A NOTE ON ORIENTATIONS OF THE INFINITE RANDOM GRAPH

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Abstract. We answer a question of P. Cameron’s by giving examples of $2^{\aleph_0}$ many non-isomorphic acyclic orientations of the infinite random graph with a topological ordering that do not have the pigeonhole property. Our examples also embed each countable linear ordering.

A graph is $n$-existentially closed or $n$-e.c. if for each $n$-subset $S$ of vertices, and each subset $T$ of $S$ (possibly empty), there is a vertex not in $S$, joined to each vertex of $T$ and no vertex of $S \setminus T$. The infinite random graph, written $R$, is the unique (up to isomorphism) countable graph that is $n$-e.c. for all $n \geq 1$. For more on the infinite random graph, the reader is directed to [2, 3].

The infinite random graph is intimately related to a certain vertex partition property. A graph $G$ has the pigeonhole property, written $(\mathcal{P})$, if for every partition of the vertices of $G$ into two nonempty parts, the subgraph induced by some one of the parts is isomorphic to $G$. This property was introduced by P. Cameron in [2], who in [3] classified the countable graphs with $(\mathcal{P})$; there are only four up to isomorphism: the graph with one vertex, the countably infinite clique and its complement, and $R$. In particular, $R$ is the unique countable 1-e.c. graph that has $(\mathcal{P})$. The pigeonhole property may be easily generalized to any relational structure. The countable tournaments with $(\mathcal{P})$ were classified in [1]; there are $\aleph_1$ many: the countable ordinal powers of $\omega$ and their reversals, and the countably infinite random tournament. A problem of [1] that has resisted solution is the following.

Problem 1: Classify the countable oriented graphs with $(\mathcal{P})$.

As proved in [1], a countable oriented graph with $(\mathcal{P})$ that is neither a tournament nor the infinite random oriented graph $O$, must be an orientation of $R$. Cameron [4] was the first to notice that any such
orientation must be acyclic (that is, contains no directed cycles), have infinitely many sources or infinitely many sinks, and admits a homomorphism into a countable ordinal.

A topological order of the vertices of an oriented acyclic graph \( D = (V, E) \) is a linear order \( \preceq \) on \( V \) such that if \((x, y) \in E\), then \( x \preceq y \). As an intermediate step towards the solution of Problem 1, Cameron [4] posed the following problem.

**Problem 2:** Are there \( 2^{\aleph_0} \) (that is, cardinality of the real numbers) many non-isomorphic acyclic orientations of the infinite random graph with a topological ordering that do not have \((P)\)?

We say that an orientation of \( R \) as described in Problem 2 is bad.

The goal of this short note is to answer Problem 2 affirmatively. We actually prove a stronger assertion, as stated in the following theorem, which we think is of interest in its own right.

**Theorem 1.** There are \( 2^{\aleph_0} \) many non-isomorphic bad orientations of \( R \) that embed each countable linear ordering.

We consider only countable simple graphs and oriented graphs, which we refer to as orgraphs. Directed edges are written \((x, y)\) and we say that \( x \) is joined to \( y \) and \( y \) is joined from \( x \). If \((x, y)\) is a directed edge in an orgraph, then we forbid \((y, x)\) to be a directed edge. If \( G \) is a graph or orgraph, then \( V(G) \) is the set of vertices of \( G \); the set \( E(G) \) are the edges of \( G \) if \( G \) is a graph, and the directed edges of \( G \) if \( G \) is an orgraph. If \( B \subseteq V(G) \), then we write \( G \upharpoonright B \) for the subgraph or suborgraph induced by \( B \); if \( H \) is an induced subgraph or suborgraph of \( G \), then we write \( H \leq G \). We write \( G \cong H \) if \( G \) and \( H \) are isomorphic. We say that \( G \) embeds in \( H \) if \( G \) is isomorphic to an induced subgraph or suborgraph of \( H \). If \( D \) is an orgraph, then the graph of \( D \), written \( G(D) \), is the graph with vertices \( V(D) \) and with edge set the symmetric closure of \( E(D) \). A vertex \( u \) is a source in \( D \) if for every \( v \in V(D) \), we have that \((v, u) \notin E(D)\); a sink is defined dually.

**Proof of Theorem 1.** For each integer \( i \geq 3 \), let \( L_i \) be the \( i \)-vertex linear order, and let \( \Phi = \{L_i : i \geq 3\} \). Fix \( X \), an infinite, co-infinite subset of \( \Phi \), and enumerate \( X = \{L_{i_1}, L_{i_2}, \ldots\} \) and \( \Phi \setminus X = \{L_{j_1}, L_{j_2}, \ldots\} \).

