Bounds on the Burning Number

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Abstract

Motivated by a graph theoretic process intended to measure the speed of the spread of contagion in a graph, Bonato, Janssen, and Roshanbin [Burning a Graph as a Model of Social Contagion, Lecture Notes in Computer Science 8882 (2014) 13-22] define the burning number $b(G)$ of a graph G as the smallest integer k for which there are vertices $x_1, \ldots, x_k$ such that for every vertex u of G, there is some $i \in \{1, \ldots, k\}$ with $\text{dist}_G(u, x_i) \leq k - i$, and $\text{dist}_G(x_i, x_j) \geq j - i$ for every $i, j \in \{1, \ldots, k\}$.

For a connected graph G of order n, they prove that $b(G) \leq 2\lceil\sqrt{n}\rceil - 1$, and conjecture $b(G) \leq \lceil\sqrt{n}\rceil$. We show that $b(G) \leq \sqrt{\frac{32}{19} \cdot \frac{n}{1-\epsilon}} + \sqrt{\frac{27}{19\epsilon}}$ and $b(G) \leq \sqrt{\frac{12n}{7}} + 3 \approx 1.309\sqrt{n} + 3$ for every connected graph G of order n and every $0 < \epsilon < 1$. For a tree T of order n with n_2 vertices of degree 2, and $n_{\geq 3}$ vertices of degree at least 3, we show $b(T) \leq \left\lceil \sqrt{(n + n_2) + \frac{1}{4}} + \frac{1}{2} \right\rceil$ and $b(T) \leq \lceil\sqrt{n}\rceil + n_{\geq 3}$. Furthermore, we characterize the binary trees of depth r that have burning number $r + 1$.

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1 Introduction

Motivated by a graph theoretic process intended to measure the speed of the spread of contagion in a graph, Bonato, Janssen, and Roshanbin [3, 4] define a *burning sequence* of a graph G as a sequence $(x_1, \ldots, x_k)$ of vertices of G such that

$$\forall u \in V(G) : \exists i \in [k] : \quad \text{dist}_G(u, x_i) \leq \quad k - i \text{ and} \tag{1}$$

$$\forall i, j \in [k] : \quad \text{dist}_G(x_i, x_j) \geq \quad j - i, \tag{2}$$

where $[k]$ denotes the set of the positive integers at most k. Furthermore, they define the *burning number* $b(G)$ of G as the length of a shortest burning sequence of G.

A burning sequence is supposed to model the expansion of a fire within a graph: At each discrete time step, first a new fire starts at a vertex that is not already burning, and then the fire spreads from burning vertices to all their neighbors that are not already burning. Condition (1) ensures that putting fire to the vertices of a burning sequence $(x_1, \ldots, x_k)$ in the order $x_1, \ldots, x_k$, all vertices of G are burning after k steps. Condition (2) ensures that one never puts fire to a vertex that is already burning.

We consider only finite, simple, and undirected graphs, and use standard terminology and notation [6]. For a graph G, a vertex u of G, and an integer k, let $N_G^k[u] = \{v \in V(G) : \text{dist}_G(u, v) \leq k\}$. Note that $N_G^0[u] = \{u\}$ and $N_G^1[u] = N_G[u] = \{u\} \cup N_G(u)$.

With this notation (1) is equivalent to

$$V(G) \quad = \quad N_G^{k-1}[x_1] \cup N_G^{k-2}[x_2] \cup \cdots \cup N_G^0[x_k]. \tag{3}$$

As previously said, condition (2) is motivated by the considered graph process, which in each step puts fire to a vertex that is not already burning. Our first result is that condition (2) is redundant.

Lemma 1 *The burning number of a graph G is the minimum length of a sequence $(x_1, \ldots, x_k)$ of vertices of G satisfying (3).*

Proof: Let k be the minimum length of a sequence satisfying (3). By definition, $b(G) \geq k$. It remains to show equality. For a contradiction, suppose $b(G) > k$. Let the sequence $s = (x_1, \ldots, x_k)$ be chosen such that (3) holds, and $j(s) = \min\{j \in [k] : \text{dist}_G(x_i, x_j) < j - i \text{ for some } i \in [j-1]\}$ is as large as possible. Since $b(G) > k$, the index $j(s)$ is well defined. Let $i(s) \in [j(s) - 1]$ be such that $\text{dist}_G(x_{i(s)}, x_{j(s)}) < j(s) - i(s)$. Since $k > j(s) - 1$, there is a vertex y in

$$V(G) \setminus \left(N_G^{(j(s)-1)-1}[x_1] \cup N_G^{(j(s)-1)-2}[x_2] \cup \cdots \cup N_G^0[x_{j(s)-1}]\right).$$

