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The good, the bad, and the great: Homomorphisms and cores of random graphs

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ABSTRACT

We consider homomorphism properties of a random graph $G(n, p)$, where p is a function of n. A core H is great if for all $e \in E(H)$, there is some homomorphism from $H - e$ to H that is not onto. Great cores arise in the study of uniquely H-colourable graphs, where two inequivalent definitions arise for general cores H. For a large range of p, we prove that with probability tending to 1 as $n \to \infty$, $G \in G(n, p)$ is a core that is not great. Further, we give a construction of infinitely many non-great cores where the two definitions of uniquely H-colourable coincide.

1. Introduction

In recent years there has been much research interest in homomorphisms of graphs. The book [3] is both an excellent reference for background on graph homomorphism theory, and a record of the growing corpus of work on the subject. In the present article, our focus is on uniquely H-colourable graphs, where H is a finite core (that is, every homomorphism from H to itself is onto, and so is an automorphism). There are two natural definitions of a uniquely H-colourable graph. Following [1,6,7], a graph G is *uniquely H-colourable* if G is H-colourable so that every homomorphism from G to H is onto, and for all homomorphisms f, h from G to H, there is an automorphism g of H so that $f = gh$. On the other hand, a graph G is *weakly uniquely H-colourable* if a similar definition holds, but with g only required to be a bijection from $V(H)$ to itself. The class of weakly uniquely H-colourable graphs is written $C_{wu}(H)$. For many familiar cores H such as cliques, odd cycles, odd wheels, and the Petersen graph, the two notions of uniquely H-colourable coincide. However, as discussed in [1], there are infinitely many examples of graphs H where the class of weakly uniquely H-colourable graphs strictly contains the class of uniquely H-colourable graphs. Following the notation in [1], a core H is *good* if the two notions uniquely colourable coincide; H is *great* if for all $e \in E(H)$, there is some homomorphism from $H - e$ to H that is not onto (or equivalently, not injective).

In [1], it was proved that every great core is good, but the converse fails for the Petersen graph. Great cores have the following algebraic characterization related to the first homomorphism theorem. We first give some notation. Define $\mathrm{Hom}(H, G)$ to be the set of homomorphisms from H into G. Given $f \in \mathrm{Hom}(G, H)$, define $\ker(f) = \{(x, y) \in V(G) \times V(G) : f(x) = f(y)\}$. Then $\ker(f)$ is an equivalence relation whose equivalence classes, called *colour blocks*, are independent sets partitioning $V(G)$. If $f \in \mathrm{Hom}(G, H)$ is surjective, then the *quotient graph* $G/\ker(f)$ has vertices the colour blocks of $\ker(f)$,

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and two colour blocks B and C are joined if and only if there is some vertex in B joined to some vertex in C. The *natural map* $\eta_f : V(G/\ker(f)) \to V(H)$ defined by $\eta_f(f^{-1}(x)) = x$ is a well-defined homomorphism. The class $C_{wu}(H)$ satisfies the *first homomorphism theorem* if for all $G \in C_{wu}(H)$ and all $f : \mathrm{Hom}(G, H)$, the homomorphism $\eta_f : V(G/\ker(f)) \to V(H)$ is an isomorphism.

Theorem 1 ([1]). *Let H be a core graph.*

(1) *The class $C_{wu}(H)$ satisfies the first homomorphism theorem if and only if H is great.*
(2) *If H is great, then H is good.*

Despite Theorem 1, the classification of which cores are great seems difficult. A more tractable problem is the classification of great cores restricted to *random* graphs. As proved in [3], with probability tending to 1 as n tends to infinity $G \in G(n, 1/2)$ is a core. Hence, a natural problem is to determine which random graphs are great cores. This problem applies more generally to random graphs $G(n, p)$, where p is a function of n.

We consider the problem of which $G(n, p)$ are great with probability tending to 1 as $n \to \infty$. Our first result is Theorem 2, which proves that if $n^{-1/3}\log^2 n < p = p(n) < 1 - n^{-1/3}\log^2 n$, then with probability tending to 1 as $n \to \infty$, $G \in G(n, p)$ is a core that is not great. This result is somewhat surprising, since most examples of well-known cores are great. The fact that the random graph is a core in this range of p generalizes the result for $p = 1/2$ proved in [3]. Our methods do not determine the probability that $G \in G(n, 1/2)$ is a good core. We leave this as an open problem.