Define an orgraph \( D(X) \) inductively, as follows. As we proceed with the induction, each vertex will be assigned exactly one colour, red or blue. Once a vertex has been assigned a colour, it will have that colour throughout the induction. If an induced suborgraph \( H \) has each of its vertices red (blue), then we say that \( H \) is red (blue). The orgraph \( D(X)_1 \) consists of two disjoint vertices with no directed edges between
them, with one vertex red and the other blue. The suborgraph $Blue(1)$ is this blue vertex. Assume that $D(X)_n$ is defined, finite, and if $n \geq 2$, then $D(X)_{n-1} \leq D(X)_n$. We will also assume that there is at least one blue vertex in $D(X)_n$, and the suborgraph $Blue(n)$ induced by the set of all the blue vertices of $D(X)_n$ is a finite linear order. The orgraph $D(X)_{n+1}$ is defined in four stages.

If $M$ is a linear order, then a vertex is called special for $M$ if $x$ is joined to the first and last vertices of $M$ only, and $x$ is not joined from any vertex of $M$. For each red copy $L$ of any one of $L_{i_1}, L_{i_2}, \ldots, L_{i_n}$ in $D(X)_n$ whose initial vertex is not a source in $D(X)_n$, add a new red vertex $x_L$ that is special for $L$. The resulting orgraph is called $D(X)'_{n+1}$.

For each pair $u, v \in Blue(n)$ so that $u < v$ in the linear order $Blue(n)$, add a new blue vertex $b_{uv}$ so that $u < b_{uv} < v$. The vertices $b_{uv}$ are not directed to or from any red vertex. Observe that there are $\binom{|Blue(n)|}{2}$ new blue vertices of the form $b_{uv}$ added at this stage. After these blue vertices are added, add a new blue vertex $a$ strictly less all the blue vertices, and a new blue vertex $z$ strictly greater than all the blue vertices; the vertices $a$ and $z$ are not joined to or from any red vertex. The orgraph with these additional blue vertices $Blue(n + 1) = \{a, z\} \cup \{b_{uv} : u, v \in Blue(n), u < v\}$ is called $D(X)'_{n+1, blue}$.

To $D(X)'_{n+1, blue}$, add a disjoint red copy of $L_{i_{n+1}}$ with an additional red vertex $x_{n+1}$ that is special for $L_{i_{n+1}}$, and add a disjoint red copy of $L_{j_{n+1}}$ that has an additional red vertex $y_{n+1}$ that is joined to the initial vertex of $L_{j_{n+1}}$ only and is not joined from any vertex of $L_{j_{n+1}}$. The resulting orgraph is called $D(X)''_{n+1}$.

Consider the finite undirected graph $G = G(D(X)''_{n+1})$. For each subset $S$ of $V = V(G)$, and each subset $T$ of $S$ (possibly empty), add a new red vertex which is joined only to vertices of $T$. Orient the edges so that if a vertex $x \notin V$ is joined to $y \in V$, then $(y, x)$ is a directed edge (in other words, edges are directed out of $G$). Give $G$ back its orientation from $D(X)''_{n+1}$. This new orgraph is called $D(X)_{n+1}$. Observe that $Blue(n + 1) \leq D(X)_{n+1}$ forms a linear order.

Define

$$D(X) = \bigcup_{n \geq 1} D(X)_n.$$ 

Hence, $D(X)$ is the union of the chain $D(X)_1 \leq D(X)_2 \leq D(X)_3 \leq \ldots$.

It follows that $H = G(D(X)) \cong R$, since $H$ is n-e.c. for all $n \geq 1$. To see this, fix $S \subseteq V(H)$ and $T$ a subset of $S$. Then $S \subseteq V(D(X)_m)$ for
some positive integer $m$. A vertex not in $S$ joined to vertices of $T$ but not $S \setminus T$ may be found in $V(D(X)_{m+1})$ (by construction of $D(X)_{m+1}$).

Let $Blue(X)$ be the suborgraph induced by the blue vertices of $D(X)$. The orgraph $Blue(X)$ is then the union of the chain $Blue(1) \leq Blue(2) \leq Blue(3) \leq \ldots$ and is a linear order. It is not hard to see that $Blue(X)$ is dense and has no endpoints. Therefore, $Blue(X)$ is isomorphic to the order type of the rational numbers (see Theorem 2.8 of [5]). From the fact that the order type of the rational numbers embeds each countable linear order (see Theorem 2.5 of [5]), it follows that $D(X)$ embeds each countable linear order.