Since $N_G^{k-j(s)}[x_{j(s)}] \subseteq N_G^{k-i(s)}[x_{i(s)}]$, the sequence $s' = (x_1, \ldots, x_{j(s)-1}, y, x_{j(s)+1}, \ldots, x_k)$ satisfies (3) and $j(s') > j(s)$, which is a contradiction. $\Box$

In view of Lemma 1, the burning number can be considered a variation (but distinct from) of well known distance domination parameters [7]. For a graph G and an integer k, a set D of vertices of G is a *distance-k-dominating set* of G if $\bigcup_{x \in D} N_G^k[x] = V(G)$. The *distance-$k$-domination number* $\gamma_k(G)$ of G is the minimum cardinality of a distance-k-dominating set of G.

The following bound on the distance-k-domination number will be of interest.

Theorem 2 (Meir and Moon [8]) *If G is a connected graph of order n at least $k+1$, then $\gamma_k(G) \leq \frac{n}{k+1}$.*

As observed in [3, 4] the burning number can be bounded above in terms of the distance-k-domination number. In fact, if $\{x_1, \ldots, x_\gamma\}$ is a distance-k-dominating set of G, then

$$\begin{aligned} V(G) \quad &= \quad N_G^k[x_1] \cup N_G^k[x_2] \cup \cdots \cup N_G^k[x_\gamma] \\ &= \quad N_G^{k+\gamma-1}[x_1] \cup N_G^{k+\gamma-2}[x_2] \cup \cdots \cup N_G^k[x_\gamma]. \end{aligned}$$

Appending any k vertices to the sequence $(x_1, \ldots, x_\gamma)$ yields a sequence of length $k + \gamma$ satisfying (3), which, by Lemma 1, implies $b(G) \leq \gamma_k(G) + k$. Using Theorem 2 and choosing $k = \lceil \sqrt{n} \rceil - 1$, this implies the following.

Theorem 3 (Bonato, Janssen, and Roshanbin [3, 4]) *If G is a connected graph of order n, then $b(G) \leq 2\lceil \sqrt{n} \rceil - 1$.*

One of the most interesting open problems concerning the burning number is the following.

Conjecture 4 (Bonato, Janssen, and Roshanbin [3, 4]) *If G is a connected graph of order n, then $b(G) \leq \lceil \sqrt{n} \rceil$.*

Since the path P_n of order n has burning number $\lceil \sqrt{n} \rceil$ [3, 4], the bound in Conjecture 4 would be tight.

Let $\mathrm{rad}(G)$ denote the radius of a graph G. Since $V(G) = N_G^{\mathrm{rad}(G)}[x]$ for every connected graph G and every vertex x of G of minimum eccentricity, Lemma 1 implies the following.

Theorem 5 (Bonato, Janssen, and Roshanbin [4]) *If G is a connected graph, then $b(G) \leq \mathrm{rad}(G) + 1$.*

In the present note, we improve the bound of Theorem 3 by showing several upper bounds on the burning number, thereby contributing to Conjecture 4. Furthermore, we characterize the extremal binary trees for Theorem 5.

2 Results

We begin with two straightforward results that lead to a first improvement of Theorem 3, and rely on arguments that are typically used to prove Theorem 2. For a vertex u of a rooted tree T, let T_u denote the subtree of T rooted in u that contains u as well as all descendants of u. Recall that the height of T_u is the eccentricity of u in T_u.

The order of a graph G is denoted by $n(G)$.