In the final section, we consider in Theorem 3 a new deterministic construction of a large class of good cores that are not great. Theorem 3 indicates that the classification of the good cores is far from complete.

All graphs in this article are finite, undirected, and simple. If G is an induced subgraph of H, then we write $G \le H$. Let $N_G(x)$ be the neighbour set of x in G. The image of a mapping f is written $Im(f)$. If A is a set, then $\mathrm{Sym}(A)$ is the set of bijective maps from A onto A. The set of automorphisms of G is denoted by $\mathrm{Aut}(G)$ (with the identity automorphism written as id_G). The set of endomorphisms of G is $\mathrm{End}(G) = \mathrm{Hom}(G, G)$. We write $\log x$ for the natural logarithm of x.

2. Almost all graphs are cores that are not great

An event holds *asymptotically almost surely* (a.a.s.) if it holds with probability tending to 1 as $n \to \infty$, and holds *with extreme probability* (w.e.p.) if it holds with probability at least $1 - \exp(-\Theta(\log^2 n))$ as $n \to \infty$. We will use the stronger notion of w.e.p. in favour of the more commonly used a.a.s., since it simplifies some of our proofs. If we consider a polynomial number of events that each holds w.e.p., then w.e.p. all events hold. To combine this notion with other asymptotic notation such as $O(\cdot)$ and $o(\cdot)$, we follow the conventions in [5]. The main result for this section is the following theorem.

Theorem 2. *If $n^{-1/3}\log^2 n < p < 1 - n^{-1/3}\log^2 n$, then w.e.p. the random graph $G \in G(n, p)$ is a core that is not great.*

To prove Theorem 2, following the proof of [3], we need several properties of $G(n, p)$ as described in Lemma 1 below. Some of the properties (namely, properties (a), (b), and (c)) are weaker versions of known properties of $G(n, p)$. Since the proofs are short we include them for completeness.

Lemma 1. *If $p = p(n) > n^{-1/3}\log^2 n$, then w.e.p. $G \in G(n, p)$ has the following properties. Let H be either G or $G - e$, where e is a fixed edge of G.*

(a) *The degree of every vertex of H is equal to*

$$pn + O(\sqrt{pn}\log n) = pn(1 + o(1)).$$

(b) *Every pair of distinct vertices of H have*

$$p^2 n + O(\sqrt{p^2 n}\log n) = p^2 n(1 + o(1)),$$

many common neighbours.

(c) *All independent sets of H have less than $n^{1/3}$ vertices.*
(d) *Each set of k vertices, where $k > k_0 = k_0(n) = 0.5n^{1/3}\log^2 n$, induces a subgraph in H with $p\binom{k}{2}(1 + O(\log^{-1} n))$ edges.*
(e) *In each set of k disjoint pairs of vertices (v_i, w_i), $i = 1, 2, \ldots, k$, where $k > k_0 = k_0(n) = 0.5n^{1/3}\log^2 n$, there are $(1 - (1 - p)^4)\binom{k}{2}(1 + O(\log^{-1} n))$ pairs (i, j) such that at least one of v_iv_j, v_iw_j, v_jw_i, w_iw_j is an edge of H.*

To prove that a property is satisfied by $G \in G(n, p)$ w.e.p., we use the following approach which we illustrate with an example. Let Y be the number of vertices with degree either strictly greater than $pn + \sqrt{np}\log n$ or strictly less than $pn - \sqrt{np}\log n$. We show that $\mathbb{E}Y$ tends to zero faster than the function $\exp(-\Theta(\log^2 n))$ as $n \to \infty$. By Markov's inequality

$$\mathbb{P}(Y = 0) = 1 - \mathbb{P}(Y \ge 1) \ge 1 - \mathbb{E}Y > 1 - \exp(-\Theta(\log^2 n)).$$

We employ the well-known *Chernoff inequalities*. For a binomially distributed random variable $X \in Bi(n, p)$ with $\mathbb{E}X = np$

$$\mathbb{P}\left(|X - \mathbb{E}X| \ge \varepsilon\mathbb{E}X\right) \le 2\exp\left(-\frac{1}{3}\varepsilon^2\mathbb{E}X\right), \tag{2.1}$$

where $\varepsilon \le 3/2$. See, for example, [4].

