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Mutually embeddable graphs and the tree alternative conjecture ☆

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Abstract

We prove that if a rayless tree T is mutually embeddable and non-isomorphic with another rayless tree, then T is mutually embeddable and non-isomorphic with infinitely many rayless trees. The proof relies on a fixed element theorem of Halin, which states that every rayless tree has either a vertex or an edge that is fixed by every self-embedding. We state a conjecture that proposes an extension of our result to all trees.

Keywords: Rayless tree; Mutually embeddable; Self-embedding

1. Introduction

A graph G *embeds* in a graph H if G is isomorphic to an induced subgraph of H. If G and H are graphs, then we write $G \leqslant H$ if G embeds in H. We write $G \sim H$ if $G \leqslant H$ and $H \leqslant G$, and we say that G and H are *mutually embeddable*.

Mutually embeddable finite graphs are necessarily isomorphic, but this is no longer the case for infinite graphs. For example, if the graph G is a disjoint union of cliques, one of each finite cardinality, then G is mutually embeddable with the graph consisting of a disjoint union of cliques with every even cardinality. In [1], we give many examples of mutually embeddable non-

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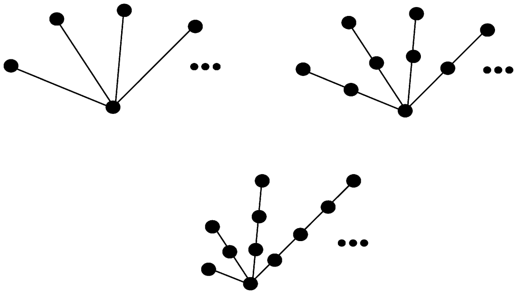

Fig. 1. Examples of countably infinite trees T with $m(T) = 1$, $\aleph_0$, and $2^{\aleph_0}$, respectively.

isomorphic graphs satisfying strong structural properties. On the other hand, the infinite two-way path is not mutually embeddable with any graph not isomorphic to it.

Define $ME(G)$ to be the set of isomorphism types of graphs H so that $G \sim H$. Define the cardinal $m(G) = |ME(G)|$. Note that $|ME(G)| \leqslant 2^{|V(G)|}$, so that $m(G)$ is well-defined. For instance, with $|V(G)| = \aleph_0$ (that is, the cardinality of the set of natural numbers), there are examples of graphs where $m(G)$ is one of 1, $\aleph_0$, or $2^{\aleph_0}$. See Fig. 1. As first stated in [1], we do not know of any example with $m(G)$ finite but larger than 1. The structure of such graphs may prove to be intriguing if they exist.

If G and H are mutually embeddable, then composing an embedding of G into H with an embedding of H into G gives a self-embedding of G. Thus, the structure of the monoid of self-embeddings of G may help us to determine the value of $m(G)$. A tree is *rayless* if it does not embed an infinite path. For example, each tree in Fig. 1 is rayless. Self-embeddings, automorphisms, and various fixed element properties of rayless trees have been well-studied; for example, see [2–4,6]. Using such properties we are able to prove the following result for rayless trees, and we in fact conjecture an extension to all trees.

Theorem 1. *If T is a rayless tree, then $m(T)$ is* 1 *or infinite.*

Tree Alternative Conjecture. *If T is a tree, then $m(T)$ is* 1 *or infinite.*

The rest of the paper is organized as follows. In Section 2, we prove a version of the Tree Alternative Conjecture for rooted rayless trees; see Theorem 2. In the final section we use a fixed element theorem of Halin's to derive Theorem 1 from Theorem 2. This suggests that if for all graphs G we have that $m(G) = 1$ or $m(G) \geqslant \aleph_0$, then a proof may use interesting fixed element properties of graphs.

All the graphs we consider are undirected and simple. If graphs G and H are isomorphic, then we write $G \cong H$. We use the notation of [5] for graph theory. We work within ZFC; no additional set-theoretic axioms will be assumed. The set of natural numbers, considered as an ordinal, will be written as ω.

