International Conference on Mathematical and Computational Models
PSG College of Technology, Coimbatore

CATCH ME IF YOU CAN: COPS AND ROBBERS ON GRAPHS

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Abstract

Vertex pursuit games are widely studied by both graph theorists and computer scientists. Cops and Robbers is a vertex pursuit game played on a graph, where some set of agents (or Cops) attempts to capture a robber. The cop number is the minimum number of cops needed to win. While cop number of a graph has been studied for over 25 years, it is not well understood, and has few connections with existing graph parameters. In this survey, we highlight some of the main results on bounding the cop number, and discuss Meyniel's conjecture on an upper bound for the cop number of connected graphs. We include a new proof of the fact that outerplanar graphs have cop number at most two.

Key Words: *graphs, Cops and Robbers, cop number, vertex pursuit games, outerplanar graphs, planar graphs, Meyniel's conjecture*

1. INTRODUCTION

Graph searching is an active research area in graph theory and theoretical computer science. At their core, graph searching games model capturing intruders in a network, and usually involve optimizing the number of agents needed for capture. Cops and Robbers is one aspect of graph searching that has received much recent attention. In this two-player game of perfect information, a set of cops tries to capture a robber by moving at unit speed from vertex-to-vertex. More precisely, *Cops and Robbers* is a game played on a reflexive graph (that is, there is a loop at each vertex). There are two players consisting of a set of *cops* and a single *robber*. The game is played over a sequence of discrete time-steps or *rounds*, with the cops going first in round 0 and then playing alternate time-steps. The cops and robber occupy vertices; for simplicity, the player is identified with the vertex they occupy.

The author acknowledges support from NSERC, MITACS, and Ryerson University.

The set of cops is referred to as C and the robber as R. When a player is ready to move in a round they must move to a neighbouring vertex. Because of the loops, players can *pass*, or remain on their own vertex. Observe that any subset of C may move in a given round.

The cops win if after some finite number of rounds, one of them can occupy the same vertex as the robber (in a reflexive graph, this is equivalent to the cop landing on the robber). This is called a *capture*. The robber wins if he can evade capture indefinitely. A *winning strategy for the cops* is a set of rules that if followed, result in a win for the cops. A *winning strategy for the robber* is defined analogously.

If a cop is placed at each vertex, then the cops are guaranteed to win. Therefore, the minimum number of cops required to win in a graph G is a well-defined positive integer, named the *cop number* (or *copnumber*) of the graph G. The notation $c(G)$ is used for the cop number of a graph G. If $c(G) = k$, then G is *k-cop-win*. In the special case $k = 1$, G is *cop-win* (or *copwin*). For example, the reader can check directly that the Petersen graph is 3-cop-win. As was recently shown in [6], the Petersen is the unique isomorphism type of 3-cop-win graph with the smallest possible order.

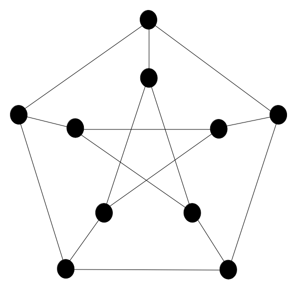

Figure 1. The Petersen graph.

The game of Cops and Robbers was first considered by Quilliot [20] in his doctoral thesis, and was independently considered by Nowakowski and Winkler [19]. Both [19,20] refer only to one cop. The introduction of the cop number came in 1984 with Aigner and Fromme [1]. Many papers have now been written on cop number since these three early works; see the surveys [2,14] and the recent book of Bonato and Nowakowski [8]. Cops and Robbers has even found recent application in artificial intelligence and so-called *moving target search*; see Isaza et al. [15] and Moldenhauer et al. [18].

The goal of this brief survey is to highlight some of the work on finding bounds on the cop number. Computing the cop number of a graph is **NP**-hard [11], and so finding bounds becomes a worthwhile endeavour. We focus on upper bounds and Meyniel's conjecture in Section 2. The cop number of planar graphs and outerplanar graphs are discussed in Section 3. A new proof is given that the cop number of an outerplanar graph is at most two. We finish with some open problems in the final section.

The graphs we consider are all finite, reflexive (as stated earlier) and undirected, and with no multiple edges. See [25] for further background on graph theory.