Lemma 6 *Let T be a tree. If the non-negative integer d is such that $N_T^d[u] \neq V(T)$ for every vertex u of T, then there is a vertex x of T and a subtree T' of T with $n(T') \leq n(T) - (d+1)$ and $V(T) \setminus V(T') \subseteq N_T^d[x]$.*

Proof: Root T at a vertex r. Since $N_T^d[r] \neq V(T)$, the height of T is at least $d+1$. The desired properties follow for a vertex x such that T_x has height exactly d and the tree $T' = T - V(T_x)$. $\square$

Theorem 7 *Let T be a tree. If the non-negative integers $d_1, \ldots, d_k$ are such that $\sum\limits_{i=1}^{k}(d_i + 1) \geq n(T)$, then there are vertices $x_1, \ldots, x_k$ of T such that $\bigcup\limits_{i=1}^{k} N_T^{d_i}[x_i] = V(T)$.*

Proof: For a contradiction, suppose that such vertices do no exist. Repeatedly applying Lemma 6, yields a sequence $x_1, \ldots, x_k$ of vertices of T as well as a sequence $T_1, \ldots, T_k$ of subtrees of T such that $n(T_i) \leq n(T_{i-1}) - (d_i + 1)$ and $V(T_{i-1}) \setminus V(T_i) \subseteq N_{T_{i-1}}^{d_i}[x_i] \subseteq N_T^{d_i}[x_i]$ for every $i \in [k]$, where $T_0 = T$. Note that after $j - 1 < k$ applications of Lemma 6, our assumption implies that $N_{T_{j-1}}^{d_j}[u] \neq V(T_{j-1})$ for every vertex u of T_{j-1}, because otherwise

$$\begin{aligned} V(T) &\subseteq (V(T_0) \setminus V(T_1)) \cup (V(T_1) \setminus V(T_2)) \cup \cdots \cup (V(T_{j-2}) \setminus V(T_{j-1})) \cup V(T_{j-1}) \\ &\subseteq \bigcup_{i=1}^{j-1} N_T^{d_i}[x_i] \cup N_{T_{j-1}}^{d_j}[u] \\ &\subseteq \bigcup_{i=1}^{j-1} N_T^{d_i}[x_i] \cup N_T^{d_j}[u] \end{aligned}$$

for some vertex u of T, contradicting our assumption. Therefore, the hypothesis of Lemma 6 remains satisfied throughout its repeated applications. Now, $V(T) \setminus V(T_k) \subseteq \bigcup_{i=1}^{k} N_T^{d_i}[x_i]$. Since $n(T_k) \leq n(T) - \sum_{i=1}^{k}(d_i+1) \leq 0$, it follows that $V(T_k)$ is empty, again contradicting our assumption. □

The previous result already allows to improve Theorem 3.

Corollary 8 *If G is a connected graph of order n, then $b(G) \leq \left\lceil \sqrt{2n+\frac{1}{4}} - \frac{1}{2} \right\rceil$.*

Proof: If H is a spanning subgraph of G, then $b(G) \leq b(H)$. Hence, we may assume that G is a tree. If $k = \left\lceil \sqrt{2n+\frac{1}{4}} - \frac{1}{2} \right\rceil$, then $((k-1)+1)+((k-2)+1)+\cdots+(0+1) = \binom{k+1}{2} \geq n(G)$. By Theorem 7, there are vertices $x_1,\ldots,x_k$ in G with $\bigcup\limits_{i=1}^{k} N_G^{k-i}[x_i] = V(G)$. By Lemma 1, $b(G) \leq k$. □

Note that Theorem 2 is tight for any graph that arises by attaching a path of order k to each vertex of a connected graph. In fact, also Theorem 7 is tight for the same kind of graph. Therefore, in order to further improve Theorem 3, one really has to leverage the full spectrum of different distances associated with the different vertices in a burning sequence. The following lemma offers some way of doing this.

Lemma 9 *Let T be a tree. If the positive integers d_1 and d_2 are such that $d_2 \geq \left\lceil \frac{3d_1}{2} \right\rceil$ and $N_T^{d_1}[u] \cup N_T^{d_2}[v] \neq V(T)$ for every two vertices u and v of T, then there are two vertices x and z of T and a subtree T' of T with $n(T') \leq n(T) - \left(\left\lceil \frac{3d_1}{2} \right\rceil + d_2 + 2 \right)$ and $V(T) \setminus V(T') \subseteq N_T^{d_1}[x] \cup N_T^{d_2}[z]$.*

Proof: Root T at a vertex r. Since $N_G^{d_2}[r] \neq V(T)$, the height of T is at least d_2+1. Let the vertex z be such that T_z has height exactly d_2. Note that $V(T_z) \subseteq N_T^{d_2}[z]$ and $|V(T_z)| \geq d_2+1$. Let x be a descendant of z such that $\mathrm{dist}_T(x,z) = d_2 - d_1$ and T_x has height exactly d_1. Note that $d_2 - d_1 \geq \left\lceil \frac{d_1}{2} \right\rceil$. Let the vertex y on the path in T between x and z be such that $\mathrm{dist}_T(x,y) = \left\lceil \frac{d_1}{2} \right\rceil$.