Proof. The proofs for H equalling G or $G-e$ are similar; we just give the proofs for $G-e$. For part (a), fix any vertex v of $G-e$. Then $\mathbb{E}\deg(v) = p(n-1) - O(1) = pn - O(1)$. Using (2.1) with $\varepsilon = 0.5\log n/\sqrt{\mathbb{E}\deg(v)}$, we have that

$$\mathbb{P}(|\deg(v) - \mathbb{E}\deg(v)| \geq \varepsilon\mathbb{E}\deg(v)) \leq \exp\left(-\Omega(\log^2 n)\right).$$

It follows that the expected number of vertices of degree greater than $pn + \sqrt{pn}\log n$ or of degree smaller than $pn - \sqrt{np}\log n$ is less than $n\exp\left(-\Omega(\log^2 n)\right) = \exp\left(-\Omega(\log^2 n)\right)$. Thus, w.e.p. all vertices have degree $pn + O(\sqrt{pn}\log n)$ by Markov's inequality.

For (b), let $v_1, v_2, v_1 \neq v_2$ be any two vertices of $G-e$. The expected number of common neighbours of v_1 and v_2 in $G-e$ is equal to $\mathbb{E}X = p^2(n-2) - O(1) = p^2 n - O(1)$. Now, using (2.1) with $\varepsilon = 0.5\log n/\sqrt{\mathbb{E}X}$, we see that w.e.p. $|X - p^2 n| \leq \sqrt{p^2 n}\log n$. Thus, the expected number of pairs of vertices having more than $p^2 n + \sqrt{p^2 n}\log n$, or less than $p^2 n - \sqrt{p^2 n}\log n$ common neighbours is bounded from above by $O(n^2)\exp\left(-\Omega(\log^2 n)\right) = \exp\left(-\Omega(\log^2 n)\right)$ which finishes the proof of (b).

For (c), note first that it is enough to show that w.e.p. there is no independent set of order $k_0 = n^{1/3}$. Let $K = \{v_1, v_2, \ldots, v_{k_0}\}$ be any set of k_0 vertices of $G-e$. The probability that set K forms an independent set is equal to $(1-p)^{\binom{k_0}{2}-O(1)}$. So the expected number of independent sets of order k_0 is bounded from above by

$$\begin{aligned}\binom{n}{k_0}(1-p)^{\binom{k_0}{2}-O(1)} &\leq \left(\frac{ne}{k_0}\right)^{k_0}(1-n^{-1/3}\log^2 n)^{\frac{1}{2}k_0^2(1+o(1))} \\ &\sim \exp\left(n^{1/3}\left(\frac{2}{3}\log n + 1 - \frac{1}{2}(1+o(1))\log^2 n\right)\right) \\ &< \exp\left(-\Omega(\log^2 n)\right),\end{aligned}$$

and the assertion follows from Markov's inequality.

For (d), let $k > 0.5n^{1/3}\log^2 n$. The expected number of edges among any set of k vertices of G is equal to $\mathbb{E}Y = p\binom{k}{2} - O(1)$. Thus, using (2.1) with $\varepsilon = 0.5/\log n$, the expected number of graphs induced by the sets of k vertices containing more than $\mathbb{E}Y(1+\varepsilon)$, or less than $\mathbb{E}Y(1-\varepsilon)$ edges is bounded from above by

$$\begin{aligned}2\binom{n}{k}\exp\left(-\frac{p\binom{k}{2}-O(1)}{12\log^2 n}\right) &< 2\left(\frac{ne}{k}\right)^k\exp\left(-0.04k^2 n^{-1/3}\right) \\ &= 2\exp\left(k(\log n + 1 - \log k - 0.04kn^{-1/3})\right) \\ &< \mathrm{e}^{-k}.\end{aligned}$$

Thus, the expected number of graphs induced by the sets of k vertices containing more than $p\binom{k}{2}(1+2\varepsilon)$, or less than $p\binom{k}{2}(1-2\varepsilon)$ edges is bounded from above by $\exp(-k)$ as well.