2. Mutually embeddability of rooted rayless trees

The class of *rooted rayless trees* consists of all pairs (T, r), where T is a rayless tree and r is some fixed vertex of T called the *root* of T. An *embedding of rooted trees* $f : (T, r) \to (T', r')$ is an embedding of T into T' so that $f(r) = r'$; we write $(T, r) \leqslant (T', r')$. An *isomorphism of* *rooted trees* is a bijective embedding of rooted trees. If there is an isomorphism of rooted trees (T, r) and (T', r'), then we write $(T, r) \cong (T', r')$. The cardinal $m(T, r)$ is defined in the obvious way. The main goal of this section is to prove the following theorem.

Theorem 2. *If (T, r) is a rayless rooted tree, then $m(T, r)$ is either* 1 *or is infinite.*

Before we give a proof of Theorem 2, we first introduce the following notation that will simplify matters. Let $\{(T_i, r_i): i \in I\}$ be a family of rayless rooted trees, and let r be a vertex not in $V(T_i)$, for all $i \in I$. Define

$$\sum_{i\in I}(T_i, r_i)$$

to be the rooted tree (T, r) which has as its root the vertex r, so that r is joined to each root r_i of T_i, for all $i \in I$. We say that (T, r) is the *sum* of the (T_i, r_i), and each (T_i, r_i) is a *summand* of (T, r).

Note that if (T, r) is a rooted tree, then

$$(T, r) = \sum_{i\in I}(T_i, r_i),$$

where the summands T_i are the connected components of $T - r$, and r_i is the unique vertex of T_i joined to r. Further, this representation of (T, r) is unique, up to a permutation of the summands. Clearly, (T, r) is rayless if and only if each summand of (T, r) is rayless.

If

$$f: \sum_{i\in I}(T_i, r_i) \to \sum_{j\in J}(T_j, r_j)$$

is an embedding, then f induces an injection from I into J, written $\hat{f}$, defined so that if $i \in I$, $\hat{f}(i)$ is the unique $j \in J$ such that $f(T_i, r_i)$ is contained in (T_j, r_j) (hence, we assume that $f(r_i) = r_j$). If f is an isomorphism, then $\hat{f}$ is a bijection.

We next prove two lemmas about rooted trees that will be used in the proof of Theorem 2.

Lemma 1. *Let*

$$(T, r) = \sum_{i\in I}(T_i, r_i)$$

be a rooted tree such that for some $k \in I$, $m(T_k, r_k) = \alpha \geqslant \aleph_0$. Then $m(T, r) \geqslant \alpha$.

Proof. Let

$$ME(T_k, r_k) = \big\{(T_{k,n}, r_{k,n}): n \in \alpha\big\}.$$

Define the rayless rooted tree

$$(T_n, r) = \sum_{i\in I}(X_i, x_i),$$

where

$$(X_i, x_i) = \begin{cases} (T_i, r_i) & \text{if } (T_i, r_i) \nsim (T_k, r_k); \\ (T_{k,n}, r_{k,n}) & \text{if } (T_i, r_i) \sim (T_k, r_k). \end{cases}$$

Note that for all $n \in \alpha$, we have $(T_n, r) \sim (T, r)$ since $(X_i, x_i) \sim (T_i, r_i)$ for all $i \in I$. For $m \neq n$, $(T_m, r) \ncong (T_n, r)$, since (T_n, r) contains summands of the form $(T_{k,n}, r_{k,n})$ and (T_m, r) does not. Hence, the family $\{(T_n, r)\colon n \in \alpha\}$ is a witness to $m(T, r) \geqslant \alpha$. $\square$

Lemma 2. *Let*

$$(T, r) = \sum_{i \in I} (T_i, r_i)$$

be a rooted tree such that $m(T_i, r_i) = 1$, *for every* $i \in I$. *Then* $m(T, r) = 1$ *or* $m(T, r) \geqslant \aleph_0$.