If $V(T_y) \subseteq N_T^{d_1}[x]$, then Lemma 6 applied to the tree $\tilde{T} = T - V(T_y)$ and the value d_2 implies the existence of a vertex z' and a subtree T' of $\tilde{T}$ with $n(T') \leq n(\tilde{T}) - (d_2+1)$ and $V(\tilde{T}) \setminus V(T') \subseteq N_{\tilde{T}}^{d_2}[z']$. Now, we have that

$$\begin{aligned} n(T') &\leq n(\tilde{T}) - (d_2+1) \\ &= n(T) - |V(T_y)| - (d_2+1) \\ &\leq n(T) - \left(\left\lceil \frac{3d_1}{2} \right\rceil + d_2 + 2 \right) \end{aligned}$$

and

$$\begin{aligned} V(T) \setminus V(T') &= (V(T) \setminus V(\tilde{T})) \cup (V(\tilde{T}) \setminus V(T')) \\ &\subseteq V(T_y) \cup N_{\tilde{T}}^{d_2}[z'] \\ &\subseteq N_T^{d_1}[x] \cup N_T^{d_2}[z']. \end{aligned}$$

Hence, we may assume that $V(T_y) \not\subseteq N_T^{d_1}[x]$. This implies the existence of a descendant y' of y that is not a descendant of x and satisfies $\mathrm{dist}_T(x,y') > d_1$. By the choice of x, y, and z, this implies $|V(T_z)| \geq d_2 + 1 + \left\lceil \frac{d_1}{2} \right\rceil$. Lemma 6 applied to the tree $\tilde{T} = T - V(T_z)$ and the value d_1 implies the existence of a vertex x' and a subtree T' of $\tilde{T}$ with $n(T') \leq n(\tilde{T}) - (d_1+1)$ and $V(\tilde{T}) \setminus V(T') \subseteq N_{\tilde{T}}^{d_1}[x']$. Now, we have that

$$\begin{aligned} n(T') &\leq n(\tilde{T}) - (d_1+1) \\ &= n(T) - |V(T_z)| - (d_1+1) \\ &\leq n(T) - \left(\left\lceil \frac{3d_1}{2} \right\rceil + d_2 + 2 \right) \end{aligned}$$

and

$$\begin{aligned} V(T)\setminus V(T') &= (V(T)\setminus V(\tilde{T}))\cup(V(\tilde{T})\setminus V(T'))\\ &\subseteq V(T_z)\cup N_{\tilde{T}}^{d_1}[x']\\ &\subseteq N_T^{d_1}[x']\cup N_T^{d_2}[z], \end{aligned}$$

which completes the proof. □

Theorem 10 *If G is a connected graph and $0<\epsilon<1$, then* $b(G)\le\sqrt{\frac{32}{19}\cdot\frac{n(G)}{1-\epsilon}}+\sqrt{\frac{27}{19\epsilon}}$.

Proof: As in the proof of Corollary 8, we may assume that G is a tree T.

Let $\ell=\left\lceil\log_9\left(\frac{3}{19\epsilon}\right)\right\rceil$. Note that $\left(1-\frac{3}{19}\cdot\left(\frac{1}{9}\right)^\ell\right)\ge 1-\epsilon$ and $3^\ell<\sqrt{\frac{27}{19\epsilon}}$. Let k be the smallest integer such that $(1-\epsilon)\cdot\frac{19k^2}{32}+(1-\epsilon)\cdot\frac{3k}{8}\ge n(T)$ and $k\equiv 0 \pmod{3^\ell}$. Note that

$$k\le\left\lceil\sqrt{\frac{32}{19}\cdot\frac{n(T)}{1-\epsilon}+\left(\frac{6}{19}\right)^2}-\frac{6}{19}+3^\ell-1\right\rceil\le\sqrt{\frac{32}{19}\cdot\frac{n(T)}{1-\epsilon}}+\sqrt{\frac{27}{19\epsilon}}.$$

For a contradiction, suppose that $b(G)>k$.