2. BOUNDS ON THE COP NUMBER AND MEYNIEL'S CONJECTURE

The *closed neighbor set* of a vertex x, written $N[x]$, is the set of vertices joined to x (including x itself). A vertex u is a *corner* if there is some vertex v such that $N[u] \subseteq N[v]$. A graph is *dismantlable* if some sequence of deleting corners results in the trivial graph K_1. For example, each tree is dismantlable: delete end-vertices repeatedly until a single vertex remains. The same approach holds with chordal graphs, which always contains at least two simplicial vertices (that is, vertices whose neighbor sets are cliques). The following result characterizes cop-win graphs.

Theorem 2.1. [19,20] A graph is cop-win if and only if it is dismantlable.

Cop-win (or dismantlable) graphs have a recursive structure, made explicit in the following sense. Observe that a graph is dismantlable if the vertices can be labeled by positive integers $[n] = \{1,2,\ldots, n\}$ in such a way that for each $i < n$, the vertex i is a corner in the subgraph induced by $\{i,i+1,\ldots, n\}$. This ordering of $V(G)$ a *cop-win ordering*. See Figure 2 for a graph with vertices labeled by a cop-win ordering.

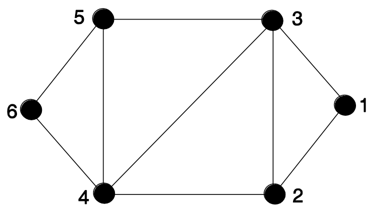

Figure 2. A cop-win ordering of a cop-win graph.

Cop-win orderings are sometimes called *elimination orderings*, as vertices from lower to higher index are deleted until only vertex n remains.

Let H be an induced subgraph of G formed by deleting one vertex. The graph H is a *retract* of G if there is a homomorphism f from G onto H so that $f(x) = x$ for $x \in V(H)$; that is, f is the identity on H. The map f is called a *retraction* (or 1-*point retraction*). For example, the subgraph formed by deleting an end-vertex is a retract. If u is a corner dominated by v, then the mapping sending u to v and fixing everything else is a retraction (recall that our graphs are reflexive, so edges may map to a single vertex).

Retracts play an important role in characterizing cop-win graphs. The next theorem, due to Berarducci and Intrigila, shows that the cop number of a retract never increases.

Theorem 2.2. [7] If H is a retract of G, then $c(H) \leq c(G)$.

While the proof of Theorem 2.2 is elementary, it gives good insight into the cop number especially for cop-win graphs. For example, as an immediate corollary, a retract of a cop-win graph is also cop-win.

2.1 Upper bounds

An elementary upper bound for the cop number is $c(G) \le \gamma(G)$, where $\gamma(G)$ is the domination number of G. In general graphs, this inequality is far from tight (consider a path, for example). We do not know how large the cop number of a connected graph can be as a function of its order. For n a positive integer, let $c(n)$ be the maximum value of $c(G)$, where G is of order n. *Meyniel's conjecture* states that

$$c(n) = O\left(\sqrt{n}\right).$$

The conjecture was mentioned in Frankl's paper [12] as a personal communication to him by Henri Meyniel in 1985 (see page 301 of [12] and reference [8] in that paper). Meyniel's conjecture stands out as one of the deepest on the cop number.

For many years, the best known upper bound for general graphs was the one proved by Frankl [12].

Theorem 2.3. [12] If n is a positive integer, then

$$c(n) = O\left(n\frac{\log\log n}{\log n}\right).$$

The key to proving Theorem 2.3 is the notion of cops guarding an isometric path. For a fixed positive integer k, an induced subgraph H of G is *k-guardable* if, after finitely many moves, k cops can move only in the vertices of H in such a way that if the robber moves into H at round t, then he will be captured at round t+1. For example, a clique or a closed neighbor set (that is, a vertex along with its neighbors) in a graph are 1-guardable.

Given a connected graph G, the *distance* between vertices u and v in G, is denoted $d_G(u,v)$. A path P in G is *isometric* if for all vertices u and v of P,

$$d_P(u,v) = d_G(u,v).$$

The following theorem of Aigner and Fromme [1] on guarding isometric paths has found a number of applications.

Theorem 2.4. [1] An isometric path is 1-guardable.

In 2008 Chinifooroshan [9] gave an improved upper bound once again using a guarding argument.

Theorem 2.4. [9] For n a positive integer

$$c(n) = O\left(\frac{n}{\log n}\right).$$

The bound in Theorem 2.4 therefore, represents the first important step forward in proving Meyniel's conjecture in over 25 years. The key to proving this bound comes again from the notion of guarding an induced subgraph.

