introduce another parameter related to the lengths of games. If $G$ is a graph, then let $c_t(G)$ be the smallest number of cops needed to capture of robber in no more than $t$ rounds. Then $c_0(G) = |V(G)|$, and $c_1(G)$ is the domination number of $G$.

The following result contrasts with the complexity results shown in Section 2 for the capture time. Consider the following problem.

BOUNDED-TIME COP NUMBER

INSTANCE: A graph $G$ and a nonnegative integer $k$.

QUESTION: Is $c_t(G) \leq k$?

**Theorem 7.** *For each positive integer* $t$, *the* BOUNDED-TIME COP NUMBER *problem is NP-complete.*

**Proof.** The problem is in NP since for a given strategy of cops we can construct a directed rooted tree of all possible moves of the robber (where the first move is a choice of the initial position) and corresponding moves of the cops. For every vertex of this tree the current positions of the cops and robber are stored (only initial positions of cops are stored for the root). This tree is a rooted tree of restricted size (out-degrees of vertices are restricted by $|V(G)|$, and distance from the root to leaves is no more than $t$). Hence, given the tree it may be checked in polynomial time whether the strategy of cops is winning.

For the NP-completeness, we reduce the SATISFIABILITY problem (see [8]) to our problem. Let $C$ be a boolean formula in conjunctive normal form with variables $x_1, x_2, \dots, x_k$ and clauses $C_1, C_2, \dots, C_m$. From the formula $C$ we construct the graph $G(C)$ as follows. For every variable $x_i$, introduce vertices $x_i$ and $\overline{x_i}$ which are connected by an edge. A path $P_t$ with one endpoint $u_i$ is added, and vertex $u_i$ is joined by edges with $x_i$ and $\overline{x_i}$. For every clause $C_j$ add a path $P_t$ with endpoints $y_j$ and $z_j$. For every literal from $C_j$, join the vertex $y_j$ to $x_i$ if this literal contains a positive occurrence of this variable, and to $\overline{x_i}$ if this literal contains a negative occurrence.

Suppose that $k$ cops can capture the robber in no more than $t$ steps. Then for every $i = 1, 2, \dots, k$, the cops must be placed on either $x_i$ or $\overline{x_i}$, or on one of the vertices of the path with endpoint $u_i$. If a cop occupies $x_i$, then let $x_i =$ true, and $x_i =$ false if a cop occupies $\overline{x_i}$. If a cop is placed on any other vertex, then the value of $x_i$ is chosen arbitrarily. If some clause $C_j$ is not satisfied by this truth assignment, then clearly the robber can occupy the vertex $z_i$ and avoid capture in the first $t$ rounds. Conversely, suppose that variables $x_1, x_2, \dots, x_k$ have a truth assignment for which $C$ has value true. If $x_i =$ true, then at the beginning of the game we place a cop on vertex $x_i$, otherwise a cop is placed on $\overline{x_i}$ for $i = 1, 2, \dots, k$. Since each clause contains a positive literal, each $y_i$ is joined to a vertex with a cop. The strategy for the cops to win is now clear and captures the robber in at most $t$ rounds, no matter what his strategy is. □

The BOUNDED-TIME COP NUMBER remains NP-complete for planar graphs and for chordal graphs. The proof of the NP-complexity of the problem for chordal graphs follows easily from the proof of our theorem. Indeed, we only need to connect all the vertices $x_i$ and all the $\bar{x}_i$ for $i = 1, 2, \dots, k$ in our construction (that is, we construct a clique with these vertices). Clearly, the graph $G(C)$ becomes chordal after this operation. By the same arguments as were used above, $k$ cops can capture the robber in no more than $t$ steps if and only if $C$ can be satisfied. For planar graphs, we need a bit more preliminary work.

Let $C$ be a boolean formula in conjunctive normal form with variables $x_1, x_2, \dots, x_k$. Define $H(C)$ to be the bipartite graph with vertices $x_1, x_2, \dots, x_k$ and $C_1, C_2, \dots, C_m$ such that $x_i$ and $C_j$ are joined if and only if clause $C_j$ contains the literal $x_i$ or $\overline{x_i}$. It is known (see [16]) that SATISFIABILITY (and 3-SATISFIABILITY) remains NP-complete if the graph $H(C)$ is planar and every variable occurs in no more than four clauses. We require the following lemma.

**Lemma 8.** SATISFIABILITY *remains NP-complete even when $H(C)$ has a plane embedding such that if clauses $C_r$ and $C_{r'}$ contain $x_i$, and $C_s$ and $C_{s'}$ contain $\overline{x_i}$, then the edges $x_iC_r$ and $x_iC_{r'}$ are edges in the boundary of one face.*

**Proof.** Consider a fixed planar embedding of $H(C)$. Suppose that the condition of the claim is not fulfilled for a fixed vertex $x_i$. We add a new variable $x_i'$ and replace $x_i$ by $x_i'$ in $C_{r'}$ and $C_{s'}$. Then clauses $x_i \vee \overline{x'}$ and $\overline{x_i} \vee x_i'$ are added. Let $C'$ be the resulting boolean formula. It can be easily seen that the given embedding of $H(C)$ can be replaced by a planar embedding of $H(C')$, for which the condition of the claim for $x_i$ and $x_i'$ is satisfied, without violating the condition for the other vertices. Since $(x_i \vee \overline{x'}) \wedge (\overline{x_i} \vee x_i') =$ true if and only if variables $x_i$ and $x_i'$ have same values, the formula $C$ can be satisfied if and only if $C'$ can be satisfied. □

It is not difficult to see that if $H(C)$ satisfies the conditions of Lemma 8, then $G(C)$ is planar. Thus, the BOUNDED-TIME COP NUMBER problem remains NP-complete when $G(C)$ is planar.

Theorem 7 contrasts with the complexity of computation of the cop number (see [2,10,13]) and for capture time for both chordal and planar graphs.


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