For $j\in[\ell]$, let $I_j=\left[\frac{2k}{3^j}-1\right]\setminus\left[\frac{k}{3^j}-1\right]=\left\{\frac{k}{3^j},\frac{k}{3^j}+1,\ldots,\frac{2k}{3^j}-1\right\}$. Since $\frac{k}{3^j}$ is an integer, it follows that $\left\lceil\frac{3d}{2}\right\rceil\le\frac{k}{3^j}+d$ for every $d\in I_j$. Repeatedly applying Lemma 9 to the $\left(1-\frac{1}{3^\ell}\right)\frac{k}{2}$ disjoint pairs $\left\{d,\frac{k}{3^j}+d\right\}$ for $j\in[\ell]$ and $d\in I_j$, yields pairs of vertices $\left\{x_d,x_{\frac{k}{3^j}+d}\right\}$ as well as a subtree T' of T such that

$$\begin{aligned} n(T') &\le n(T)-\sum_{j=1}^{\ell}\sum_{d=\frac{k}{3^j}}^{\frac{2k}{3^j}-1}\left(\left\lceil\frac{3d}{2}\right\rceil+\left(\frac{k}{3^j}+d\right)+2\right)\\ &\le n(T)-\sum_{j=1}^{\ell}\sum_{d=\frac{k}{3^j}}^{\frac{2k}{3^j}-1}\left(\frac{5d}{2}+\frac{k}{3^j}+2\right)\\ &= n(T)-\sum_{j=1}^{\ell}\left(\frac{1}{9^{j-1}}\cdot\frac{19k^2}{36}+\frac{1}{3^{j-1}}\cdot\frac{k}{4}\right)\\ &= n(T)-\left(1-\left(\frac{1}{9}\right)^\ell\right)\cdot\frac{19k^2}{32}-\left(1-\left(\frac{1}{3}\right)^\ell\right)\cdot\frac{3k}{8} \end{aligned}$$

and

$$\begin{aligned} V(T)\setminus V(T') &\subseteq \bigcup_{j=1}^{\ell}\bigcup_{d=\frac{k}{3^j}}^{\frac{2k}{3^j}-1}\left(N_T^d[x_d]\cup N_T^{\left(\frac{k}{3^j}+d\right)}\left[x_{\frac{k}{3^j}+d}\right]\right)\\ &= \bigcup_{i=\frac{k}{3^\ell}}^{k-1}N_T^i[x_i]. \end{aligned}$$

Note that, similarly as in the proof of Theorem 7, the assumption $b(G)>k$ implies that the hypothesis of Lemma 9 remains satisfied throughout its repeated applications.

Now, repeatedly applying Lemma 6 for all $\frac{k}{3^\ell}$ values d in $\{0\}\cup[\frac{k}{3^\ell}-1]$, yields vertices $x_0,\ldots,x_{\frac{k}{3^\ell}-1}$ and a subtree T'' of T' such that

$$\begin{aligned} n(T'') &\leq n(T') - \sum_{d=0}^{\frac{k}{3^\ell}-1}(d+1) \\ &= n(T') - \left(\frac{1}{9}\right)^\ell \cdot \frac{k^2}{2} - \left(\frac{1}{3}\right)^\ell \cdot \frac{k}{2} \end{aligned}$$

and

$$V(T')\setminus V(T'') \subseteq \bigcup_{i=0}^{\frac{k}{3^\ell}-1} N_T^i[x_i].$$

Altogether, the vertices $x_0,\ldots,x_{k-1}$ satisfy

$$V(T)\setminus V(T'') \subseteq \bigcup_{i=0}^{k-1} N_T^i[x_i].$$

Since

$$\begin{aligned} n(T'') &\leq n(T) - \left(1-\left(\frac{1}{9}\right)^\ell\right)\cdot\frac{19k^2}{32} - \left(1-\left(\frac{1}{3}\right)^\ell\right)\cdot\frac{3k}{8} - \left(\frac{1}{9}\right)^\ell \cdot \frac{k^2}{2} - \left(\frac{1}{3}\right)^\ell \cdot \frac{k}{2} \\ &= n(T) - \left(1-\frac{3}{19}\cdot\left(\frac{1}{9}\right)^\ell\right)\cdot\frac{19k^2}{32} - \left(1+\left(\frac{1}{3}\right)^{\ell+1}\right)\cdot\frac{3k}{8} \\ &\leq n(T) - (1-\epsilon)\cdot\frac{19k^2}{32} - (1-\epsilon)\cdot\frac{3k}{8} \\ &\leq 0, \end{aligned}$$

it follows that $V(T'')$ is empty, which implies the contradiction $b(T)\leq k$. $\square$