Finally, the expected number of graphs induced by the sets of $k > k_0(n)$ vertices containing more than $p\binom{k}{2}(1+2\varepsilon)$, or less than $p\binom{k}{2}(1-2\varepsilon)$ edges is bounded from above by

$$\sum_{k=k_0}^{n}\mathrm{e}^{-k} < \sum_{k=k_0}^{\infty}\mathrm{e}^{-k} = \frac{\mathrm{e}^{-k_0}}{1-1/e} < \exp(-\Omega(\log^2 n)),$$

which completes the proof of (d). Property (e) can be proved using a similar approach used in the proof of property (d), and so we omit the proof. □

Proof of Theorem 2. For a contradiction, suppose that G is great. Then for all $e \in E(V)$, there is a homomorphism $f \in \mathrm{Hom}(G-e, G)$ such that $f(x) = f(y) = z$ for some distinct vertices $x, y \in V(G)$. So the edges incident with vertices x or y in $G-e$ must be mapped to edges incident with z in G; that is, $f(A) \subseteq B$ where $A = N_{G-e}(x) \cup N_{G-e}(y)$, $B = N_G(z)$.

Note that, using properties (a) and (b) from Lemma 1 in the case $H = G-e$, w.e.p.

$$\begin{aligned}|A| &= 2pn(1+o(1)) - p^2 n(1+o(1)) = p(2-p)n(1+o(1)) \\ |f(A)| &\leq |B| = pn(1+o(1)).\end{aligned}$$

Thus, w.e.p.

$$|A| - |f(A)| \geq p(1-p)n(1+o(1)) \geq (1+o(1))n^{2/3}\log^2 n, \tag{2.2}$$

since the function $h(p) = p(1-p)$ is minimized for $p = n^{-1/3}\log^2 n$ (or $p = 1 - n^{-1/3}\log^2 n$).

For a vertex $v \in V(G)$, the set $f^{-1}(v)$ is an independent set in $G-e$. So w.e.p. $|f^{-1}(v)| < n^{1/3}$ for any vertex of G by property (c) in Lemma 1. Thus, using this fact and (2.2), it follows that w.e.p. there are

$$k > \frac{|A| - |f(A)|}{n^{1/3}} > \frac{1}{2}n^{1/3}\log^2 n$$

vertices $v_1, v_2, \ldots, v_k \in f(A)$ such that $|f^{-1}(v_i)| \geq 2$. To see this, consider placing $|A|$ balls (vertices of A) in $|f(A)|$ bins (sets $f^{-1}(v)$, where $v \in f(A)$). Each bin contains at least one ball, and so there are $|A| - |f(A)|$ remaining balls. Since at most $n^{1/3}$ balls may go into any one bin, there are at least $\frac{|A|-|f(A)|}{n^{1/3}}$ many bins which have two or more balls. According to property (e) from Lemma 1, we have that w.e.p. there are at least $(1-(1-p)^4)\binom{k}{2}(1+o(1))$ edges each of which span two distinct colour classes $f^{-1}(v_i)$. As f is a homomorphism, this gives $(1-(1-p)^4)\binom{k}{2}(1+o(1))$ many edges among the vertices $v_1, v_2, \ldots, v_k$. Property (d) from the lemma implies that w.e.p. there are at most $p\binom{k}{2}(1+o(1))$ such edges. This gives a contradiction since

$$(1-(1-p)^4)(1+o(1)) = (4p+O(p^2))(1+o(1)) = 4p(1+o(1)) > p(1+o(1)).$$

holds when p is tending to zero with n, and holds for $p = \Theta(1)$.

To prove that G is w.e.p. a core, we may proceed in a similar way to the proof above that G is not great. For that we can use Lemma 1 with $H = G$. □

If $p = 1$, then w.e.p. $G \in G(n, p)$ is a clique, and so is a great core. However, we think that the conclusions of Theorem 2 hold for other values of $p = p(n)$. This direction will be further explored in the sequel. Let $\delta(G)$ and $\Delta(G)$ be the minimum and maximum degrees of vertices of G, respectively. If $\delta(G) \leq 1$, or G has n vertices and G contains a vertex of degree $n-2$, then G is not a core. We think (although we cannot prove it at present) that for any $p = p(n)$ such that w.e.p. a random graph $G \in G(n, p)$ satisfies $\delta(G) \geq 2$ and $\Delta(G) \leq n-3$, w.e.p. G is a core that is not great.

3. A new construction of good but not great cores

The Petersen graph, written P, is an example of a good but not great core. The graph P was the only such example given in [1]. We demonstrate that there are infinitely many good but not great cores in this section. A nontrivial core H is *fair* if the following properties hold.