Proof. Suppose for a contradiction that $1 < m(T, r) < \aleph_0$, and let

$$(T', r') = \sum_{j \in J} (T'_j, r'_j)$$

be a rooted tree not isomorphic to (T, r) such that $(T', r') \sim (T, r)$. Fix embeddings $f : (T, r) \to (T', r')$, $g : (T', r') \to (T, r)$, and consider the injections $\hat{f} : I \to J$, $\hat{g} : J \to I$. We first prove the following claim.

Claim. *If for some* $j^* \in J$, *the summand* (T'_{j^*}, r'_{j^*}) *of* (T', r') *is not isomorphic as a rooted tree to any summand of* (T, r), *then* $m(T, r) \geqslant \aleph_0$.

Proof. Without loss of generality, we may assume that $I \cap \omega = \emptyset$. For an integer $n \geqslant 1$, define $L_n = \{1, \ldots, n\}$, and let $L_0 = \emptyset$. Define for $n \in \omega$

$$(S_n, r) = \sum_{i \in (I \cup L_n)} (X_i, x_i),$$

where $(X_i, x_i) = (T_i, r_i)$ if $i \in I$, and $(X_i, x_i) = (T'_{j^*}, r'_{j^*})$ if $i \in L_n$. Then $(S_0, r) = (T, r)$ and $(S_n, r) \leqslant (S_{n+1}, r)$ for all $n \geqslant 0$. We show that $(S_{n+1}, r) \leqslant (S_n, r)$.

Let $I' = \{i_k\colon k \in \omega\}$ be the subset of I defined by $i_0 = \hat{g}(j^*)$, and $i_m = \hat{g}\hat{f}(i_{m-1})$ for $m \geqslant 1$. If for some $m > 0$ we have that $i_m = i_0$, then g and $f(gf)^{m-1}$ induce mutual embeddings between the non-isomorphic rooted trees (T'_{j^*}, r'_{j^*}) and $(T_{\hat{g}(j^*)}, r_{\hat{g}(j^*)})$, contradicting the hypothesis that $m(T_{\hat{g}(j^*)}, r_{\hat{g}(j^*)}) = 1$. Thus, $i_m \neq i_0$ for all $m > 0$, and since $\hat{g}\hat{f} : I \to I$ is injective, we have that $i_m \neq i_{m'}$ for all $m \neq m'$. Therefore, we can combine the restriction of gf to

$$\sum_{i \in I'} (T_i, r_i)$$

with the restriction of g to $(X_{n+1}, x_{n+1}) = (T'_{j^*}, r'_{j^*})$, and the identity on the remainder of (S_{n+1}, r) to obtain an embedding of (S_{n+1}, r) in (S_n, r). Thus, we have that $(S_n, r) \sim (S_0, r) = (T, r)$ for all $n \geqslant 0$. Since (S_n, r) contains exactly n summands isomorphic to (T'_{j^*}, r'_{j^*}), the rooted trees (S_n, r), $n \in \omega$, are pairwise non-isomorphic. The claim follows. $\square$

Consider the set $\{(X_k, x_k)\colon k \in K\}$ of isomorphism types of the rooted trees (T_i, r_i), and let $p : I \to K$ be the surjection defined by $(T_i, r_i) \cong (X_{p(i)}, x_{p(i)})$. By the claim, there is a map $q : J \to K$ such that $(T'_j, r'_j) \cong (X_{q(j)}, x_{q(j)})$, for all $j \in J$. Therefore, $m(T'_j, r'_j) = 1$, for all $j \in J$. If q were not surjective, then reversing the role of (T, r) and (T', r'), the claim would give that

$$m(T, r) = m(T', r') \geqslant \aleph_0.$$

Thus, q is surjective since $1 < m(T,r) < \aleph_0$ by assumption.

We have that

$$(T,r) = \sum_{i \in I} (X_{p(i)}, x_{p(i)})$$

and

$$(T',r') = \sum_{j \in J} (X_{q(j)}, x_{q(j)}).$$

Since these two rooted trees are not isomorphic, there exists some $k \in K$ such that $|p^{-1}(k)| \neq |q^{-1}(k)|$. Without loss of generality, we may assume that $|p^{-1}(k)| < |q^{-1}(k)|$. Define

$$(T'',r) = \sum_{i \in (I \setminus p^{-1}(k))} (T_i, r_i).$$

We will show that $(T'',r) \sim (T,r)$.