Choosing in the above proof $\ell=1$, and k as the smallest multiple of 3 that satisfies $\frac{7}{12}k^2+\frac{5}{12}k\geq n(T)$, allows to deduce a similar contradiction, which implies $b(G)\leq\sqrt{\frac{12n(G)}{7}}+3\approx 1.309\sqrt{n(G)}+3$ for every connected graph G.

The following results generalize the equality $b(P_n)=\lceil\sqrt{n}\rceil$, and establish approximate versions of Conjecture 4 under additional restrictions.

Lemma 11 *If $n_1,\ldots,n_p$ and k are positive integers such that $n_1+\cdots+n_p+k(p-1)\leq k^2$, then $b(P_{n_1}\cup\cdots\cup P_{n_p})\leq k$.*

Proof: The proof is by induction on $n=n_1+\cdots+n_p$. Let $G=P_{n_1}\cup\cdots\cup P_{n_p}$ and $n_1\leq\ldots\leq n_p$. Note that $p\leq k$. If $n_p\leq k-p+1$, let the set $\{x_1,\ldots,x_p\}$ contain a vertex from each component of G. We have $V(G)=N_G^{k-1}[x_1]\cup N_G^{k-2}[x_2]\cup\cdots\cup N_G^{k-p}[x_p]$, and Lemma 1 implies $b(G)\leq k$. Hence, we may assume that $n_p\geq k-p+2$, which implies $n\geq(p-1)+(k-p+2)=k+1$.

If $n_p\geq 2k$, let x_1 be a vertex at distance $k-1$ from an endvertex of a component of G of order n_p. The graph $G'=G-N_G^{k-1}[x_1]$ has p components and $|N_G^{k-1}[x_1]|=2k-1$. Since

$$\begin{aligned} n_1+\cdots+n_{p-1}+(n_p-(2k-1))+(k-1)(p-1) &\leq n_1+\cdots+n_{p-1}+(n_p-(2k-1))+k(p-1) \\ &\leq k^2-(2k-1) \\ &= (k-1)^2, \end{aligned}$$

there are, by induction, vertices $x_2, \ldots, x_k$ such that

$$V(G') = N_{G'}^{(k-1)-1}[x_2] \cup N_{G'}^{(k-1)-2}[x_3] \cup \cdots \cup N_{G'}^{0}[x_k]. \tag{4}$$

This implies (3), and Lemma 1 implies $b(G) \leq k$.

Hence, we may assume that $n_p \leq 2k-1$. In this case we choose as x_1 a vertex of minimal eccentricity in a component of G of order n_p. This implies that $G' = G - N_G^{k-1}[x_1]$ has $p-1$ components. Since

$$\begin{aligned} n_1 + \cdots + n_{p-1} + (k-1)(p-2) &\leq k^2 - n_p - (k(p-1) - (k-1)(p-2)) \\ &\leq k^2 - (k-p+2) - (k+p-2) \\ &= k^2 - 2k \\ &< (k-1)^2, \end{aligned}$$

there are, by induction, vertices $x_2, \ldots, x_k$ that satisfy (4), which again implies $b(G) \leq k$. $\Box$

Since $n + (\lceil\sqrt{n}\rceil + (p-1))(p-1) \leq (\lceil\sqrt{n}\rceil + (p-1))^2$ for positive integers n and p, Lemma 11 implies the following.

Corollary 12 (Roshanbin [9]) *If the forest T of order n is the union of p paths, then $b(T) \leq \lceil\sqrt{n}\rceil + (p-1)$.*

We derive further consequences of Lemma 11.