(F1) The graph H is symmetric (that is vertex- and edge-transitive).
(F2) For all $e \in E(H)$, $H - e$ is a core, and $\chi(H) = \chi(H - e)$.
(F3) For all distinct $e, f \in E(H)$, the graph $H - \{e, f\}$ formed by deleting e and f from H admits a non-onto homomorphism into H.

It can be shown that P is fair. For two graphs G and H, $G + H$ is the graph formed by adding all edges between disjoint copies of G and H. By the results of [2], if G and H are cores, then so is $G + H$.

Let G and H be graphs and let $f \in \mathrm{Hom}(G, H)$ be onto. If G is weakly uniquely H-colourable, then η_f is a bijection and $G/\ker(f)$ is isomorphic to a spanning subgraph of H. The main result of this section is the following.

Theorem 3. *If H is a fair core, then each of H and the cores $H + K_n$, for $n \geq 1$, are good but not great.*

Proof. To show that H is not great, we argue by contradiction. For a fixed $e = ab \in E(H)$ assume that the homomorphism $f : H - e \to H$ is not onto. We find a non-onto endomorphism of $H - e$, contradicting (F2). Fix $x \notin Im(f)$. By vertex-transitivity there is an automorphism g of H so that $g(x) = a$. Then $a \notin Im(gf)$. Hence, $gf : H - e \to H$ has an image which induces a subgraph A of $H - a$. Note that $A \leq H - e$; let i be the inclusion homomorphism from A into $H - e$. Then igf is a non-onto homomorphism of $H - e$ into $H - e$.

We show that H is good. Let G be a weakly uniquely H-colourable graph, and let $f, h \in \mathrm{Hom}(G, H)$. Then there is a $g \in \mathrm{Sym}(V(H))$ so that $f = gh$. We know that $G/\ker(f) = G/\ker(h)$.

Case (1) η_f or n_h is an isomorphism.

We consider when n_h is an isomorphism; the case for n_f is similar and so is omitted. We show $g \in \mathrm{Aut}(H)$; as H is a core, it is enough to show that $g \in \mathrm{End}(H)$. To see this, fix $xy \in E(H)$. Then by assumption there are $a \in h^{-1}(x)$ and $b \in h^{-1}(y)$ so that $ab \in E(G)$. Hence, $f(a)f(b) \in E(H)$, as f is a homomorphism, but $f(a) = g(h(a)) = g(x)$ and $f(b) = g(y)$. But then $g(x)g(y) \in E(H)$.

Case (2) $G/\ker(f)$ is isomorphic via η_f to $H - e$, for some $e \in E(H)$, and $G/\ker(h)$ is isomorphic via η_h to $H - e'$, for some $e' \in E(H)$.

As H is edge-transitive by (F1) there is an $\alpha \in \mathrm{Aut}(H)$ so that $e = \alpha(e')$; in particular, α is an isomorphism from $H - e'$ to $H - e$. The following facts hold.

(1) $f, \alpha h \in \mathrm{Hom}(G, H - e)$ and αh is onto $H - e$;
(2) $\ker(\alpha h) = \ker(h) = \ker(f)$.

By fact 2 there is a $g' \in \mathrm{Sym}(V(H - e))$ so that

$$f = g'\alpha h. \tag{3.1}$$

We first show that $g' \in \mathrm{Aut}(H - e)$. As $H - e$ is a core by (F2), it is enough to show that $g' \in \mathrm{End}(H - e)$. Let $xy \in E(H - e)$. Then $\alpha^{-1}(x)\alpha^{-1}(y) \in E(H - e')$ as α is an isomorphism. But then by the hypotheses of Case 2, $h^{-1}\alpha^{-1}(x)h^{-1}\alpha^{-1}(y) \in E(G/\ker(h))$

which is equivalent to $(\alpha h)^{-1}(x)(\alpha h)^{-1}(y) \in E(G/\ker(h))$. Let $a \in (\alpha h)^{-1}(x)$ and $b \in (\alpha h)^{-1}(y)$ be chosen so that $ab \in E(G)$. As $f \in \mathrm{Hom}(G, H - e)$, $f(a)f(b) \in E(H - e)$. By (3.1) $f(a) = g'(x)$, $f(b) = g'(y)$ so that $g'(x)g'(y) \in E(H - e)$.