Define

$$J_0 = \{ j \in J\colon\ q(j) = k \text{ and } p(\hat{g}(j)) \neq k \}.$$

Since $|p^{-1}(k)| < |q^{-1}(k)|$, we have that $J_0 \neq \emptyset$. We define the sets $I_0 = \hat{g}(J_0) \subseteq I$ and $I_m = \hat{g}\hat{f}(I_{m-1}) \subseteq I$ for $m \geqslant 1$. Let

$$I' = \bigcup_{i \in \omega} I_m.$$

By reasoning similar to that given in the proof of the claim, we have that $I_m \cap I_{m'} = \emptyset$, whenever $m \neq m'$. Sequences of composition of the maps f and g demonstrate that $(X_k, x_k) \leqslant (T_i, r_i)$ whenever $i \in I'$. Moreover for some m, we have that

$$\left| \bigcup_{0 \leqslant j \leqslant m-1} I_j \right| \geqslant |p^{-1}(k)|.$$

Indeed if $|p^{-1}(k)| < \aleph_0$ we can put $m = |p^{-1}(k)|$, and if $|p^{-1}(k)| \geqslant \aleph_0$, then since $|q^{-1}(k) \setminus J_0| \leqslant |p^{-1}(k)|$, we have $|J_0| = |q^{-1}(k)|$ whence

$$|I_0| = |J_0| = |q^{-1}(k)| > |p^{-1}(k)|.$$

For this integer m, define

$$I'' = \bigcup_{0 \leqslant j \leqslant m-1} I_j$$

and fix an injection $\phi : p^{-1}(k) \to I''$. We may then combine embeddings $h_i : (T_i, r_i) \to (T_{\phi(i)}, r_{\phi(i)})$, where $i \in p^{-1}(k)$, with the restriction of $(gf)^m$ to

$$\sum_{i \in I'} (T_i, r_i)$$

and the identity on the remainder of (T,r) to define an embedding of (T,r) in (T'',r). Since $(T'',r) \leqslant (T,r)$, we then have $(T'',r) \sim (T,r)$. However, since (T,r) has summands isomorphic to (X_k, x_k) and (T'',r) does not, the claim applied to (T'',r) gives that $m(T,r) = m(T'',r) \geqslant \aleph_0$, which contradicts our assumption that $m(T,r) < \aleph_0$. $\square$

With Lemmas 1 and 2 in hand, we now turn to the proof of Theorem 2.

Proof of Theorem 2. Suppose for a contradiction that there exists a rooted rayless tree (T, r) such that $1 < m(T, r) < \aleph_0$. By Lemmas 1 and 2, there is some summand (T_1, r_1) of (T, r) satisfying $m(T_1, r_1) \in (1, \aleph_0)$. By repeated application of Lemmas 1 and 2, we may recursively choose a sequence $((T_i, r_i)\colon i \in \omega)$, with $(T_0, r_0) = (T, r)$, and where (T_{i+1}, r_{i+1}) is a summand of (T_i, r_i) such that $m(T_{i+1}, r_{i+1}) \in (1, \aleph_0)$. But then the path in T beginning with r_0 and whose remaining vertices are the r_i constitutes a ray in T, which is a contradiction. □

3. Mutual embeddability of rayless trees

Define a *fixed vertex* u of a graph G to be one with the property that for all self-embeddings f of G, $f(u) = u$. Define a *fixed edge* uv of G to be one with the property that for all self-embeddings of G, $\{f(u), f(v)\} = \{u, v\}$. The following "fixed element" theorem was first proved by Halin [2], and will be used in the proof of Theorem 1.