We prove that $D(X)$ is acyclic by induction on $n$. Clearly, $D(X)_1$ is acyclic. Assume that $D(X)_n$ is acyclic. In $D(X)_{n+1}$, the addition of no vertex $x_L$ creates a directed cycle. No blue vertex added in $D(X)_{n+1, blue}$ is joined to or from a red vertex; further, $Blue(n+1)$ is a linear order and hence, acyclic. Therefore, there is no directed cycle in $D(X)_{n+1, blue}$. In $D(X)_{n+1}$, the addition of neither $x_{n+1}$ nor $y_{n+1}$ creates a directed cycle. Since each $z$ added in $V(D(X)_{n+1}) \setminus V(D(X)_{n+1})$ is a sink in $D(X)_{n+1}$, there are no directed cycles in $D(X)_{n+1}$.

Define the subset $A$ of the vertices of $D(X)$ to be all the “primed vertices”; in particular, $A_1$ is the unique red vertex of $D(X)_1$ that we name $v$, and $A_n = V(D(X)_n) \setminus V(D(X)_{n-1})$ with $A = \bigcup_{n \geq 1} A_n$. The set $A$ is stable, that is, there are no directed edges between vertices of $A$. To see this, we prove the following stronger claim.

Claim 1: The vertex $v$, and each vertex $x_L$ added in at stage $D(X)_n$, for each $n > 1$, are sources in $D(X)$.

To prove the claim, we prove by induction on $s$ that $v$ is a source in each $D(X)_s$ for $s \geq 1$, and each vertex $x_L$ is a source in $D(X)_s$ for every $s \geq n$. Since the arguments used for the cases of $v$ and $x_L$ are similar, we consider only the argument for $x_L$.

If $s = n$, there is nothing to prove. Assume that $x_L$ is a source in $D(X)_s$. If there is a vertex joined to $x_L$ in $D(X)_s$, then this vertex is of the form $x_L'$, where $L'$ is a linear order in $D(X)_s$ whose initial vertex is not a source. But then $x_L$ is a vertex of $L'$ and so must be the initial vertex of $L'$, which gives a contradiction. Clearly, $x_L$ is a source in $D(X)_{n+1, blue}$, $D(X)_{n+1}$, and $D(X)_{n+1}$.

For $n \geq 2$, define $B_n$ to be the red vertices in $V(D(X)_n) \setminus V(D(X)_{n-1})$, and let $B = \bigcup_{n \geq 2} B_n$. Then $A$, $Blue(X)$, and $B$ forms a partition of $V(D(X))$. We linearly order the vertices of $D(X)$ by first enumerating $A$ with $v$ as the first vertex, then listing the vertices of $A_2$ in some linear order, and then listing the vertices of $A_3$ in some order, and so on. Observe that each $A_i$ is finite. Now include $Blue(X)$ in the linear
order, respecting the linear ordering of $Blue(X)$. Hence, each vertex of $A$ is less every vertex of $Blue(X)$. Next, list the vertices of $B$ by first listing $B_1, B_2, \ldots$. Within each $B_n$, list first $x_n$, and then list vertices of $L_{in}$, respecting the linear order of $L_{in}$. Next list $y_n$ and then the vertices of $L_{jn}$, respecting the linear order of $L_{jn}$. Finally, list the vertices of $V(D(X)_n) \backslash V(D(X)_n^u)$ in any way. Hence, each vertex of $A \cup Blue(X)$ is less than every vertex of $B$. Name this linear ordering of $V(D(X))$ by $L = (V(D(X)), \preceq)$.

The linear order $L$ is topological. To see this, note that this holds for vertices in $A$, since there are no directed edges between vertices of $A$. Since $Blue(X)$ is itself a linear order, the ordering there is topological. In $B$, the topological property is satisfied by our choice of ordering of the vertices of $B$. As the vertices of $A$ are all sources in $D(X)$, and since $A$ forms an initial segment of $L$, we need only check the topological property for directed edges $(c, d)$, where either $c \in Blue(X)$ and $d \in B$ or $c \in B$ and $d \in Blue(X)$. However, since there are no directed edges $(c, d)$ with $c$ red and $d$ blue, this follows immediately.

The linear order $L$ has order type $\omega + \eta + \omega$, where $\omega$ is the order type of the natural numbers, and $\eta$ is the order type of the rational numbers. We now prove the following claim.

Claim 2: The orgraph $D(X)$ does not have $(P)$.