We now claim that $g' \in \mathrm{Aut}(H)$. Let $e = ab$. As $g' \in \mathrm{Aut}(H - e)$, g' preserves degrees. Since H is vertex-transitive, it is regular say of degree d. But then a and b are the only vertices of $H - e$ with degree $d - 1$. Hence, in $H - e$, $\{g'(a), g'(b)\} = \{a, b\}$. But then g' preserves edges in H: the edge ab is preserved, and edges of the form xy, where $\{x, y\} \neq \{a, b\}$, are preserved (as $g' \in \mathrm{Aut}(H - e)$). Hence, $g' \in \mathrm{End}(H) = \mathrm{Aut}(H)$. But then $f = (g'\alpha)h$, where $g'\alpha \in \mathrm{Aut}(H)$.

Case (3) $H/\ker(f)$ is isomorphic to $H - S$, where S is a set of edges, $|S| \geq 2$.

In this case, G is not even weakly uniquely H-colourable. To see this, note that by assumption, $f \in \mathrm{Hom}(G, H - S)$. By (F3), there is a non-surjective homomorphism $k : H - S \to H$. But then $kf \in \mathrm{Hom}(G, H)$ is non-surjective. This contradiction finishes the proof that H is good.

Fix $n \geq 1$. We show that $J = H + K_n$ is good but not great. To see that J is not great, since H is not great, there is an edge $e = ab \in E(H)$ so that each element of $\mathrm{Hom}(H - e, H)$ is onto. Fix $f \in \mathrm{Hom}(J - e, J)$; we show f is onto. To the contrary, assume that there is an $x \notin Im(f)$. If $x \in V(H)$, then we use the vertex-transitivity of H to find $g \in \mathrm{Aut}(H)$ so that $g(x) = a$. Note that $g' = (g \cup id_{K_n}) \in \mathrm{Aut}(J)$ and $a \notin Im(g'f)$. Now using an argument parallel to the one which showed that H is not great, we can find an endomorphism of $J - e$ which is not onto, contradicting the fact that $J - e$ is a core by (F2) (note that $J - e \cong (H - e) + K_n$). If $x \in V(K_n)$, then f is a homomorphism of $J - e$ into $H + K_{n-1}$. But then $\chi(H - e) + n = \chi(J - e) \leq \chi(H + K_{n-1}) = \chi(H) + n - 1$, contradicting the fact that $\chi(H - e) = \chi(H)$ (by (F2)).

To show J is good, we use the same case analysis as in the proof that H is good. We use the fact that if $g \in \mathrm{Aut}(H)$ then $(g \cup id_{K_n}) \in \mathrm{Aut}(J)$.

Case 1 is similar to Case 1 for H. In Case 2 the first subcase is if e or e' are not in H (we use the notation as in the argument above that H is good). Then we can find a non-surjective homomorphism from G into J, which contradicts that G is weakly uniquely J-colourable. (Observe that if we delete an edge between H and K_n, or in K_n, the resulting graph is not a core.) The second subcase occurs when $e, e' \in E(H)$. This subcase is related to the argument for H. Define $\alpha \in \mathrm{Aut}(H)$ as before, and let $\alpha' = (\alpha \cup id_{K_n}) \in \mathrm{Aut}(J)$. Then α' is an isomorphism from $J - e'$ to $J - e$. There is a $g' \in \mathrm{Sym}(V(J - e))$ so that $f = g'\alpha'h$. The same argument as before shows that $g' \in \mathrm{Aut}(J - e)$. We argue that $g' \in \mathrm{Aut}(J)$. Recall that H is d-regular, so $d \leq |V(H)| - 1$, and if $d = |V(H)| - 1$ then H would be a complete graph, which is impossible since no complete graph is fair. Therefore, $d < |V(H)| - 1$. If $e = ab$, then $\deg(a) = \deg(b) = d - 1 + n$. If $x \in V(H) \setminus \{a, b\}$, then $\deg(x) = d + n$. If $y \in V(K_n)$, then $\deg(y) = |V(H)| + n - 1 > d - 1 + n$. Hence, a, b are the only vertices of $J - e$ of degree $d - 1 + n$, so $\{g'(a), g'(b)\} = \{a, b\}$. The rest of the argument now runs parallel to the argument in Case 2 in the proof that H is good. Case 3 fails in a similar fashion to the first subcase of Case 2 and so is omitted. □

As P is fair, by Theorem 3, each of the graphs $P + K_n$ are good but not great cores. Theorem 3 helps demonstrate that the classification of the good cores is far from complete. For example, we do not know if the Kneser graphs are good.

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