Theorem 3. *If T is a rayless tree, then there is either a vertex or an edge fixed by every self-embedding of T.*

Note that the maps that we refer to as *self-embeddings* are referred to as *endomorphisms* in [2].

Proof of Theorem 1. Suppose that $m(T) \geqslant 2$. By Theorem 3, there exists a fixed vertex u or a fixed edge $e = uv$ of T. Consider the rooted tree (T, u). We will use Theorems 2 and 3 to prove that in both cases we have that:

(a) $m(T, u) \geqslant \aleph_0$.
(b) If $\{(T_i, u_i)\colon i \in \omega\}$ is a family of pairwise non-isomorphic rooted trees mutually embeddable with (T, u), then $\{T_i\colon i \in \omega\}$ is a family of rayless trees mutually embeddable with T, with the additional property that for all $i \in \omega$, there is *at most* one $j \in \omega$ such that $i \neq j$ and $T_i \cong T_j$.

Once items (a) and (b) are proven, it will follow that $m(T)$ is infinite, and our proof of Theorem 1 will be concluded.

To prove item (a), we argue as follows. As $m(T) \geqslant 2$, let T' be a rayless tree that is non-isomorphic and mutually embeddable with T. Then there exist embeddings $f : T \to T'$ and $g : T' \to T$. If $gf(u) = u$, then f and g act as mutual embeddings between the non-isomorphic rooted trees (T, u) and $(T', f(u))$. Hence, $m(T, u) \geqslant \aleph_0$ by Theorem 2.

Otherwise, since gf is a self-embedding of T and $gf(u) \neq u$, we are dealing with the case where uv is an edge fixed by all self-embeddings of T, where $gf(u) = v$ and $gf(v) = u$. Therefore, f and gfg act as mutual embeddings between the two rooted trees (T, u) and $(T', f(u))$, which again implies that $m(T) \geqslant \aleph_0$.

We prove item (b) by contradiction, assuming that there are distinct $i, j, k \in \omega$ such that there exist isomorphisms $h_{ij} : T_i \to T_j$ and $h_{ik} : T_i \to T_k$. Since (T_i, u_i), (T_j, u_j), and (T_k, u_k) are mutually embeddable with (T, u), there exist embeddings $f_i : T \to T_i$, $g_j : T_j \to T$, and $g_k : T_k \to T$ such that

$$f_i(u) = u_i, \qquad g_j(u_j) = u, \qquad g_k(u_k) = u. \tag{1}$$

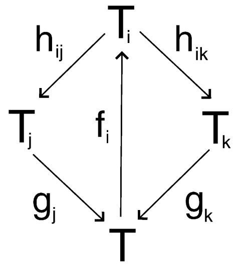

Fig. 2. Maps in the proof of Theorem 1.

See Fig. 2.

Since (T_i, u_i), (T_j, u_j), and (T_k, u_k) are pairwise non-isomorphic as rooted trees, we have that $h_{ij}(u_i) \neq u_j$ and $h_{ik}(u_i) \neq u_k$. This implies by (1) that $g_j h_{ij} f_i(u) \neq u$, and that $g_k h_{ik} f_i(u) \neq u$. Therefore, we are in the case when uv is a fixed edge of T, and both self-embeddings $g_j h_{ij} f_i$ and $g_k h_{ik} f_i$ interchange u and v. Hence,

$$g_j h_{ij} f_i(v) = u, \qquad g_k h_{ik} f_i(v) = u. \tag{2}$$

Equations (1) and (2) imply that

$$h_{ij}\big(f_i(v)\big) = {g_j}^{-1}(u) = u_j, \qquad h_{ik}\big(f_i(v)\big) = {g_k}^{-1}(u) = u_k. \tag{3}$$

Equations (1)–(3) together imply that

$$h_{ik} {h_{ij}}^{-1}(u_j) = u_k.$$

Therefore, $h_{ik} {h_{ij}}^{-1}$ is an isomorphism from T_j to T_k which maps u_j to u_k, contradicting the fact that (T_j, u_j) and (T_k, u_k) are non-isomorphic as rooted trees. □

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