An improvement exists to the bound in Theorem 2.4. The following theorem was proved independently by three sets of authors. We note that each proof uses the probabilistic method.

Theorem 2.5. [13,17,23] For n a positive integer

$$c(n) = O\left(\frac{n}{2^{(1-o(1))\sqrt{\log_2 n}}}\right).$$

The bound in Theorem 2.5 is currently the best upper bound for general graphs that is known, but it is still far from proving Meyniel's conjecture or even the soft version of the conjecture. Meyniel's conjecture was recently proved for random graphs [22]. See the survey [6] for further background on the conjecture.

2.2. Lower bounds

A useful theorem of Aigner and Fromme [1] is the following. The *girth* of a graph is the length of minimum order cycle. The *minimum degree* of G is written $\delta(G)$.

Theorem 2.6. [1] If G has girth at least 5, then $c(G) \geq \delta(G)$.

Frankl [12] proved the following theorem generalizing Theorem 2.6 (which is the case $t = 1$).

Theorem 2.7. [12] For a fixed integer $t \geq 1$, if G has girth at least $8t-3$ and $\delta(G) > d$, then

$$c(G) > \delta(G)^t.$$

Meyniel's conjecture states that the cop number is at most approximately $\sqrt{n}$. For graphs with large cop number near the conjectured bound, consider projective planes. A *projective plane* consists of a set of points and lines satisfying the following axioms.

1. There is exactly one line incident with every pair of distinct points.
2. There is exactly one point incident with every pair of distinct lines.
3. There are four points such that no line is incident with more than two of them.

Finite projective planes have $q^2 + q + 1$ points for some integer $q > 0$ (called the *order* of the plane). All known projective planes have prime power order, and these are conjectured to be the only permissible orders. For a given projective plane P, define $G(P)$ to be the bipartite graph with red vertices the points of P, and the blue vertices represent the lines. Vertices of different colors are joined if they are incident. This is the *incidence graph* of P. See Figure 3 for the incidence graph of the well-known *Fano plane*: the unique projective plane of order 2.

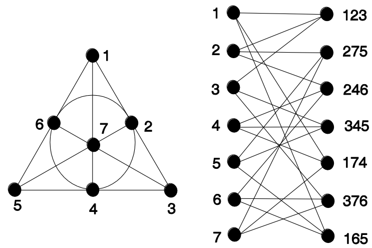

Figure 3. The Fano plane and its incidence graph.

Theorem 2.6 proves for a projective plane P that $c(G(P)) \geq q+1$. As proven in Prałat [21], $c(G(P)) = q+1$. Using a result from number theory, it was shown in Prałat [21] that for sufficiently large n,

$$c(n) \geq \sqrt{\frac{n}{2} - n^{0.2625}}.$$

3. PLANAR AND OUTERPLANAR GRAPHS

Planar graphs have inspired some of the deepest results in graph theory, most notably the Four Color Theorem (which states that every planar graph has chromatic number at most 4; see [5]. A graph is *planar* if it can be drawn in the plane without any two of its edges crossing. For example, a cycle is planar, and so is K_4. The graph K_5 is not planar, and neither is the complete bipartite graph $K_{3,3}$. As Kuratowski proved in [16], K_5 and $K_{3,3}$ are, in a certain sense, the only obstructions to being non-planar. We *subdivide* an edge uv by replacing it by a two-path uxv, where x is a new vertex of degree 2. A *subdivision* of a graph results by subdividing some subset of its edges.

Theorem 3.1. [16] A graph is planar if and only if it does not contain a subgraph which is a subdivision of K_5 or $K_{3,3}$.

At first glance, there is no reason to believe that a fixed constant number of cops can guard every planar graph. However, Aigner and Fromme [1] showed in fact that planar graphs require no more than three cops (and that some actually require three; see Figure 4).

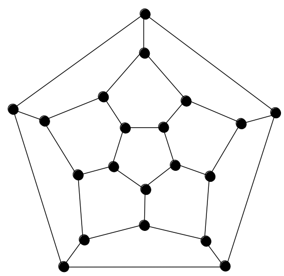

Figure 4. The dodecahedron is planar with cop number 3.

Theorem 3.2. [1] If G is a planar graph, then $c(G) \leq 3$.