Theorem 13 *If T is a tree of order n that has $n_{\geq 3}$ vertices of degree at least 3, then $b(T) \leq \lceil\sqrt{n}\rceil + n_{\geq 3}$.*

Proof: By Corollary 12, we may assume that $n_{\geq 3} \geq 1$. Let $k = \lceil\sqrt{n}\rceil + n_{\geq 3}$. Let $x_1, \ldots, x_{n_{\geq 3}}$ be the vertices of degree at least 3. Let $T' = T - \{x_1, \ldots, x_{n_{\geq 3}}\}$, and let $T'' = T - N_T^{k-1}[x_1] \cup \cdots \cup N_T^{k-n_{\geq 3}}[x_{n_{\geq 3}}]$. Every component of T' is a path P such that at least one endvertex of P has a neighbor in $\{x_1, \ldots, x_{n_{\geq 3}}\}$. Therefore, the distinct components of T'' arise by removing at least $k - n_{\geq 3} = \lceil\sqrt{n}\rceil$ vertices from distinct components of T'. This implies that if $T'' = P_{n_1} \cup \cdots \cup P_{n_p}$, then

$$n_1 + \cdots + n_p + \lceil\sqrt{n}\rceil (p-1) < \left(n_1 + \lceil\sqrt{n}\rceil\right) + \cdots + \left(n_p + \lceil\sqrt{n}\rceil\right) \leq n - n_{\geq 3} < \lceil\sqrt{n}\rceil^2.$$

Now, Lemma 11 implies the existence of vertices $y_1, \ldots, y_{\lceil\sqrt{n}\rceil}$ such that

$$V(T'') = N_{T''}^{\lceil\sqrt{n}\rceil-1}[y_1] \cup \cdots \cup N_{T''}^{0}[y_{\lceil\sqrt{n}\rceil}].$$

We obtain

$$V(T) = N_T^{k-1}[x_1] \cup \cdots \cup N_T^{\lceil\sqrt{n}\rceil}[x_{n_{\geq 3}}] \cup N_T^{\lceil\sqrt{n}\rceil-1}[y_1] \cup \cdots \cup N_T^{0}[y_{\lceil\sqrt{n}\rceil}],$$

and Lemma 1 implies $b(T) \leq k$. $\Box$

Theorem 14 *If T is a tree of order n that has n_2 vertices of degree 2, then*

$$b(T) \leq \left\lceil \sqrt{(n+n_2) + \frac{1}{4}} + \frac{1}{2} \right\rceil.$$

Proof: Let $k = \left\lceil \sqrt{(n+n_2) + \frac{1}{4}} + \frac{1}{2} \right\rceil$. Note that $k(k-1) \geq n + n_2$. For a contradiction, suppose that $b(T) > k$. Root T at a vertex r. As before, we obtain that the height of T is at least k. Let x_d be a vertex of T such that T_{x_d} has height exactly d for some $d \in \{0\} \cup [k-1]$. Let $V(T_{x_d})$ contain exactly p_d vertices that have degree 2 in T. If P is a path of length d between x_d and a leaf of T_{x_d}, then at least $d-p_d$ vertices of P have a child that does not lie on P. Therefore, $|V(T_{x_d})\setminus\{x_d\}|\geq 2d-p_d$, and $T'=T-(V(T_{x_d})\setminus\{x_d\})$ is a tree with n_2-p_d vertices of degree 2 such that $V(T)\setminus V(T')\subseteq N_T^d[x_d]$. Note that x_d has degree 1 in T'. Iteratively repeating this argument similarly as in the previous proofs, we obtain vertices $x_0,\ldots,x_{k-1}$ and integers $p_0,\ldots,p_{k-1}$ such that $p_0+\cdots+p_{k-1}\leq n_2$ and $\sum_{d=0}^{k-1}(2d-p_d)\leq n$. Since $\sum_{d=0}^{k-1}(2d-p_d)\geq k(k-1)-n_2\geq n$, we obtain $V(T)=N_T^0[x_0]\cup\cdots\cup N_T^{k-1}[x_{k-1}]$, which implies the contradiction $b(G)\leq k$. $\Box$

In view of the simple argument that shows Theorem 5, the extremal graphs for this bound might have a rather special structure. Our final result supports this intuition for binary trees.

Recall that a rooted tree is *binary* if every vertex has at most two children, and that a binary tree is *perfect* if every non-leaf vertex has exactly two children, and all leaves have the same depth, that is, the same distance from the root. Let T_1 be the rooted tree of order 2, and, for an integer r at least 2, let T_r be the rooted tree that arises from the perfect binary tree of depth $r-1$ by subdividing all edges that are incident with a leaf. Alternatively, T_r arises by attaching a new leaf to each of the 2^{r-1} leaves of the perfect binary tree of depth $r-1$.