Let $L \in X$ be fixed, and let $L'$ be the copy of $L$ added in disjointly to $D(X)_m^u$ for some $m$. Recall that in $D(X)_m^u$, the vertex $x_m$ is special for $L'$. Let the initial vertex of $L'$ be named $a'$. Let $C = \{x \in B \setminus \{x_m\} : x \preceq a'\}$. Observe that $C$ is finite, since $B$ has order type $\omega$ in $L$. Furthermore, all the vertices of $D(X)$ different from $x_m$ which are joined to $a'$ are in $A \cup C$.

Let $A' = A \cup C \cup Blue(X)$, and let $B' = V(D(X)) \setminus A'$. The orgraph $A'$ cannot be isomorphic to $D(X)$. Otherwise, $G = G(D(X) \upharpoonright (A'))$ would be isomorphic to $R$. Since $A$ is a stable set and $Blue(X)$ is a linear order, $G$ consists of the union of an infinite empty graph (the subgraph induced by the vertices of $A$), an infinite complete graph (the subgraph induced by the vertices of $Blue(X)$) and a finite graph (the subgraph induced by the vertices of $C$). But then $G$ does not have property $(P)$, which is a contradiction.

We show that $B'' = D(X) \upharpoonright B'$ cannot be isomorphic to $D(X)$ using the back-and-forth game or method, which in our case is a two player game of perfect information played in countably many steps on two countable orgraphs $D_0$ and $D_1$. The players are named the duplicator and the spoiler. (The names come from the facts that the duplicator is trying to show the structures are alike, while the spoiler is trying to
show they are different.) A move consists of a choice of a vertex from either structure, and the spoiler makes the first move. The players take turns choosing vertices from the $V(D_i)$, so that if one player chooses a vertex from $V(D_i)$, the other must choose a vertex of $V(D_{i+1})$ (the indices are mod 2). Players cannot choose previously chosen vertices. After $n$ rounds, this gives rise to a list of vertices $U_n = \{a_i : 1 \leq i \leq n\}$ from $D_0$ and $V_n = \{b_i : 1 \leq i \leq n\}$ from $D_1$. The duplicator wins if for every $n \geq 1$, the suborgraph induced by $U_n$ is isomorphic to the suborgraph induced by $V_n$. Otherwise, the spoiler wins. From this it follows that the duplicator has a winning strategy if and only if $D_0$ and $D_1$ are isomorphic. See [2] for more on the back-and-forth method.

Now the spoiler chooses in $B''$ the vertex $x_m$ and the vertices of $L'$ in succession. The duplicator must respond with $|V(L')|+1$ corresponding vertices in $D(X)$ that give rise to a linear order $L''$ and a vertex $x''_m$ which is joined to the first vertex of $L''$, which we name $\alpha$. Since $(x''_m, \alpha)$ is a directed edge in $D(X)$, $\alpha$ is a not a source and so there is a $z \in V(D(X)) \setminus \{x''_m\}$ that is joined to $\alpha$. The spoiler can win in the next round by choosing $z$. To see this, note that the duplicator cannot now choose an appropriate vertex of $B''$, since the spoiler already has chosen all the vertices of $B''$ which are joined to $\alpha$. Claim 2 follows.

It is not hard to show that there are $2^{\aleph_0}$ many distinct infinite, co-infinite subsets of the natural numbers. To finish the proof of the theorem, we use this fact in conjunction with the following claim.

**Claim 3:** If $X \neq Y$, then $D(X) \not\cong D(Y)$.

Without loss of generality, there is an $L_i \in X \setminus Y$; name the first vertex of $L_i$ $a$ and the last vertex $z$. In $D(X)$, for every copy of $L_i$ so that $a$ is not a source, there is a vertex that is joined to both $a$ and $z$. This is clear for the red vertices by construction, and for the blue vertices by the fact that $Blue(X)$ is a dense linear order without endpoints.

We show that this property fails for $D(Y)$, which will prove Claim 3. Consider a fixed copy of $L_i$ added at some stage $D(Y)'_{m+1}$. Since $(y_m, a)$ is a directed edge, the vertex $a$ is not a source. However, there is no vertex $x' \in V(D(Y))$ which is joined to both $a$ and $z$, although there is in $D(X)$. If there were such an $x'$ in $D(Y)$, then $x'$ must be a vertex added in at some stage $D(Y)'_{i+1}$. But the vertices of $D(Y)'_{i+1}$ not among the vertices of $D(Y)_r$ are special only for linear orders isomorphic to those in $Y$. Hence, if $(x', a)$ is a directed edge, then $(x', z)$ is not a directed edge, otherwise, $L_i \in Y$. (We are tacitly using here the fact that there is, up to isomorphism, exactly one linear order on $n$ vertices, if $n$ is a positive integer.) □
REFERENCES


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