The proof of Theorem 3.2, is much more involved than in the outerplanar case which we discuss below, making heavy use of isometric paths and Theorem 2.4. The original proof in [1], while correct, was not entirely readable. A new, more readable proof of Theorem 2.4 may be found in Chapter 4 of [8]. We note that Andreae generalized the result on planar graphs by proving that the cop number of a K_5-minor-free graph (or $K_{3,3}$ -minor-free graph) is at most 3 (see also [4]).

A graph G is *outerplanar* if it has an embedding in the plane with the following properties.

1. Every vertex lies on a circle.
2. Every edge of G either joins two consecutive vertices around the circle or is a chord across the circle.
3. If two chords intersect, then they do so at a vertex.

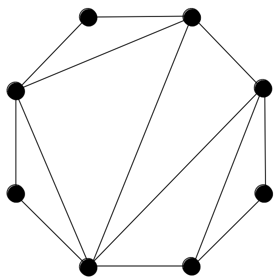

Figure 5. An outerplanar graph.

Often the edges are drawn on the outside of the circle of vertices but it is equivalent to have them on the inside. See Figure 5. We will label the vertices clockwise around the circle v_0, $v_1,\ldots,v_{n-1}$. Outerplanar graphs are of course planar, and have a characterization analogous to Kuratowski's Theorem 3.1: a graph is outerplanar if and only if it does not contain a subgraph which is a subdivision of K_4 or $K_{2,3}$.

Nancy Clarke proved the next result in her doctoral thesis.

Theorem 3.3. [10] If G be an outerplanar graph, then $c(G) \leq 2$.

Since a cycle is an outerplanar graph, not all outerplanar graphs are cop-win, so we need an algorithm to show that two cops suffice. We give a new proof of this fact below, which appears in Chapter 4 of [8].

Proof of Theorem 3.3. Assume first that G has no cut vertices. Suppose that for a given i that v_i is not adjacent to v_{i+1}. We can renumber the subscripts so that $i = 0$. Since G is connected and the degree of v_0 is at least two, then let v_j be the vertex of least index which is adjacent to v_0. The edge v_0v_j prevents any vertex in $\{v_1,\ldots,v_{j-1}\}$ from being adjacent to any vertex of the set $\{v_{j+1}, v_{j+2},\ldots,v_{n-1}\}$ and no vertex of $\{v_1, v_2,\ldots,v_{j-1}\}$ is adjacent to v_0.Therefore, v_j is a cut vertex, which is a contradiction. Hence, we may assume that for all i, v_i is adjacent to both v_{i-1} and v_{i+1}, with subscripts taken modulo n.

If the embedding contains no chords, then it is a cycle and two cops suffice to capture the robber. Let $a_0, a_1, \ldots, a_k$ be the vertices of degree at least 3 in order around the circle. Note that vertices on the cycle between a_i and a_{i+1} are of degree 2 and so *the path* between a_i and a_{i+1} is well-defined.

Place the cops C_1 and C_2 on the vertex v_0. If v_0 has degree 2, then it is on a path between a_0 and a_k (renumbering the a_i if necessary); if it has degree 3 or more, then renumber so that $a_0 = v_0$. In either case, we can move C_1 to a_0 and C_2 to a_k so that the robber is not on the path between these two vertices.

We now assume more generally that C_1 and C_2 are on a_i and a_j for some $i < j$, respectively, and that the robber is not on and cannot move to any vertex in $\{v_p, v_{p+1}, \ldots, v_0, \ldots, v_q\}$, where $v_p = a_j$ and $v_q = a_i$. That is, every path from the robber to a vertex in $\{v_p, v_{p+1}, \ldots, v_0, \ldots, v_q\}$ passes through a_i or a_j. Such an area is called *the cop territory*. The idea of the proof now is to show that the cops can increase the cop territory, so that it is eventually all of G and the cops win.

Suppose a_i has a chord to a vertex in the robber territory. Let a_r be a vertex adjacent to a_i which is closest to a_j. If the robber is on the arc of the circle from a_i to a_r, then he cannot move off that arc if C_1 on a_i does not move. Therefore, C_2 can be moved to a_r and the cop territory has increased. If the robber is between a_j and a_r, then C_1 moves from a_i to a_r again increasing the cop territory. A similar analysis holds for a_j. Hence, the only case to consider is when neither a_i nor a_j have an interior edge to the robber territory. In this case, the only paths to the cop territory from the robber are the ones along the cycle incident to a_i and a_j. That is, every path from the robber to the cop territory passes through a_{i+1} or a_j. Hence, moving C_1 along the path from a_i to a_{i+1} does not allow the robber to move into the cop territory, and the cop territory has increased. We will refer to this as the *no-cut-vertex strategy*.