Theorem 15 *If r is a positive integer and T is a binary tree of depth r, then $b(T)=r+1$ if and only if T contains T_r as a subtree.*

Proof: Since the statement is trivial for $r=1$, we may assume that $r\geq 2$.

First, we show that $T=T_r$ has burning number $r+1$. For a contradiction, suppose that $b(T)\leq r$. Let u be the root of T, and let v^1 and v^2 be the two children of u. For $i\in[2]$, let T^i be the subtree of T rooted in v^i that contains v^i as well as all descendants of v^i in T. By Lemma 1, there are vertices $x_1,x_2,\ldots,x_r$ with $V(T)=N_T^{r-1}[x_1]\cup N_T^{r-2}[x_2]\cup\cdots\cup N_T^0[x_r]$. By symmetry, we may assume that $x_1\not\in V(T^1)$. Let L be the set of leaves of T that belong to T^1. Since T^1 is isomorphic to T_{r-1}, we have $|L|=2^{r-2}$. Note that $N_T^{r-1}[x_1]$ does not intersect L. Furthermore, for every $i\in[r-1]\setminus\{1\}$, the set $N_T^{r-i}[x_i]$ contains at most 2^{r-i-1} vertices from L. In fact, the set $N_T^{r-i}[x_i]$ contains exactly 2^{r-i-1} vertices from L if and only if $x_i\in V(T^1)$ and x_i has depth i in T. Since $N_T^0[x_r]=\{x_r\}$, the set $N_T^0[x_r]$ contains at most one vertex from L. Since $|L|=2^{r-2}=\sum_{i=2}^{r-1}2^{r-i-1}+1$, every vertex in L belongs to exactly one of the sets $N_T^{r-i}[x_i]$ for $i=[r]\setminus\{1\}$. This implies that $x_2,\ldots,x_r\in V(T^1)$, x_i has depth i in T for $i\in[r-1]\setminus\{1\}$, and x_r is a leaf of T. Let $u_0\ldots u_r$ be the path in T from the root $u=u_0$ to the leaf $x_r=u_r$. Note that $u_1=v^1$. Since x_2 belongs to T^1, x_2 has depth 2 in T, and $x_r\not\in N_T^{r-2}[x_2]$, the vertex x_2 is the child of u_1 distinct from u_2. Moreover, as every vertex of L belongs to exactly one of the sets $N_T^{r-i}[x_i]$ for $i\in[r]\setminus\{1\}$, no vertex x_i with $i\in[r]\setminus\{1,2\}$ is a descendant of x_2. Iterating these arguments, it follows that, for every $i\in[r-1]\setminus\{1\}$, the vertex x_i is the child of u_{i-1} distinct from u_i. However, this implies the contradiction that $u_{r-1}\not\in N_T^{r-1}[x_1]\cup N_T^{r-2}[x_2]\cup\cdots\cup N_T^0[x_r]$. Altogether, we obtain that T_r has burning number $r+1$. Together with Theorem 5, this implies that a binary tree T of depth r has burning number $r+1$ if T contains T_r as a subtree.

For the converse, we assume that T is a binary tree of depth r that does not contain T_r as a subtree. It follows that T has a leaf of depth less than r or that T has a vertex of depth less than $r-1$ that has only one child. In both cases we will show that $b(T)\leq r$. First, we assume that T has a leaf at depth less than r. Let d be the minimum depth of a leaf of T. Let $u_0\ldots u_d$ be a path in T between the root u_0 and a leaf u_d. By assumption, we have $d<r$. For $i\in[d]$, let x_i be the child of u_{i-1} that is distinct from u_i. Note that the subtree of T rooted in x_i that contains x_i as well as all descendants of x_i in T has depth at most $r-i$. This implies that $V(T)=N_T^{r-1}[x_1]\cup N_T^{r-2}[x_2]\cup\cdots\cup N_T^{r-d}[x_d]\cup N_T^0[u_d]$, and, by Lemma 1, we obtain $b(T)\leq r$. Next, we assume that T has a vertex x of depth less than $r-1$ that has only one child. Let T' arise from T by adding a new leaf y as a child of x. Clearly, T' is a binary tree of depth r that has a leaf of depth less than r, and, hence, $b(T)\leq b(T')\leq r$. $\Box$

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