Now suppose that G has at least one cut vertex. Let $B(G) = \{G_1, G_2, \ldots, G_m\}$ be the set of maximal induced subgraphs of G such that each G_i itself has no cut vertices. Note that each G_i will contain a vertex which is a cut vertex of G, and each G_i has at least two vertices. For example, see Figure 6.

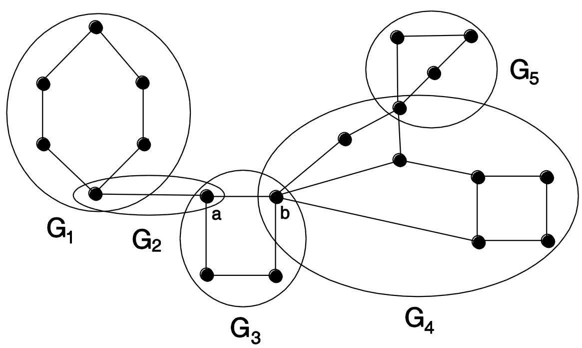

Figure 6. The induced subgraphs G_i in an outerplanar graph with cut vertices.

We can retract G onto G_i, for any i, by the mapping described as follows. Let $x \in V(G_i)$ and x be a cut vertex of G. All vertices of G that are disconnected from G_i by the deletion of x are mapped to x. Vertices of G_i are mapped to themselves. (Consider G_3 in Figure 6. Vertices of G_1 and G_2 are mapped to a and those of G_4 and G_5 are mapped to b.) Let G_i' denote this retract. Since G_i is a subgraph of an outerplanar graph, G_i' is also outerplanar. Fix an embedding for each G_i.

Choose some G_i and place the two cops on two vertices in G_i as in the case with no cut vertices. Employing the strategy of the case with no cut vertices, the cops will capture the robber's image on G_i'. Since $|V(G_i)| \geq 2$, at least one more vertex and one more element of $B(G)$ is added to the cop territory. If the robber is actually on G_i, then he has been caught. If not, then the cops have captured the robber's shadow on a vertex x whose deletion separates G_i from the G_k where the robber presently resides. This cut vertex also lies in some G_j that either contains the robber (that is, $j = k$) or contains a cut vertex y distinct from x whose deletion separates G_j from G_k (and j is unique). Fix an outerplanar embedding of G_j'. The cops now execute the no-cut-vertex strategy on G_j'. Hence, the cops eliminate the subgraphs in $B(G)$, and eventually they capture the robber (rather than just his image). □

A graph is *series-parallel* if it has no subgraph isomorphic to a subdivision of K_4. Hence, every outerplanar graph is series-parallel. In [24], Theorem 3.3 was generalized to show that every series-parallel graph has cop number at most 2.

4. FUTURE DIRECTIONS

The most difficult open problem now surrounding the cop number is Meyniel's conjecture. Even proving that for some $\varepsilon > 0$,

$$c(n) = O(n^{1-\varepsilon})$$

is open. Define m_k to be the minimum order of a connected graph G satisfying $c(G) \geq k$. As shown in [6], proving Meyniel's conjecture is equivalent to proving that

$$m_k = \Omega(k^2).$$

Our understanding of the minimum orders of graphs with a given cop number is therefore intimately tied with Meyniel's conjecture. We only know three values: $m_1 = 1$, $m_2 = 2$, and $m_3 = 3$. As mentioned in the introduction, the Petersen graph is 3-cop-win. Using a computer search, it was found in [6] that the Petersen graph is the unique isomorphism type of minimum order 3-cop-win graph. Hence, It would be nice to find a proof of this fact without recourse to a computer search.

The cop number of digraphs is defined analogously as in the undirected case (for oriented edges, players simply follow the direction of the edges). Surprisingly, we have no characterization for cop-win digraphs analogous to the one of Theorem 2.1 for graphs.

Finally, we do not have a characterization of the cop-win planar graphs (recall that from Theorem 3.1, planar graphs have cop number 1, 2, or 3). Finding such a characterization of this graph class is a tantalizing open problem.

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